Support-based lower bounds

for the

positive semidefinite rank

of a nonnegative matrix

Troy Lee
Centre for Quantum Technologies
Singapore

Dirk Oliver Theis
University of Tartu
Estonia

UNIVERSITY OF TARTU
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

ICCOPT'13 Almada, Pt
July 27 – Aug 1, 2013

Outline

Example: Max-Cut problem

The polyhedral/convexity question

Connection with (combinatorial) matrix theory

What we can prove for Max-Cut & TSP

Outlook

The Max-Cut problem

Max-Cut problem

On a complete graph K_n with edge-weights, find an edge cut maximizing the sum of the weights of the edges in the cut

Polyhedral approach:

Typical situation:

The question:

The Max-Cut problem

Max-Cut problem

On a complete graph K_n with edge-weights, find an edge cut maximizing the sum of the weights of the edges in the cut

Polyhedral approach:

- P Cut polytope convex hull of incidence vectors of cuts
- Solving Max-Cut $\hat{=}$ optimize linear function over P
- Use Linear Programming to do that

Typical situation:

The question:

The Max-Cut problem

Max-Cut problem

On a complete graph K_n with edge-weights, find an edge cut maximizing the sum of the weights of the edges in the cut

Polyhedral approach:

- P Cut polytope convex hull of incidence vectors of cuts
- Solving Max-Cut $\hat{=}$ optimize linear function over P
- Use Linear Programming to do that
- Problem: P is impossibly complicated to describe

Typical situation:

The question:

The Max-Cut problem

Max-Cut problem

On a complete graph K_n with edge-weights, find an edge cut maximizing the sum of the weights of the edges in the cut

Polyhedral approach:

- P Cut polytope convex hull of incidence vectors of cuts
- Solving Max-Cut $\hat{=}$ optimize linear function over P
- Use Linear Programming to do that
- Problem: P is impossibly complicated to describe
- Approach: Use a relaxation $P' \supset P$,
  P' solution set of some system of linear inequalities

Typical situation:

The question:

The Max-Cut problem

Max-Cut problem

On a complete graph K_n with edge-weights, find an edge cut maximizing the sum of the weights of the edges in the cut

Polyhedral approach:

- P Cut polytope convex hull of incidence vectors of cuts
- Solving Max-Cut $\hat{=}$ optimize linear function over P
- Use Linear Programming to do that
- Problem: P is impossibly complicated to describe
- Approach: Use a relaxation $P' \supset P$,
  P' solution set of some system of linear inequalities

Typical situation:

- The relaxation $P \subset P'$ is quite good, but the system describing P' is too large.

The question:

The Max-Cut problem

Max-Cut problem

On a complete graph K_n with edge-weights, find an edge cut maximizing the sum of the weights of the edges in the cut

Polyhedral approach:

- P Cut polytope convex hull of incidence vectors of cuts
- Solving Max-Cut $\hat{=}$ optimize linear function over P
- Use Linear Programming to do that
- Problem: P is impossibly complicated to describe
- Approach: Use a relaxation $P' \supset P$,
  P' solution set of some system of linear inequalities

Typical situation:

- The relaxation $P \subset P'$ is quite good, but the system describing P' is too large.

The question:

- Can I find something at least as good as P', but "smaller"?
- Linear / positive semidefinite — doesn't matter
- May use additional variables, if it helps

In general

The situation:

- Polyhedra $P \subset P'$
- About P we known vertices $\hat{=}$ feasible solutions
- About P' we know the facets

The question:

In general

The situation:

- Polyhedra $P \subset P'$
- About P we known vertices $\hat{=}$ feasible solutions
- About P' we know the facets

The question:

- Can we find some (convex) object Q such that
  - Q "easier" ($\hat{=}$ fewer constraints) than P'
  - Ok to use additional variables $\hat{=}$ higher dimension (if it helps):

$$\begin{array}{ccccc} & & Q & & \\ & & \big\downarrow{\scriptstyle \pi} & & \\ P & \subset & \pi(Q) & \subset & P' \end{array}$$

  - $P \subset \pi(Q)$ (i.e., no feasible solution lost)
  - $\pi(Q) \subset P'$ (i.e., Q is at least as "good" as P')

" Q *dominates* P' "

Linear/PSD extensions

$$\begin{array}{ccccc} & & Q & & \\ & & \downarrow \pi & & \\ P & \subset & \pi(Q) & \subset & P' \end{array}$$

- Q polyhedron:
  - **linear extension** of $P \subset P'$
  - size $:=$ number of facets of Q

