Nondeterministic Communication Complexity of Random 01 Matrices

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What this talk is about

Complexity Theory		Combinatorics	Combinatorial Matrix Theory
Nondeterministic Communication Complexity of a Boolean function f	Rectangle Covering Number of a 01 matrix M	Biclique Covering Number of a bipartite graph H	Boolean Rank of a Boolean matrix
...	smallest number of 1-rectangles needed to cover all 1-entries in M	smallest number of of blicliques needed to cover all edges of H	...

Def.: *1-Rectangle* in M: $K \times L$ w/ $M_{k,\ell} = 1 \ \forall (k,\ell) \in K \times L$

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- $\mathrm{rc}(M)$ $\updownarrow$ chromatic number of "Lovász-Saks rectangle graph of M" $G_{\boxtimes}(M)$

  Random 01-matrices $M^{n,p}$ (entries Bernoulli w/ parameter p) $\updownarrow$ special type of random graphs $G_{\boxtimes}^{n,p}$

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- $\mathrm{rc}(M)$ — Random 01-matrices $M^{n,p}$ (entries Bernoulli w/ parameter p)

$\updownarrow$ chromatic number of "Lovász-Saks rectangle graph of M" $G_{\boxtimes}(M)$

$\updownarrow$ special type of random graphs $G_{\boxtimes}^{n,p}$

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This talk:

- Bound $\mathrm{rc}(M^{n,p}) = \chi(G_{\boxtimes}^{n,p})$

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This talk:

- Bound $\mathrm{rc}(M^{n,p}) = \chi(G_\boxtimes^{n,p})$
- Bound other parameters related to χ,
  e.g., clique number, independence number.

“Why are you doing this?”

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- Nondet. Communication Complexity lower bd to other CC measures.
- Questions on other CC measures are equiv. to versions about nondet.CC (e.g., Log-Rank Conjecture).

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Boolean Rank is the simplest "factorization rank".

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Combinatorialists:

Biclique covering
$\hat{=}$ 2-dimension of some posets;
$\hat{=}$ strong isometric dimension of some graphs;
$\hat{=}$ cover-free families (=generalizations of Sperner families).

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Rectangle covering n° lower-bounds number of inequalities of Linear Programming formulations for a given problem, e.g.:

(a) How many ieqs in an LP for Traveling Salesman Problem on N cities?

(b) How many ieqs in an LP for Minimum Spanning Tree Problem in a complete graph with N vertices?

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Exciting recent developments, breakthroughs
+ still several important open questions. E.g. (b)

Outline

Lovász-Saks Rectangle Graph $G_{\boxtimes}(M)$

Basics on the random graphs $G_{\boxtimes}^{n,p}$

The clique number ($=$ fooling set size)

The independence number ($=$ largest 1-rectangle)

The chromatic number ($= \mathrm{rc}$, $\hat{=}$ NdCC)

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01 matrix M $\longrightarrow$ Graph $G_{\boxtimes}(M)$

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01 matrix M $\longrightarrow$ Graph $G_{\boxtimes}(M)$

Vertices of $G_{\boxtimes}(M)$: 1-entries

$$V = \Big\{(k,\ell) \mid M_{k,\ell} = 1\Big\}$$

$$\begin{pmatrix} * & * & * & * & * \\ * & \mathbf{1} & * & * & * \\ * & * & * & * & * \\ * & * & * & \mathbf{1} & * \\ * & * & * & * & * \end{pmatrix}$$

Lovász-Saks Rectangle Graph $G_\boxtimes(M)$

01 matrix M $\longrightarrow$ Graph $G_\boxtimes(M)$

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Edges of $G_\boxtimes(M)$:

"Two 1-entries (= vertices) are adjacent,
if they span a 2×2 rectangle containing a 0."

$$E = \Big\{\{(k,\ell),(k',\ell')\} \mid M_{k,\ell}M_{k',\ell'} = 1 \quad \& \quad M_{k',\ell}M_{k,\ell'} = 0\Big\}$$

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01 matrix M $\longrightarrow$ Graph $G_{\boxtimes}(M)$

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$$V = \left\{(k,\ell) \mid M_{k,\ell} = 1\right\}$$

$$\begin{pmatrix} * & * & * & * & * \\ * & 1 & * & 0 & * \\ * & * & * & * & * \\ * & 1 & * & 1 & * \\ * & * & * & * & * \end{pmatrix}$$

adjacent

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Rectangle Graph — Example

Definition of rectangle graph $G_{\boxtimes}(M)$:

$$V := \Big\{ (k,\ell) \;\Big|\; M_{k,\ell} = 1 \Big\} = \text{set of 1-entries of } M$$

$$E\colon (k,\ell) \sim (k',\ell') \quad \text{iff} \quad M_{k,\ell'} M_{k',\ell} = 0$$

