Some notes on bobs-only Grandsire Triples

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Abstract. This paper presents a proof of W H Thompson's Theorem from 1886, that there is no bobs-only extent of 5040 Grandsire Triples, as background to a criticism of his argument that there is no bobs-only touch of Grandsire Triples of length greater than 4998. An example is given showing why his argument is incorrect. This opens up the possibility that there is such a touch, of length 5012 or 5026, and also the possibility that there is a more complex, but correct, argument for the non-existence of such a touch. Beyond simple notions of oddness and evenness of permutations, no group theory is assumed or used; but the results are easily interpreted as statements about certain generators of the alternating group Alt(6).

1 Introduction

Our main object is to show that an argument of W. H. Thompson from 1886 about long bobs-only[1] touches of Grandsire Triples is incorrect; as background we briefly prove his main result, that a bobs-only extent of 5040 Grandsire Triples is impossible. Wilson [13] is a good reference for campanological terms not defined here.

The main argument follows Thompson's paper [12], but is more mathematical and with fewer definitions, since some of Thompson's are now standard. Background information on parity (aka "nature", i.e. oddness or evenness) of permutations and full disjoint cycle notation[2] for permutations is assumed. No group theory is assumed or used. The following elementary result is essential:

Lemma. Let θ be a permutation of $\{1, 2, \ldots, n\}$, expressed in full disjoint cycle notation using m cycles. Then θ is an even[3] permutation iff the number[4] $n - m$ is even.

Proof. The cycles of even order make an even contribution to n; so the number of cycles of odd order is odd if and only if n is odd. These odd order cycles however are even permutations. Those of even order are odd permutations; their number determines the parity of θ. **QED**

An example may help: take $n = 5$, and consider the two permutations (1)(23)(45) and (1234)(5), respectively even and odd. The first has $m = 3$ and so $n - m = 2$, an even number; the second has $m = 2$, and so $n - m = 3$, an odd number.

Using full disjoint cycle notation, this gives a simple way to determine the parity of a permutation. Note that, as we are just interested in the parity, we can use $n + m$ rather than $n - m$, since $2m$ is even.

Full disjoint cycle notation can be interpreted in two ways. For example, the permutation (1)(23)(45)(6)(7) could mean "leave 1, 6 and 7 fixed, interchange the numbers 2 and 3, and also interchange the numbers 4 and 5". Or it could mean, after adopting a standard representation with the things being permuted in order, "leave the first thing and the last two things where they are, interchange the items in positions 2 and 3, and also those in positions 4 and 5". In the very special case where the representation is 1234567, the two approaches yield the same answer; but if, after some permutations, the representation is now 1752634, then the first interpretation yields 1743625 and the second, less easy to do in one's head, yields 1576234. We will use only the second method of interpretation. We talk of "positions" rather than "places" to avoid ambiguity; methods are in part made up of bells "making places".

From now on we are only concerned with Grandsire Triples.

By a *lead* we mean[5] the 14 rows from the backstroke blow of the treble (i.e. bell 1) in position 1 to its next handstroke blow in position 1, inclusive[6].

[1] We'd prefer to say "single-free"; but the terminology is already fixed.

[2] "Full' is our terminology for the idea that cycles of length 1 are fully exploited, as in (1)(2)(34)(56)(7) rather than just (34)(56). "Disjoint" means that our representation avoids overlaps such as (345)(456).

[3] [12] says "in course".

[4] The *discriminant* of θ.

[5] [12] means the 14 rows from the treble's handstroke blow in position 2 to its next backstroke blow in position 1, inclusive. The difference is inessential in what follows. In fact, two chapters of [12] adopt, for greater clarity and convenience, our preferred terminology. We are also avoiding the ambiguity that would be introduced by referring to position 1 as the "lead".

[6] For example, these 14 rows, starting from rounds, are 1234567, 2135476, 2314567, 3241657, 3426175, 4362715, 4637251, 6473521, 6745312, 7654132, 7561423, 5716243, 5172634, 1527364, with the last being 1576243 in the case of a Bob. .