Linear/PSD extensions

$$
\begin{array}{ccccc}
 & & Q & & \\
 & & \big\downarrow \pi & & \\
P & \subset & \pi(Q) & \subset & P'
\end{array}
$$

- Q polyhedron:
  - **linear extension** of $P \subset P'$
  - size $:=$ number of facets of Q
- Q intersection of affine space w/ set of PSD matrices:
  - **PSD extension** of $P \subset P'$
  - size $:=$ order of PSD matrices
    (i.e., q, if matrix is $q \times q$)

Example: Max-Cut

- $P :=$ Cut polytope
- $P' := \left\{x \mid x(E(U)) \leq |U|^2/4\right\}$ "clique relaxation"

Example: Max-Cut

- $P :=$ Cut polytope
- $P' := \left\{ x \;\middle|\; x(E(U)) \le |U|^2/4 \right\}$ "clique relaxation"

Theorem (Lee-T.)

Every linear extension of $P \subset P'$ has $2^{\Omega(n)}$ facets.

Example: Max-Cut

- $P :=$ Cut polytope
- $P' := \{x \mid x(E(U)) \leq |U|^2/4\}$ "clique relaxation"

Theorem (Lee-T.)

Every linear extension of $P \subset P'$ has $2^{\Omega(n)}$ facets.

Theorem (Goemans-Williamson)

There exists a PSD extension of $P \subset P'$ with matrices of order n.

Example: Max-Cut

- $P :=$ Cut polytope
- $P' := \{x \mid x(E(U)) \leq |U|^2/4\}$ "clique relaxation"

Theorem (Lee-T.)

Every linear extension of $P \subset P'$ has $2^{\Omega(n)}$ facets.

Theorem (Goemans-Williamson)

There exists a PSD extension of $P \subset P'$ with matrices of order n.

Question.

- Is there a smaller PSD extension of $P \subset P'$ than Goemans-Williamson?

Example: Max-Cut

- $P :=$ Cut polytope
- $P' := \{x \mid x(E(U)) \leq |U|^2/4\}$ "clique relaxation"
- $P'' := \{x \mid x(E(U)) \leq \lfloor |U|^2/4 \rfloor\}$ Barahona-Mahjoub

Theorem (Lee-T.)

Every linear extension of $P \subset P'$ has $2^{\Omega(n)}$ facets.

Theorem (Goemans-Williamson)

There exists a PSD extension of $P \subset P'$ with matrices of order n.

Question.

- Is there a smaller PSD extension of $P \subset P'$ than Goemans-Williamson?

Example: Max-Cut

- $P :=$ Cut polytope
- $P' := \{x \mid x(E(U)) \leq |U|^2/4\}$ "clique relaxation"
- $P'' := \{x \mid x(E(U)) \leq \lfloor |U|^2/4 \rfloor\}$ Barahona-Mahjoub

Theorem (Lee-T.)

Every linear extension of $P \subset P'$ has $2^{\Omega(n)}$ facets.

Theorem (Goemans-Williamson)

There exists a PSD extension of $P \subset P'$ with matrices of order n.

Question.

- Is there a smaller PSD extension of $P \subset P'$ than Goemans-Williamson?
- What about $P \subset P''$: Does there exist a PSD extension with matrices of order $O(n)$? $O(n^{3/2})$? $O(n^2/\log n)$? $O(n^2)$? $O(n^{200})$?

Connection with (combinatorial) matrix theory

The slack matrix

- Situation $P \subset P' \subset \mathbb{R}^d$ polytopes (for simplicity)

The slack matrix

- Situation $P \subset P' \subset \mathbb{R}^d$ polytopes (for simplicity)
- $P =$ convex hull of columns of matrix $X \in \mathbb{M}(d \times n)$

  $P' =$ solution set of system $Ax \geq b$, $A \in \mathbb{M}(m \times d)$, $b \in \mathbb{R}^m$

The slack matrix

- Situation $P \subset P' \subset \mathbb{R}^d$ polytopes (for simplicity)
- $P =$ convex hull of columns of matrix $X \in \mathbb{M}(d \times n)$

  $P' =$ solution set of system $Ax \geq b$, $A \in \mathbb{M}(m \times d)$, $b \in \mathbb{R}^m$

- Define **slack matrix:**

$$S_{k,\ell} := A_{k,\cdot}X_{\cdot,\ell} - b_k \quad \text{slack of } k\text{th ieq at } \ell\text{th point}$$

so $S = (b, A) \cdot \binom{1^\top}{X}$ (matrix product)