Example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

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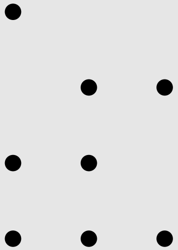

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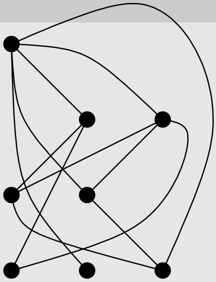

Relationship with graph coloring

Lovász-Saks (1993):

$$\mathrm{rc}(M) = \chi(G_{\boxtimes}(M))$$

Because:

Inclusion-wise maximal independent sets in $G_{\boxtimes}(M)$

$$\updownarrow$$

inclusion-wise maximal 1-rectangles in M

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inclusion-wise maximal 1-rectangles in M

Construction "goes in both directions":

- $\forall G$: G induced subgraph of $G_{\boxtimes}(\mathbb{1} - \mathrm{Adj}(G))$

$$\mathbb{1} = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$ all-1 matrix of appropriate dimensions.

The Log-Rank conjecture

Log-Rank Conjecture, Communication Complexity version

$$\text{Deterministic\_CC}(M) \leq \operatorname{polylog} \operatorname{rk}(M)$$

Log-Rank Conjecture, Graph Theory version

$$\chi(G) \leq 2^{\operatorname{polylog} \operatorname{rk}(\operatorname{Adj}(G))}$$

(The two are equivalent though the Lovász-Saks construction.)

- **This talk is not concerned with the Log-Rank conjecture.** (It's trivially true for random matrices).

Lovász-Saks construction

Properties of the Lovász-Saks construction:

Bipartite graph H	**Matrix** M	**Rectangle graph** $G_{\boxtimes}$
edges	1-entries	vertices
biclique covering n°	rectangle covering n° rc	chromatic n° χ
incl-wise max biclique	incl-wise max 1-rectangle	incl-wise max indep set
cross-free matching	"fooling set" (CC)	clique

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Bipartite graph

H

Matrix

$M = \mathrm{Adj}(H)$

Rectangle graph

$G_\boxtimes(M)$

Clique

$U \subseteq V(G_\boxtimes(M))$

$|U| = r$

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Bipartite graph	**Matrix**	**Rectangle graph**
H	$M = \mathrm{Adj}(H)$	$G_\boxtimes(M)$

- r 1-entries
- any 2 of which span a 2×2 rectangle which contains a 0

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Bipartite graph
H

Matrix
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- any 2 of which span a 2×2 rectangle which contains a 0

$$\begin{pmatrix} \mathbf{1} & A & B & D \\ A & \mathbf{1} & C & E \\ B & C & \mathbf{1} & F \\ D & E & F & \mathbf{1} \end{pmatrix}$$

At least one of the As has to be 0.
Same for B,C,D,E,F,. . .

Rectangle graph
$G_{\boxtimes}(M)$

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Bipartite graph
H

- *"Cross-free matching"*:

  r independent edges no two of which induce a $K_{2,2}$.

Matrix
$M = \mathrm{Adj}(H)$

- r 1-entries
- any 2 of which span a 2×2 rectangle which contains a 0

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Basics on the random graphs $G_{\boxtimes}^{n,p}$

The clique number (= fooling set size)

The independence number (= largest 1-rectangle)

The chromatic number (= rc, $\hat{=}$ NdCC)

The random matrices/graphs

Random matrix:

$M^{n,p}$: $n \times n$ matrix, entries independent,

$$M^{n,p}_{k,\ell} = \begin{cases} 1 & \text{w/ probability } p, \\ 0 & \text{w/ probability } 1-p. \end{cases}$$

Random graph $G^{n,p}_{\boxtimes} := G_{\boxtimes}\big(M^{n,p}\big)$

rectangle graph of the random matrix

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Random graph $\quad G^{n,p}_{\boxtimes} := G_{\boxtimes}\big(M^{n,p}\big)$

rectangle graph of the random matrix

As usual: $n \to \infty$, $p = p(n)$

- ω, α, χ known when $p = \Theta(1)$ (or $p = 1/2$)
- Not when $p = o(1)$ or $p = 1 - o(1)$.