We are concerned only with *lead heads*, i.e. with the rows that occur as the first rows in a lead that is reached without singles; since (with no singles) every change (from one row to the next) in Grandsire Triples involves three pairs of bells swapping, all of these lead heads are even permutations of the numbers 1 to 7, and all begin with 1.

It is an easy exercise to show that, provided a sequence of Plains and Bobs leads to no repetition in the lead heads, there is no repetition at all[7].

From now on *row* means "lead head".

Theorem 1. Any even permutation that fixes 1 gives us such a row, i.e. one that can be reached without using singles.

Proof. Consider an even permutation, e.g. 1732564. How can we reach it from rounds, i.e. 1234567, by Plains and Bobs? The first step is to get 7 into position 2; one Bob will suffice. Using zero or two Bobs, we can similarly ensure that 2 or 4 can be got into position 2. If we want to get one of 3, 5 or 6 there, we use Plains to move it to position 7 and then a Bob. Now a number (at most 4) of Plains will ensure we can get the right bell to position 3. The first three positions are now as they should be; there remain just 12 cases to deal with.

For clarity, and without loss of generality, we'll just deal with the cases where it is $1, 2$ and 3 that are already in position. One is trivial. Using the notation introduced below, the following touches do the job: PBBPBBPB (for 1234756), BPBBPBB (for 1234675), BPBBPBPPBPB (for 1235476), PBBPBPBB (for 1235647), PBPBPPPPB (for 1235764), BPPPBBPBPPP (for 1236457), BBPPPBB (for 1236574), PBPBPPBB (for 1236745), BBPPBPB (for 1237465), BPPB (for 1237546) and PBBPBPPPB (for 1237654). Each leaves the bells in positions 1, 2 and 3 unchanged and evenly permutes the other four bells, in one of 11 different ways. **QED**

Using longer sequences of calls, the case analysis here can easily be halved, because, for example, the calls to get from 1234567 to 1234756 will get from 1234756 to 1234675, and the calls to get from 1234567 to 1237546 will get from 1237546 to 1236574. There may be falsity, because of the preliminary Plains and Bobs; but that can just be cut out. In fact, at most 12 leads are required: 1732564 is an example, requiring[8] e.g. just PPPBBPBPPBPP.

Extending this exhaustive case analysis to Grandsire Cinques is left as an exercise for the over-enthusiastic reader; a better exercise is to find a simple argument not dependent on such a case analysis.

It is unnecessary, but perhaps helpful, to recall that a *plain course* consists[9] of the five rows

$$1234567, 1253746, 1275634, 1267453, 1246375$$

and a *bob course* consists of the three rows

$$1234567, 1752634, 1467352.$$

2 Operators

An *operator* is an even permutation of the numbers 1 to 7, leaving 1 fixed. We will express these only using full disjoint cycle notation; for example, we have the *Identity* $I = (1)(2)(3)(4)(5)(6)(7)$, the *Plain* operator $P = (1)(2)(34675)$ and the *Bob* operator $B = (1)(247)(365)$, which may be conveniently remembered by considering hours and days. There are 360 operators, since 360 is half of the number 720 of all (both even and odd) permutations of the six numbers $2, \ldots, 7$.

Operators *act* on rows; we write the operator after the row (and then calculate). The operator $P = (1)(2)(34675)$ acts on a row x (giving us the row xP) as follows: the items in positions 1 and 2 stay put, that in position 3 moves to position 4, that in position 4 moves to position 6, etc, and that in position 5 moves

[7] Each lead contains 14 rows, which are alternately even and odd. Consider any even row other than one with the treble in position 1. Its even parity determines which part of a lead it occurs; for example, if the treble is in position 3, then it is the 12th row of the lead, and the permutations used to obtain it from the even lead head are exactly the same as those used for any other 12th row. So, if the lead heads differ, the 12th rows differ. A similar argument holds for odd rows.

There is one exception to this argument: it is the 14th row, the *lead end* where the treble has just moved to position 1, by means of a permutation fixing the bell either in position 7 or in position 3 (plain and bob respectively). But such a row is always related to the next row, the lead head, by the permutation (1)(23)(45)(67) that fixes the bell in position 1 and swaps others in pairs; so, if the lead heads differ, then the lead ends differ. Note that the argument doesn't work for Plain Bob Triples, where the two rows with the treble in position 1 are of the same parity. See also [1, 8.2].