The slack matrix

- Situation $P \subset P' \subset \mathbb{R}^d$ polytopes (for simplicity)
- $P =$ convex hull of columns of matrix $X \in \mathbb{M}(d \times n)$

  $P' =$ solution set of system $Ax \geq b$, $A \in \mathbb{M}(m \times d)$, $b \in \mathbb{R}^m$

- Define **slack matrix:**

$$S_{k,\ell} := A_{k,\cdot} X_{\cdot,\ell} - b_k \qquad \text{slack of } k\text{th ieq at } \ell\text{th point}$$

  so $S = (b, A) \cdot \binom{1^\top}{X}$ (matrix product)

- Property: (entrywise) nonnegative

Faktorization ranks: nonnegative rank

Theorem (Yannakakis 1991).

Minimum number of facets in a linear extension

equals

nonnegative rank $\mathrm{rk}_+(S)$ of slack matrix S

Faktorization ranks: nonnegative rank

Theorem (Yannakakis 1991).

Minimum number of facets in a linear extension
equals
nonnegative rank $\mathrm{rk}_+(S)$ of slack matrix S

Definition.

For $S \geq 0$, nonnegative rank $\mathrm{rk}_+(S)$ is

smallest integer q such that

$\exists$ nonnegative vectors $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in \mathbb{R}^q_+$
with

$$(\xi_k \mid \eta_\ell) = S_{k,\ell}$$

Faktorization ranks: positive semidefinite rank

Theorem (Gouveia, Parrilo, Thomas 2012.)

Minimum order of matrices in a PSD extension
equals
PSD rank $\mathrm{rk}_{\oplus}(S)$ of slack matrix S

Faktorization ranks: positive semidefinite rank

Theorem (Gouveia, Parrilo, Thomas 2012.)

Minimum order of matrices in a PSD extension

equals

PSD rank $\mathrm{rk}_{\oplus}(S)$ of slack matrix S

Definition.

For $S \geq 0$, PSD rank $\mathrm{rk}_{\oplus}(S)$ is

smallest ineger q such that

$\exists$ PSD $(q \times q)$-matrices $A^{(1)}, \dots, A^{(m)}, B^{(1)}, \dots, B^{(n)}$

with

$$\mathrm{tr}(A^{(k)}B^{(\ell)}) =: \left(A^{(k)} \mid B^{(\ell)}\right) = S_{k,\ell}$$

Combinatorial bounds:
Use zero/non-zero information only

- From matrix S construct graph G:
  - Vertices: $V(G) = \{(k,\ell) \mid S_{k,\ell} \neq 0\}$
  - Edges: $(k,\ell) \sim (k',\ell') \iff S_{k',\ell}S_{k,\ell'} \neq 0$

Combinatorial bounds:
Use zero/non-zero information only

- From matrix S construct graph G:
  - Vertices: $V(G) = \{(k,\ell) \mid S_{k,\ell} \neq 0\}$
  - Edges: $(k,\ell) \sim (k',\ell') \iff S_{k',\ell} S_{k,\ell'} \neq 0$

Lemma (folklore).

Nonnegative rank of S $\geq$ vertex-chromatic number of G

$$\mathrm{rk}_+(S) \geq \chi(G)$$

Combinatorial bounds: Use zero/non-zero information only

- From matrix S construct graph G:
  - Vertices: $V(G) = \{(k, \ell) \mid S_{k,\ell} \neq 0\}$
  - Edges: $(k, \ell) \sim (k', \ell') \iff S_{k',\ell} S_{k,\ell'} \neq 0$

Lemma (folklore).

Nonnegative rank of S $\geq$ vertex-chromatic number of G

$$\mathrm{rk}_+(S) \geq \chi(G)$$

- Enormously successful lower bound for nonnegative rank
- Known under several names in different areas

Biclique covering #, Boolean rank, rectangle covering #, nondeterministic communication complexity

Combinatorial bounds:
Use zero/non-zero information only

- From matrix S construct graph G:
  - Vertices: $V(G) = \{(k,\ell) \mid S_{k,\ell} \neq 0\}$
  - Edges: $(k,\ell) \sim (k',\ell') \iff S_{k',\ell} S_{k,\ell'} \neq 0$

Lemma (folklore).

Nonnegative rank of S $\geq$ vertex-chromatic number of G

$$\mathrm{rk}_+(S) \geq \chi(G)$$

- Enormously successful lower bound for nonnegative rank
- Known under several names in different areas

  Biclique covering #, Boolean rank, rectangle covering #, nondeterministic communication complexity

- Not a lower bound on PSD rank

Zero/non-zero information for PSD rank/extension

Theorem (Lee & T.)

