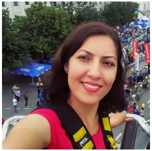
The Graph of the Pedigree Polytope is Asymptotically Almost Complete

Abdullah Makkeh

Mozhgan Pourmoradnasseri



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Outline

Pedigrees, Tours, Extensions

Our Result

Alice & Bob: The Adjacency Game

Symmetric Traveling Salesman polytopes

Definition: (Symmetric) Traveling Salesman Polytope

Convex hull of Traveling Salesman tours:

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- At the center of Applegate-Bixby-Chvatal-Cook (et al.) success in solving large-scale Traveling Salesman Problem instances.
- Lots and lots of theoretical (combinatorial, geometric) work!

Graphs of Polytopes

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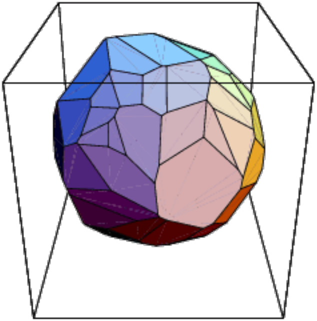
Let P be a polytope.



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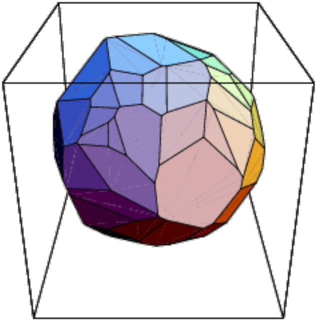

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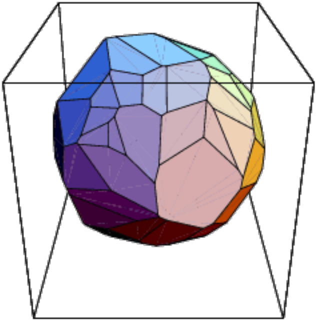

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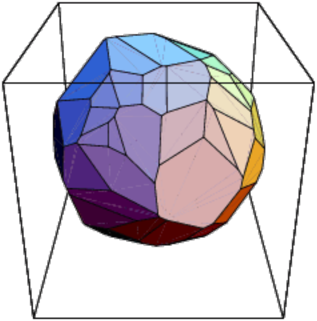

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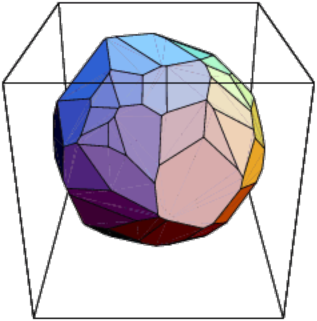

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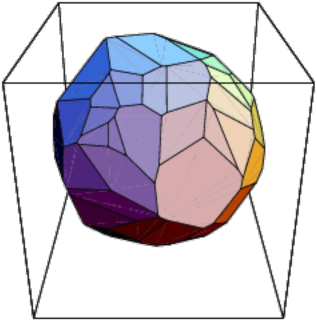

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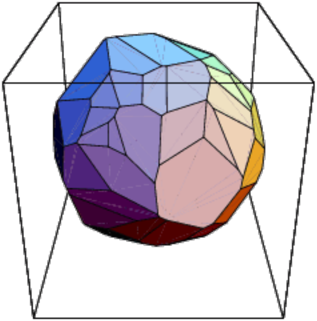

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Theorem (Steinitz).

The graphs of 3-dimensional polytopes are exactly the 3-connected planar graphs.

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Theorem (Papadimitriou).

Deciding adjacency of vertices of TSP(n) is coNP complete.

Grötschel-Padberg conjecture — “TSP-related” polytopes

- Grötschel & Padberg also state the conjecture for other Traveling-Salesman related polytops.