[8] This result follows from an exhaustive search, using Prolog software available from the author.

[9] Augmented, of course, in each lead, by the other 13 changes of the lead.

to position 3. This corresponds to the cycle of work: make thirds, 5-4 down, 7-6 down, 6-7 up, 4-5 up, make thirds again, ...

Likewise, $B = (1)(247)(365)$ acts on a row x as follows: 1 stays put, the item in position 2 moves to position 4, the item in position 4 moves to position 7, the item in position 7 moves to position 2, the item in position 3 moves to position 6, etc, giving us the row xB.

For example, if x is the row 1234567, then xP is the row 1253746 and xB is the row 1752634. Likewise, if x is the row 1532467, then xP is the row 1543726 and xB is the row 1745632. When x is written out as a row, we write (e.g.) $1532467 \cdot P$, with a $\cdot$ for typographical clarity.

Operators may be *multiplied*: XY means first apply X, then apply Y, following the algebraic rather than the analytic convention in mathematics[10]. Thus, if x is a row, $x(XY) = (xX)Y$. We write XX as X^2, and similarly for other positive indices. X^0 just means the Identity operator, I. For example, $P^2 = PP = (1)(2)(36547)$ and $PB = (1)(2456)(37)$.

An operator X may be given a negative index; these are interpreted in the usual way, with $XX^{-1} = X^{-1}X = I$. For example, $P^{-1} = (1)(2)(35764)$ and $B^{-1} = (1)(274)(356)$. Note that $(XY)^{-1} = Y^{-1}X^{-1}$.

The *order* of an operator X is then the least index $k > 0$ for which $X^k = I$. I has order 1; P has order 5; B has order 3. Some easy mathematical facts are illustrative but not essential for the argument[11].

Another notation for operators just shows the result of the action of the operator on *rounds*, i.e. the row $r = 1234567$. For example, the result of the operator P on rounds is 1253746. We shall ignore this notation.

3 Round blocks and divisions

A *round block* is a repetition-free sequence of rows (so each begins with 1), each obtained from its predecessor by one use either of P or of B. The first row is regarded as immediately following the last row, i.e. the last row is regarded as the predecessor of the first row. In such a block, we say that a row x is *Plained* if it is the predecessor of xP, and *Bobbed* if it is the predecessor of xB. The *order* of the block is just the number of elements it contains. Two round blocks are regarded as *equal* if they contain the same rows in the same order, apart from a cyclic permutation of the order.

For example, the plain course displayed above, obtained just by using P, is a round block of order 5. Likewise, the displayed bob course is a round block of order 3, obtained just by using B. Longer examples may be obtained using any bobs-only touch and an arbitrary row as starting point. For example, the sequence $P, P, B, P, P, B, P, P, B, P, P, B$ gives such a touch[12] , and hence a round block of order 12. (It may be observed that many, if not most, touches of this Triples method use singles.)

A *division*[13] is a disjoint collection $\mathcal{D}$ of round blocks whose orders add up to 360; it distributes the 360 rows into round blocks. For example, one can have 72 blocks each made up with use only of P, or 120 blocks each made up with use only of B. A disjoint collection of round blocks with total order less than 360 may or may not be extendable to a division: see Section 7 below.

A bobs-only extent (if such a thing existed) would correspond, since $5040 = 360 \times 14$, to a division consisting of a single round block of order 360, ending with *rounds*, the row 1234567. But, we shall show that no such thing exists.

4 Q-sets and Q-cycles

We follow [12] by defining the operator Q as PB^{-1}; since we are writing operators on the right rather than on the left it is PB^{-1} rather than $B^{-1}P$. Note that $QB = P$. Since $P = (1)(2)(34675)$ and $B^{-1} = (1)(274)(356)$, we can calculate that $Q = (1)(27643)(5)$. Q has order 5, i.e. (for any row x) we have $xQ^5 = x$ (and this holds for no smaller positive index).

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10 Our choice is determined by the way in which a touch such as $PPBPPBPPBPPB$ begins on the left with P rather than on the right with B.