The best possible lower bound for PSD rank based only on zero/non-zero pattern equals minimum rank of a matrix with the given zero/non-zero pattern.

Zero/non-zero information for PSD rank/extension

Theorem (Lee & T.)

The best possible lower bound for PSD rank based only on zero/non-zero pattern equals minimum rank of a matrix with the given zero/non-zero pattern.

- Bad news: zero/non-zero based bounds always $\leq \operatorname{rk} S = d + 1$

Zero/non-zero information for PSD rank/extension

Theorem (Lee & T.)

The best possible lower bound for PSD rank based only on zero/non-zero pattern equals minimum rank of a matrix with the given zero/non-zero pattern.

- Bad news: zero/non-zero based bounds always $\leq \operatorname{rk} S = d+1$

Corollary.

PSD rank is at least size of largest triangular submatrix:

If S contains

- upper/lower $t \times t$ triangular submatrix
- with non-zeros on the diagonal
- (after permuting rows and columns)

then $\operatorname{rk}_{\oplus}(S) \geq t$.

Zero/non-zero information for PSD rank/extension

Theorem (Lee & T.)

The best possible lower bound for PSD rank based only on zero/non-zero pattern equals minimum rank of a matrix with the given zero/non-zero pattern.

- Bad news: zero/non-zero based bounds always $\leq \operatorname{rk} S = d + 1$

Corollary.

PSD rank is at least size of largest triangular submatrix:

If S contains

- upper/lower $t \times t$ triangular submatrix
- with non-zeros on the diagonal
- (after permuting rows and columns)

then $\operatorname{rk}_{\oplus}(S) \geq t$.

- Trivial: If $P = P'$ then $\exists$ such a submatrix of size $t = d + 1$

What we can prove for Max-Cut & TSP

Some results for Max-Cut

- $P :=$ cut polytope for K_n
- P' various relaxations

Some results for Max-Cut

- $P :=$ cut polytope for K_n
- P' various relaxations

$P' :=$ take all clique ieqs

Some results for Max-Cut

- $P :=$ cut polytope for K_n
- P' various relaxations

$P' :=$ take all clique ieqs

- The famous Goemans-Williamson relaxation for Max-Cut is an extension of $P \subset P'$ of size n
- Every PSD extension of $P \subset P'$ has size at least n (Lee & T.)

Some results for Max-Cut

- $P :=$ cut polytope for K_n
- P' various relaxations

$P' :=$ take all clique ieqs

- The famous Goemans-Williamson relaxation for Max-Cut is an extension of $P \subset P'$ of size n
- Every PSD extension of $P \subset P'$ has size at least n (Lee & T.)

$P' :=$ take all clique ieqs w/ RHS rounded down (Barahona-Mahjoub)

- Every PSD extension $P \subset P'$ has size at least $(1 - o(1))\, n^2/2$ (Lee & T.)

Some results for Max-Cut

- $P :=$ cut polytope for K_n
- P' various relaxations

$P' :=$ take all clique ieqs

- The famous Goemans-Williamson relaxation for Max-Cut is an extension of $P \subset P'$ of size n
- Every PSD extension of $P \subset P'$ has size at least n (Lee & T.)

$P' :=$ take all clique ieqs w/ RHS rounded down (Barahona-Mahjoub)

- Every PSD extension $P \subset P'$ has size at least $(1 - o(1))\, n^2/2$ (Lee & T.)

$P' :=$ take all triangle ieqs

- Every PSD extension $P \subset P'$ has size at least $(1 - o(1))\, n^2/2$ (Lee & T.)

A result for the Traveling Salesman Problem

- $P :=$ TSP-polytope for K_n

$P' :=$ Dantzig-Fulkerson-Johnson relaxation
(nonnegativity + subtour ieqs)

A result for the Traveling Salesman Problem

- $P :=$ TSP-polytope for K_n

$P' :=$ Dantzig-Fulkerson-Johnson relaxation
(nonnegativity $+$ subtour ieqs)

- Every PSD extension $P \subset P'$ has size at least $(1 - o(1))\, n^2/2$

(Lee & T.)

Outlook — the important problems

Matching problem: (for $P = P'$)

- Is there a polynomially sized linear extension? (famous+old)

Spanning tree problem: (for $P = P'$)

Traveling Salesman w/ Dantzig-Fulkerson-Johnson relax.:

Outlook — the important problems

Matching problem: (for $P = P'$)

- Is there a polynomially sized linear extension? (famous+old)
- Is there a polynomially sized PSD extension?