Number of vertices of $G_{\boxtimes}^{n,p}$

Number N of vertices of $G_{\boxtimes}^{n,p}$ (= number of 1s in $M^{n,p}$)

- Binomial random variable $\mathrm{Bin}(n^2, p)$
- $\rightsquigarrow$ nice concentration near pn^2

BTW: $\Omega(1/n) = p = 1 - \Omega(1/n)$

(We couldn't see anything interesting happening outside of that range.)

Number of edges of $G_{\boxtimes}^{n,p}$

Number M of edges of $G_{\boxtimes}^{n,p}$

- Mean: $\mathbf{E}\, M = {\scriptstyle (1+o(1))} (1-p^2) \binom{pn^2}{2}$
- Density $\delta := 1 - p^2$
- No very good concentration...
- ...particularly when $p = 1 - o(1)$

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- Density $\delta := 1 - p^2$
- No very good concentration. . .
- . . . particularly when $p = 1 - o(1)$

What's the problem:

$$\begin{pmatrix}
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & \mathbf{1} & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & *
\end{pmatrix}$$

Changing the blue 1 to a 0 may introduce up to $(n-1)^2$ new edges.

Concentration of clique size and chromatic number

Concentration of $\omega(G_{\boxtimes}^{n,p})$

$$\omega = \mathbf{E}\,\omega + O(\sqrt{n})$$

McDiarmid's ieq no-brainer:
Changing a row can change ω by at most 1.

Concentration of clique size and chromatic number

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Concentration of $\chi(G_{\boxtimes}^{n,p})$

$$\chi = \mathbf{E}\,\chi + O(\sqrt{n})$$

(Same reason.)

Concentration of clique size and chromatic number

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(Same reason.)

Bad news:

- For $p = \Omega(1)$, $\sqrt{n}$ is large compared to $\mathbf{E}\,\omega = O(\ln n)$
- For $p = 1 - o(1)$, $\sqrt{n}$ will be large compared to $\mathbf{E}\,\chi = O((1-p)n\ln n)$

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The independence number (= largest 1-rectangle)

The chromatic number ($= \mathrm{rc}$, $\hat{=}$ NdCC)

Clique number

Theorem [PT].

Clique number $\omega(G_{\boxtimes}^{n,p})$:

(a) $p = o(1/\sqrt{n})$: $\omega = (1-o(1))\, \nu(M^{n,p})$

(with $\nu :=$ size of largest matching in the bipartite graph with adj. mtx. M)

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(b) $p = c/\sqrt{n}$: $\omega \geq \dfrac{1 - e^{-c}}{c} \cdot n$.

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(b) $p = c/\sqrt{n}$: $\omega \geq \dfrac{1-e^{-c}}{c} \cdot n$.

(c) $1/\sqrt{n} \ll p \rightsquigarrow \Theta(1)$: ω drops from $\Theta(n)$ to $\Theta(\ln n)$.

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(c) $1/\sqrt{n} \ll p \rightsquigarrow \Theta(1)$: ω drops from $\Theta(n)$ to $\Theta(\ln n)$.

(d) $p = 1 - o(1)$: $\log_{1/\delta} n - 1 \leq \omega \leq 4 \log_{1/\delta} n - 1.$

(recall $\delta := (1-p)^2$ =density of $G_{\boxtimes}^{n,p}$)

(a,b,c) Tricks using independence n° of some associated Erdős-Renyi graph
(d) UB: Union bound;
LB: Greedy algorithm

Clique number

Theorem [PT].

Clique number $\omega(G_{\boxtimes}^{n,p})$:

(a) $p = o(1/\sqrt{n})$: $\omega = (1-o(1))\,\nu(M^{n,p})$

(with ν := size of largest matching in the bipartite graph with adj. mtx. M)

(b) $p = c/\sqrt{n}$: $\omega \geq \dfrac{1-e^{-c}}{c} \cdot n.$

(c) $1/\sqrt{n} \ll p \rightsquigarrow \Theta(1)$: ω drops from $\Theta(n)$ to $\Theta(\ln n)$.

(d) $p = 1 - o(1)$: $\log_{1/\delta} n - 1 \leq \omega \leq 4\log_{1/\delta} n - 1.$

(recall $\delta := (1-p)^2$ =density of $G_{\boxtimes}^{n,p}$)

(a,b,c) Tricks using independence n° of some associated Erdős-Renyi graph
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Problems:

- The constant in (b) is probably not optimal.
- In (c), how does ω decrease?