11 Such an order must be one of $1, 2, 3, 4$ and 5. For example, the operator $(1)(23)(45)(6)(7)$ has order 2; the operator $(1)(2345)(67)$ has order 4. Only the identity operator has order 1. The number of operators of order 2 is 45, since, using full disjoint cycle notation, they must be of the form $(1)(ab)(cd)(e)(f)$, where $1, a, b, c, d, e$ and f are distinct; this can be done in $(6 \times 5 \times 4 \times 3)/(2 \times 2 \times 2)$ distinct ways. The number of operators of order 3 is 80, since $40 = 6 \times 5 \times 4/3$ are of the form $(1)(abc)(d)(e)(f)$ and 40 are of the form $(1)(abc)(def)$. The number of operators of order 4 is 90, i.e. $6 \times 5 \times 4 \times 3/4$; they are of the form $(1)(abcd)(ef)$. The number of operators of order 5 is 144, i.e. $6 \times 5 \times 4 \times 3 \times 2/5$; they are of the form $(abcde)$. Note that $1 + 45 + 80 + 90 + 144 = 360$.

12 It is 1253746, 1275634, **1462375**, 1436527, 1453762, **1274653**, 1267345, 1236574, **1452736**, 1475623, 1467352, **1234567**, where rows just reached by a Bob are in bold font. Their immediate predecessors are those that are Bobbed.

13 There is no such terminology in [12].

The Q-*set* of a row x is the set $\{xQ^i : 0 \le i \le 4\}$. Two rows x and y are Q-*equivalent*, written $x \equiv_Q y$, iff for some i we have $x = yQ^i$; the relation of being Q-equivalent is an equivalence[14] relation, and it partitions the 360 rows into $72 = 360/5$ Q-sets. For example, the Q-set containing the row 1234567 is the set $\{1234567, 1346572, 1467523, 1672534, 1723546\}$.

A Q-*cycle* is a cyclic sequence $(xQ^i : 0 \le i \le 4)$, treated as an ordered set with each xQ^i immediately followed by xQ^{i+1} (and with the convention that x immediately follows xQ^4).

It is useful to have a canonical representative from each Q-set. Note that all elements of the Q-set have the same number in position 5, so we can begin by choosing that from the numbers from 2 to 7 inclusive. Let us call it P (for "Pivot"). Then, we consider the predecessor P' of P, e.g. if P is 3 we consider 2, etc, with the convention that 7 is the predecessor of 2. [Other choices are possible here.] Among the five members of the Q-set there is exactly one that has this number P' in position 4. This is the representative we choose. It can be abbreviated by the information consisting of P and, in order, the numbers in positions 2 and 3. Rounds is, as one would expect, the representative of the Q-set to which it belongs.

That done, we can represent any row by a pair consisting of the canonical representative of its Q-set and the number of applications of Q required to reach it from that representative. For example, 1765234 is thus represented by the pair $(1347265, 2)$, since $1347265 \cdot Q^2 = 1765234$.

Here is a systematic enumeration of twelve (those with 7 and 2 in positions 4 and 5) of the 72 canonical representatives of the 72 Q-sets:

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R = [1, 3, 4, 7, 2, 6, 5] ;
R = [1, 3, 5, 7, 2, 4, 6] ;
R = [1, 3, 6, 7, 2, 5, 4] ;
R = [1, 4, 3, 7, 2, 5, 6] ;
R = [1, 4, 5, 7, 2, 6, 3] ;
R = [1, 4, 6, 7, 2, 3, 5] ;
R = [1, 5, 3, 7, 2, 6, 4] ;
R = [1, 5, 4, 7, 2, 3, 6] ;
R = [1, 5, 6, 7, 2, 4, 3] ;
R = [1, 6, 3, 7, 2, 4, 5] ;
R = [1, 6, 4, 7, 2, 5, 3] ;
R = [1, 6, 5, 7, 2, 3, 4]
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5 Thompson's Theorem

Theorem 2. There is a natural correspondence between divisions and the maps from the set $\mathcal{Q}$ of Q-sets to the set $\{P, B\}$.

Proof. Let $\mathcal{D}$ be a division. Wherever it contains some round block R which contains a row x immediately followed by xP, the row x is plained in R, and thus is *plained* in D; since D contains all the 360 rows[15], it must also contain xQ, and if this is bobbed (either in R or in some other round block R'), then $(xQ)B$, i.e. $x(QB)$, i.e. xP, will immediately follow it—so xQ must be plained in D, lest the division $\mathcal{D}$ contain a repetition. So, for each x, the Q-set of x is either all plained in D or all bobbed in D, i.e. we have a map from $\mathcal{Q}$ to $\{P, B\}$.

Conversely, suppose we have such a map $f : \mathcal{Q} \to \{P, B\}$; we construct a division $\mathcal{D}_f$ as follows. Consider any row x; it is in exactly one Q-set Q_x, and then $f(Q_x)$ equals either P or B, which determines whether x should be plained or bobbed in D_f. If the plaining (by this method) of x to xP and the bobbing of y to yB lead to $xP = yB$, then $y = xP(B^{-1}) = x(PB^{-1}) = xQ$, so y and x are in the same Q-set, and so must be both plained or both bobbed by f, contrary to hypothesis; so, as required for construction of a division, there are no repetitions. All rows are covered by this approach, so we obtain a division. **QED**

Thus, so long as we are just interested in divisions, once the nature (plained or bobbed) of each of the 72 Q-sets (or just of their canonical representatives) is determined, the nature (plained or bobbed) of each row is determined. For example, if all Q-sets are plained, we obtain a division of $72 = 360/5 = 5040/70$ disjoint round blocks, each of order 5; if all are bobbed, then we obtain a division of $120 = 360/3 = 5040/42$ disjoint round blocks, each of order 3. Note that 72 is an even number; it is coincidental (but not useful) that it is also the number of Q-sets.

Theorem 3 [12]**.** Let $\mathcal{D}$ be a division. If the Q-sets $Q_1, \ldots, Q_n$ are bobbed in $\mathcal{D}$ and the Q-sets $Q_{n+1}, \ldots, Q_{72}$ are plained in $\mathcal{D}$, then the bobbing of Q_{n+1} alters the number of round blocks in $\mathcal{D}$ by an even number.

[14] I.e. it is reflexive, symmetric and transitive.
[15] I.e. the even lead heads.

Proof.[16] Label the round blocks in D as $R_1, \ldots, R_m$ and let $Q_{n+1} = \{x_0Q^0, x_0Q^1, x_0Q^2, x_0Q^3, x_0Q^4\}$, with $x_0Q^0 = x_0$. Each of the 5 rows in Q_{n+1} belongs to a round block: let R_{j_i} be the round block to which x_0Q^i belongs. There are at most 5 such round blocks, and the bobbing of Q_{n+1} doesn't affect any other round blocks, because if R is any of the other blocks, none of its elements has its operator changed, so none has its successor changed, and so R still forms a round block.

For each $i = 0, 1, 2, 3, 4$, define the *chain* C_i to be the sequence of rows obtained from x_0Q^i, up to and including the next $x_0Q^{k_i}$ in the same block. It does not include x_0Q^i itself. We have exactly five such chains: C_0, C_1, C_2, C_3 and C_4.

Before we change operators, the *successor* of chain C_i is C_{k_i}. The last element of the chain C_i is $x_0Q^{k_i}$: now bob the Q-set Q_{n+1}. Since $QB = P$, the first element of the new successor of C_i is $(x_0Q^{k_i})B = x_0(Q^{k_i}B) = x_0(Q^{k_i-1}P) = (x_0Q^{k_i-1})P$, which is the first element of the chain C_{k_i-1}.

So whereas C_{k_i} was the successor of C_i, now it is C_{k_i-1} that is the successor. Thus we have split the five blocks $B_{j_i} : i = 0, \ldots, 4$ into chains and rearranged the chains in a different order, making up blocks in (perhaps) a different way.

Define the permutation α of $\{1, 2, 3, 4, 5\}$ by taking i to k_i, and β by $\beta(i) = \alpha(i) - 1 \pmod 5$[17]. Then, after the bobbing, the chain $C_{\beta(i)}$ immediately follows[18] the chain C_i. Let γ be the permutation (54321), i.e. the operation on the numbers $0, \ldots, 4$ that subtracts 1; then $\beta = \alpha\gamma$ (first apply α, then apply γ). Note that γ is even, being a 5-cycle, so α and β are both even or both odd.

By the Lemma, twice, the total number of cycles in α and the total number of cycles in β are both even or both odd. These numbers correspond to the numbers of round blocks occupied by the Q-set, and thus the bobbing of the Q-set does not change the evenness or oddness of the total number of round blocks. **QED**

Corollary. [12] Let D be a division. Then its size is an even number.

Corollary. [12] There is no bobs-only extent of 5040 Grandsire Triples.

Wilson [13, p. 127] mentions Thompson's result (that there is no bobs-only extent), without proof, but offers a "simple way" to show that it is "unlikely" to be false: the common addition of extra courses to a plain course, by calling "a set of bobs on the same three bells", adds (he says) two courses to the plain course, and yet 72 (=5040/(5*14)) is even[19]. We do not regard this (even if a different touch is used) as remotely convincing. Still less convincing is the claim [13, p. 128] that the same reasoning applies to Stedman: indeed, it is now well-known, in the case of Stedman, to be false, i.e. there are bobs-only extents of Stedman Triples [10, 11] and [1, p. 276].

Fletcher [4][20] comments that

> "The beauty of the proof is marred by the fact that the number of round blocks lost or gained is always even is carried out by a long and tedious process of enumeration. But it is very difficult to see any means by which this could have been avoided. The enumeration of the cosets of a group of large order is inevitably tedious, and modern processes do not seem to offer any way of reducing Thompson's labours to any marked extent."

We believe that the proof just presented avoids the "long and tedious process" referred to.

Burbidge [1, p. 263] says that

> "What must have convinced those who set themselves against the possibility of a bobs only peal must surely have been the age old problem in composition, ..., that of being able to add only an even number of blocks to a piece using bobs, and therefore starting with one block you must always end up with an odd number of blocks, whereas the peal you seek is one of an even number of blocks. This is in essence why [an extent of] Grandsire Triples is not attainable by bobs only."

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[16] This was typewritten in about 1970 and typeset in 2017, by myself in each case. I forget whether the argument was original. The basic structure is that of Thompson in [12], but, by the use of chains, we avoid his case analysis. Rankin [8] (corrected in [9]) does the same, but with much heavier use of notation and of group theory. McGuire [6, 7] presents a simplified version of Rankin's argument. I have not seen any of the work of Davies, such as [3].

[17] Note that $0 - 1 = 4 \pmod 5$.

[18] For example, suppose that before the bobbing we have the two round blocks $C_1C_3C_5$ and C_2C_4. After the bobbing, we have the two round blocks $C_1C_2C_3C_4$ and C_5. So α is (in full cycle notation) (135)(24) and β is (1234)(5). In this case we haven't changed the number of round blocks. As another example, suppose that before the bobbing we have the five round blocks C_1, C_2, C_3, C_4 and C_5. After the bobbing we have just the round block $C_1C_5C_4C_3C_2$. So α is (in full cycle notation) (1)(2)(3)(4)(5) and β is (15432); we have reduced the number of round blocks by 4.

[19] The touch PPPBPPPBPPPB seems to be what he has in mind; after 4 leads the row is 1342567, and after another 4 it is 1423567. But the length of the touch is 2.4 times the length of a plain course, rather than 3 times the length.

[20] A paper marred by the statement that "A peal on four bells is called *Singles*"; no, he means "Minimus".

Note that in fact an extent (if it existed) would be of an odd (not an "even") number of blocks, and the starting position is of an even (not an "odd") number, namely 72, of plained blocks.

6 Long Bobs-only Touches of Grandsire Triples

Thompson [12] also argues that there is no bobs-only touch of Grandsire Triples of length greater than 4998 (i.e. 5040 less 3×14) changes. This may be true: no-one has ever found one, the longest found being 4998 changes [5]. His argument is that such a touch (of length 5012 or 5026) must make a round block (of length 358 or 359), and that the remaining rows must also make a round block; but, the shortest round block is a Bob course, of length 3, i.e. too long to fit.