Examples:
Hamilton path polytope, disjoint paths polytope, Graphical TSP polytope, Pedigree polytope

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- Occasionally TSP-related polytopes (other than TSP(n)) are used by practitioners to solve TSP based on Linear Programming (Branch & Cut)

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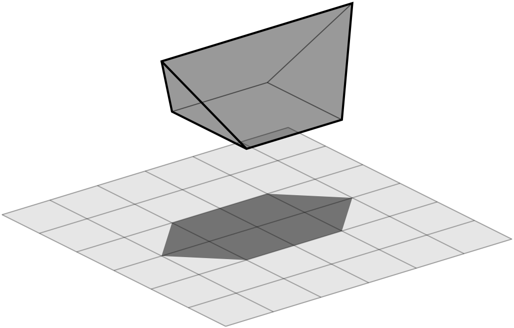
Theorem (Arthanari 2003).

For every n, the Pedigree polytope Ped(n) is an extension of the Symmetric Travaling Salesman polytope TSP(n).

Definition: Extension



(From Welte & Pashkovich (2014), with thanks)

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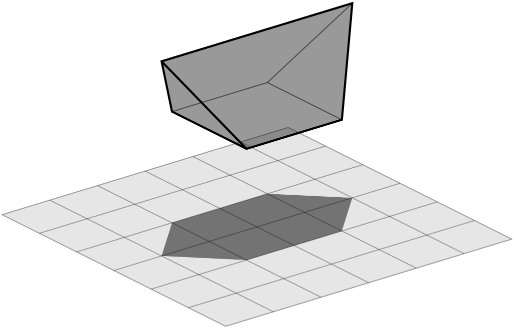

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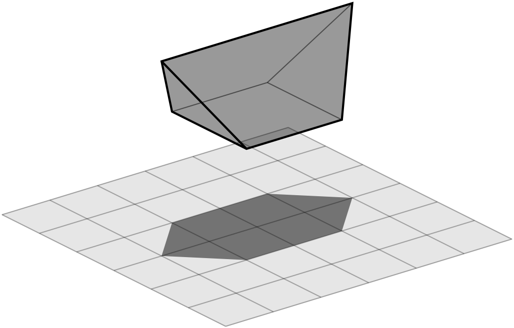

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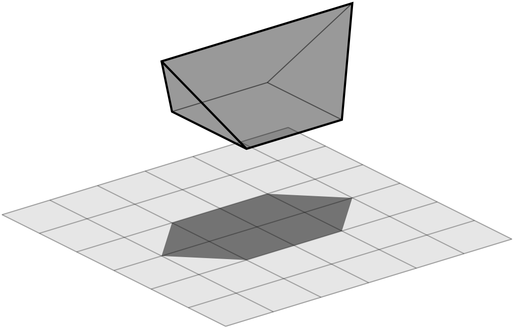

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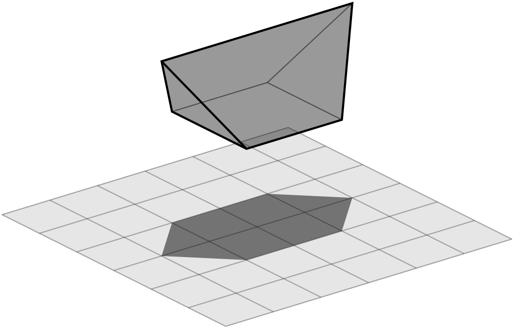

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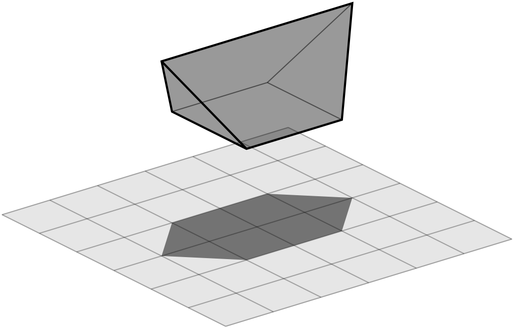

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- . . . but for Ped(n) it is not trivial.