Lovász-Saks Rectangle Graph $G_{\boxtimes}(M)$

Basics on the random graphs $G_{\boxtimes}^{n,p}$

The clique number ($=$ fooling set size)

The independence number ($=$ largest 1-rectangle)

The chromatic number ($= \mathrm{rc}$, $\hat{=}$ NdCC)

The independence number of $G_{\boxtimes}^{n,p}$

Theorem (Park & Szpankowski (2005)).

For $p = \Theta(1)$, if $e^{-1/(k+1)} < p \leq e^{-1/k}$,

Largest 1-rectangle has k rows $p^k n$ columns

(or vice versa).

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Theorem [PT].

- With $p = 1 - \lambda/n$, $\lambda = o(n)$, $\lambda \geq c$:

$$\alpha\left(G_{\boxtimes}^{n,p}\right) = (1-o(1))\,\frac{n}{e\lambda}.$$

Proof: Chernoff juggling.

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Basics on the random graphs $G_{\boxtimes}^{n,p}$

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Chromatic number

Theorem (Sherstov).

For $p = {}^1\!/_2$:

- ${}^n\!/_2 \leq \chi \leq n$.
- More careful look: $\chi = (1-o(1))\, n = \dfrac{N}{\alpha}$.

(Recall $N := |V(G_{\boxtimes}^{n,p})|$ = n° vertices of rectangle graph)

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Similar to, e.g., Erdős-Renyi random graphs: $\chi \le 2\,\dfrac{|V|}{\alpha}$.

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Question:

$$\chi(G_{\boxtimes}^{n,p}) = O\Big(\frac{N}{\alpha(G_{\boxtimes}^{n,p})}\Big)?$$

Fractional chromatic number

$$\frac{N}{\alpha} \leq \chi^* \leq \chi = O(\chi^* \cdot \ln N)$$

$\frac{N}{\alpha}$ ‖ independence ratio

χ^* ‖ fractional chromatic number

χ ‖ chromatic number

$\uparrow$ folklore fact

Recall the definition of the fractional chromatic number

- Take probability distribution μ on the independent sets
- $\rightsquigarrow \quad c(\mu) := \min_{v \in V} \mathbf{P}_{R \sim \mu}(v \in R)$
- $\rightsquigarrow \quad \min_{\mu} \frac{1}{c(\mu)} =: \chi^*$

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Questions:

$$\chi^*(G_{\boxtimes}^{n,p}) = O\Big(\frac{N}{\alpha(G_{\boxtimes}^{n,p})}\Big)?$$

$$\chi(G_{\boxtimes}^{n,p}) = O\Big(\chi^*(G_{\boxtimes}^{n,p})\Big)?$$

For Erdős-Renyi random graphs:

Yes!

Yes!

Random Rectangle Graphs:

Conjecture [PT].

For $p = 1 - o(1)$,
$$\chi^*(G_{\boxtimes}^{n,p}) = o\Big(\chi(G_{\boxtimes}^{n,p})\Big)$$

Theorem [PT].

If $\lambda = \ln^{o(1)} n$ and $p = 1 - \lambda/n$:

$$\chi^*(G_{\boxtimes}^{n,p}) = O\left(\frac{\ln n}{\ln \ln n}\right) \quad \ll \quad \log_2 n - O(1) \le \chi(G_{\boxtimes}^{n,p})$$

Chromatic number

Theorem [PT].

If $4 \ln n \leq \lambda = o(n)$ and $p = 1 - \lambda/n$:

$$\frac{N}{\alpha(G_{\boxtimes}^{n,p})} = (1-o(1))\, e\lambda = \chi^*(G_{\boxtimes}^{n,p})$$

In particular,

$$(1-o(1))\, e\lambda \leq \chi(G_{\boxtimes}^{n,p}) = O(\ln \lambda \cdot \ln n).$$

Open problems.

1. **Conjecture.** For $p = 1 - o(1)$, $\chi^*(G_{\boxtimes}^{n,p}) = o\Big(\chi(G_{\boxtimes}^{n,p})\Big)$.
   $\leadsto$ Use other LBs on χ, not based on χ^*.

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4. [Deterministic:] Characterize LS rectangle graphs!

   5.a) Is there a "combinatorial" characterization which makes sense?!?
   5.b) Can LS rectangle graphs be recognized in Poly Time?

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THANK YOU!

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

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