We are not convinced by this argument. (We have an open mind about whether or not 4998 can be exceeded.) The problem is not that the touch must make a round block (that's obvious) but that the remainder must make one; this isn't obvious. (It is the case for Holt's touch of 4998 changes, the remainder being just a Bob course. But one example doesn't prove a general rule.)

The essence of Thompson's argument about an extent is that if one row x is followed by a Bob, (i.e. is 'Bobbed') then **all** the other four rows Q-equivalent to x must, to avoid falsity of the extent, be Bobbed; this then has predictable effects on the number of round blocks. But, now consider a row x in a **touch** and the four other rows xQ, xQ^2, xQ^3 and xQ^4 in the same Q-set. Can some of these rows be Plained and others be Bobbed, without making the touch false? Well, if some of them are Absent from the touch, yes. The following two simple conditions must hold:

1. If in a touch x occurs Plained, then xP occurs next in the touch; so, if $xQ = xPB^{-1}$ is in the touch it must be Plained.

2. If in a touch x occurs Bobbed, then xB occurs next in the touch; so, if $xQ^{-1} = xBP^{-1}$ is in the touch it must be Bobbed.

The second of these is in fact an immediate consequence of the first; they are equivalent. We'll re-present the first as "if, in a touch, the row x occurs Plained, then xQ is either Plained or Absent from the touch". This proves the following result:

Theorem 4. Let T be a touch; from each Q-cycle is determined thereby a cyclic sequence (of length 5) of labels P, B and A, the last being for *Absent*. The touch being true, i.e. repetition-free, no such cyclic sequence can contain the label P immediately followed by B. **QED**

But we can have (for example) y Plained, yQ Absent, yQ^2 Absent, yQ^3 Bobbed and yQ^4 Absent. A real example is required to make matters clear. Consider the touch $PPPPBPPBPBBPPPBPPBPB$ of 280 changes (i.e. with 20 rows).

Let us number the 21 rows (with rounds added at the start) as $r_0, r_1, r_2, \ldots, r_{20}$. Row 10 is $r_{10} = 1672534$, which is Q-equivalent to $r_0 = 1234567$, with $r_{10} = r_0Q^3$. Row r_0 is Plained, but row r_{10} is Bobbed. The relevant Q-cycle is $(1234567, 1346572, 1467523, 1672534, 1723546)$; according to this touch, it is annotated with (P, A, A, B, A). In other words, 1234567 is Plained, 1346572 is Absent, 1467523 is Absent, 1672534 is Bobbed and 1723546 is Absent. The combination P, B does not occur in this Q-cycle's annotation.

In the following, the first column consists of the row numbers, the next column consists of the actual rows, the next column consists of the canonical Q-representatives of the rows, and the final number is the number of Qs used to get from the representative to the actual row. We write the symbol 'B' (for Bobbed) on the **right** rather than on the left; it means not "how the row is obtained" but "what is done to it". Other rows are, by default, 'Plained' (unless they are 'Absent'). The full touch is as follows:

```
 0  [1,2,3,4,5,6,7]    [1,2,3,4,5,6,7]  0
 1  [1,2,5,3,7,4,6]    [1,3,4,6,7,2,5]  3
 2  [1,2,7,5,6,3,4]    [1,2,7,5,6,3,4]  0
 3  [1,2,6,7,4,5,3]    [1,7,5,3,4,2,6]  3
 4  [1,2,4,6,3,7,5]    [1,7,5,2,3,4,6]  2 B
 5  [1,5,3,2,7,4,6]    [1,2,4,6,7,5,3]  3
 6  [1,5,7,3,6,2,4]    [1,2,4,5,6,7,3]  2
 7  [1,5,6,7,4,3,2]    [1,6,7,3,4,2,5]  4 B
 8  [1,2,4,5,3,6,7]    [1,6,7,2,3,4,5]  2
 9  [1,2,3,4,7,5,6]    [1,4,5,6,7,2,3]  3 B
10  [1,6,7,2,5,3,4]    [1,2,3,4,5,6,7]  3 B
```

```
11  [1,4,5,6,3,7,2]    [1,6,7,2,3,4,5]  3
12  [1,4,3,5,2,6,7]    [1,5,6,7,2,4,3]  3
13  [1,4,2,3,7,5,6]    [1,3,5,6,7,4,2]  3
14  [1,4,7,2,6,3,5]    [1,2,3,5,6,4,7]  3 B
15  [1,5,6,4,3,7,2]    [1,4,7,2,3,5,6]  3
16  [1,5,3,6,2,4,7]    [1,6,4,7,2,5,3]  3
17  [1,5,2,3,7,6,4]    [1,2,3,6,7,4,5]  4 B
18  [1,4,7,5,6,2,3]    [1,4,7,5,6,2,3]  0
19  [1,4,6,7,3,5,2]    [1,7,5,2,3,4,6]  3 B
20  [1,2,3,4,5,6,7]    [1,2,3,4,5,6,7]  0
```

We see this as a **local** counterexample[21] to the argument in [12] and elsewhere for the conjecture that there can be no touch of length greater than 357 rows (i.e. 4998 changes); that argument, if correct, would show that, in a touch, in each Q-set, all rows are either all Plained or all Bobbed, and this is now seen to be false. Whether there is a global counterexample[22] or not is another matter: nevertheless, this shows that a better argument (if there is one) is needed.

7 Coherence

We define a collection $\mathcal{C}$ of round blocks to be *coherent* iff (i) the blocks are disjoint and (ii) whenever its union intersects a Q-set, the elements of the intersection are either all Plained or all Bobbed. Otherwise it is *incoherent*. Clearly a division is coherent.

Theorem 5. If a collection $\mathcal{C}$ of round blocks is coherent, then it can be extended to a division $\mathcal{D}$.

Proof. $\mathcal{C}$ determines for some Q-sets that they are Plained and for some that they are Bobbed; any other Q-sets can be fixed as (for example) Plained. The Plaining or Bobbing of each row is now determined, so by Theorem 2 we have a division $\mathcal{D}$, which is easily seen to extend $\mathcal{C}$. **QED**.

This applies in particular to a collection consisting of a single block, and even more particularly to a touch. It is clear that an incoherent touch (such as that, of length 20 rows, analysed in Section 6) cannot be extended to a division.

Experiments show that, below length 20, all the touches (227 in all) are coherent[23]. There are 757 touches of length at most 20, of which 557 are coherent and 200 (i.e. 26%) are incoherent. For length at most 25, there are 10,924 touches, of which 7,617 are coherent and 3,307 (i.e. 30%) are incoherent. We have no experimental results beyond length 25, except that, as one also infers from the above theory, Holt's touch [2, Touch 89] of 357 rows (the last 3 rows of his 360 rows being avoided by non-use of Singles) is coherent: the remaining block needed to make a division is just a Bob course with a suitable starting point..

8 Final comments

It follows that if there is a bobs-only touch of length 5012 or 5026 (i.e. 358 or 359 rows) of Grandsire Triples, then all the Q-sets, **with just one or two exceptions**, are fully Plained or fully Bobbed, and the exceptions must obey the rule that, if they are considered as Q-cycles, then there is no occurrence (taking the cyclicity into account) of PB; but various patterns **may be** possible, such as ABBBP, ABBPP, ABPPP, AABBP, ABABP, APABP, ... Even the patterns like APPPP are not, so far as we can see, excluded. Search for such a touch might therefore begin by choosing one or two of the 359 possible rows other than rounds as Absent; this determines one or two Q-cycles. Then one picks one of this limited number of patterns for each of these one (or two) Q-cycles, and then insists that, once a Q-set (from the remaining 70 or 71) is entered, it must be fully Plained or fully Bobbed. There may be theoretical results, not yet discovered, that exclude some of these patterns. Exhaustive search is obviously impractical; however, avoidance of full coherence may open avenues not explored by Holt (and others).

Also, there remains the challenge, mentioned in a footnote, of showing that every row of Grandsire Cinques can be reached by an appropriate sequence of calls (i.e. the permutations P and B generate the group of rows).

[21] Thus; it refutes the argument; it doesn't refute the conjecture.

[22] One that refutes the conjecture.

[23] A surprise.

There is a result in group theory of which this is a consequence; the challenge is to find a simple argument, not based on group theory or an impossible exhaustive case analysis. Such an argument can be expected to apply also to Grandsire Triples.

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