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Extensions and Graphs

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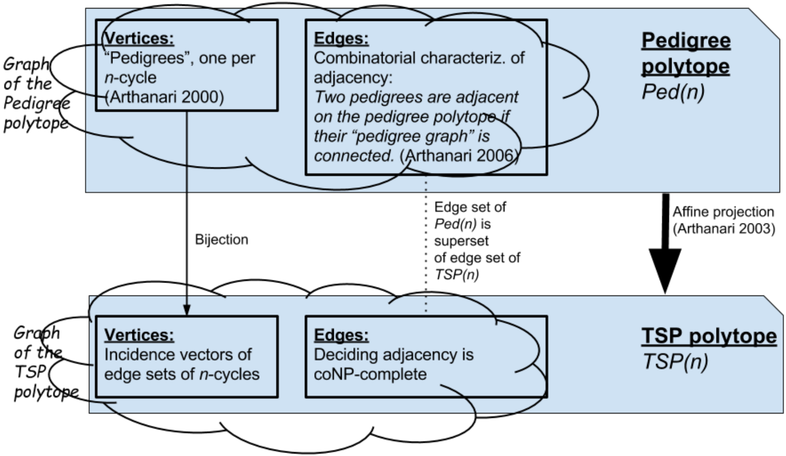
- Idea: Verify diam(Ped(n)) ≤ 2.



Graph of the Pedigree polytope
Vertices:
"Pedigrees", one per n-cycle (Arthanari 2000)
Edges:
Combinatorial characteriz. of adjacency:
Two pedigrees are adjacent on the pedigree polytope if their "pedigree graph" is connected. (Arthanari 2006)
Pedigree polytope
Ped(n)
Bijection
Edge set of Ped(n) is superset of edge set of TSP(n)
Affine projection (Arthanari 2003)
Graph of the TSP polytope
Vertices:
Incidence vectors of edge sets of n-cycles
Edges:
Deciding adjacency is coNP-complete
TSP polytope
TSP(n)

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Theorem.

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- $p > 1/2$ suffices for diam ≤ 2.

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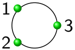
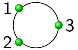
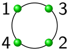
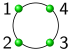
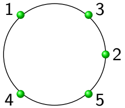
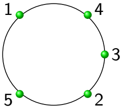
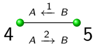
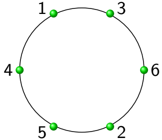
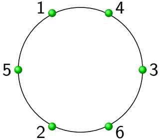
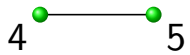
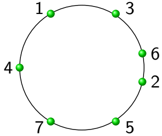
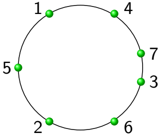
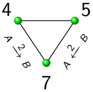
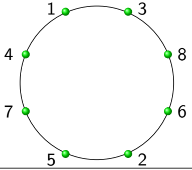
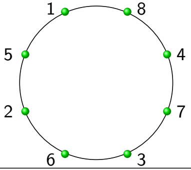
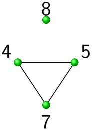
  - "Pedigree graph" G_n^{AB} is connected.
- When cycles are extended ($\hat{=}$ insert new node $n+1$ into some edge of A_n, insert new node $n+1$ into some edge of B_n)
  pedigree graph is updated: $G_n^{AB} \rightsquigarrow G_{n+1}^{AB}$.

n	A_n	B_n	G_n^{AB}
3			$G_3^{AB} = \emptyset$
4			
5			
6			
7			
8			

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"This is a stupid game! I always lose!"

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Lemma (trivial).

Bob's winning probability is $p := \dfrac{\text{minimum degree}}{\text{number of vertices}}$.

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Theorem (MPT—journal version).

Let p_n be the probability (taken over all of Bob's choices since the beginning) that Bob wins round n.

$$\liminf_{n\to\infty} \frac{\text{mindeg}(\Gamma_n)}{|\Gamma_n|} = \liminf_{n\to\infty} p_n.$$

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  as $n \to \infty$, it gets more and more difficult for Alice to win a round.
- Conclude $p_n \to 1$.

Open Questions

- What is the density of the graph of TSP(n)?!?
  - A. Sarangarajan (SIDMA, 1997) has $\log n/n$ lower bound for Asymmetric TSP.
- Find a "tighter" combinatorial condition implied by adjacency on TSP(n). Prove Grötschel-Padberg for that.
- Prove Grötschel-Padberg for other extensions of TSP(n).
  - Maybe start with: S. Onn "Geometry, complexity, and combinatorics of permutation polytopes." JCTA (1993).
- Prove the Grötschel-Padberg's conjecture.