Spanning tree problem: (for $P = P'$)

Traveling Salesman w/ Dantzig-Fulkerson-Johnson relax.:

Outlook — the important problems

Matching problem: (for $P = P'$)

- Is there a polynomially sized linear extension? (famous+old)
- Is there a polynomially sized PSD extension?
- Improve the (trivial) lower bounds of $\Omega(n^2)$ to $\Omega(n^{2+\varepsilon})$!

Spanning tree problem: (for $P = P'$)

Traveling Salesman w/ Dantzig-Fulkerson-Johnson relax.:

Outlook — the important problems

Matching problem: (for $P = P'$)

- Is there a polynomially sized linear extension? (famous+old)
- Is there a polynomially sized PSD extension?
- Improve the (trivial) lower bounds of $\Omega(n^2)$ to $\Omega(n^{2+\varepsilon})$!

Spanning tree problem: (for $P = P'$)

- Is there a linear/PSD extension of size $o(n^3)$? (Goemans)

Traveling Salesman w/ Dantzig-Fulkerson-Johnson relax.:

Outlook — the important problems

Matching problem: (for $P = P'$)

- Is there a polynomially sized linear extension? (famous+old)
- Is there a polynomially sized PSD extension?
- Improve the (trivial) lower bounds of $\Omega(n^2)$ to $\Omega(n^{2+\varepsilon})$!

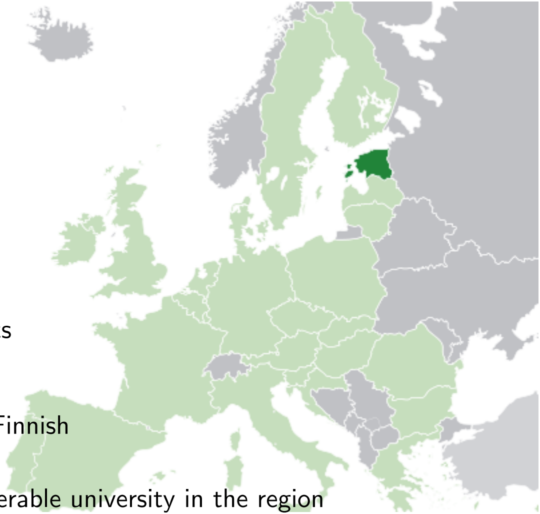
Spanning tree problem: (for $P = P'$)

- Is there a linear/PSD extension of size $o(n^3)$? (Goemans)

Traveling Salesman w/ Dantzig-Fulkerson-Johnson relax.:

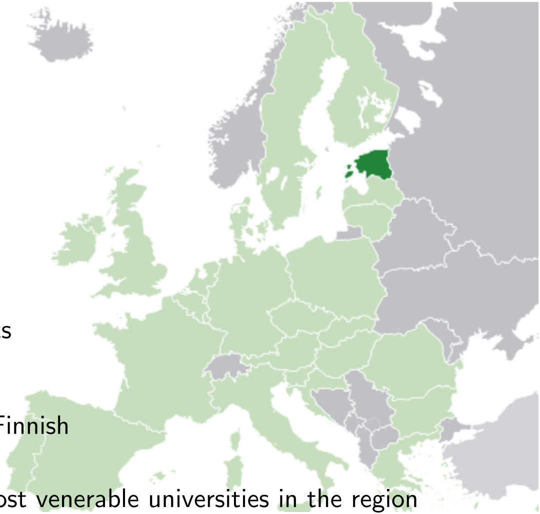
- Is there a linear/PSD extension of size $o(n^3)$?

Outlook towards the other end of the continent. . .



- Estonia:
  - 1.3 million inhabitants
  - Euro-zone since 2011
  - Bustling economy
  - Language similar to Finnish
- Tartu:
  - Houses the most venerable university in the region
  - Student city (= lots of *bars*, *restaurants*, *cafes*, . . .)
  - Climate:
    - 6 months party in the sauna (= winter)
    - 4 months party in the street (spring+summer)

Outlook towards the other end of the continent. . .



- Estonia:
  - 1.3 million inhabitants
  - Euro-zone since 2011
  - Bustling economy
  - Language similar to Finnish
- Tartu:
  - Houses one of the most venerable universities in the region
  - Student city (= lots of *bars*, *restaurants*, *cafes*, . . .)
  - Climate:
    - 6 months party in the sauna (= winter)
    - 4 months party in the street (spring+summer)

UNIVERSITY OF TARTU
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE