# Fooling-sets and Rank in Nonzero Characteristic


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# What is a fooling-set matrix?

**Definition.**

$M$ $n \times n$-matrix over field $\mathbb{k}$ is called

*fooling-set matrix of size $n$*

if

- $M_{k,k} \neq 0 \quad \forall k$
- $M_{k,\ell} M_{\ell,k} = 0 \quad \forall\, k \neq \ell$

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- Alice only gets a row idx $k$; Bob gets a column idx $\ell$
- Alice & Bob want to output the entry $A_{k,\ell}$
- How many bits do they have to exchange for that?

Log of size of a fooling-set sub-matrix in $A$
is a lower bound to the # bits.

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**Example: Polytope extension problem**

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## Example: Polytope extension problem

- $P$ a polytope
- $Q$ another polytope, $\pi$ a projective mapping taking $Q$ *onto* $P$
- Geometric problem:
  Find such $Q$ with minimum number of facets.

Size of a fooling-set sub-matrix in some matrix* $A(P)$ defined by $P$ is a lower bound to this number.

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**Theorem (Dietzfelbinger, Hromkovič, Schnitger 1996).**

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- Rank of $A$ is often generally known
- If $(\mathrm{rk}\, A)^2$ is good lower bound, go find fooling-set sub-matrix!

# Q: Can the exponent 2 be improved — or not?

**Open Problem (Dietzfelbinger, Hromkovič, Schnitger 1996)**

Can the exponent 2 in the inequality

$$n \leq (\mathrm{rk}_{\mathbb{k}}\, M)^2$$

be improved — or not?

# Our result

**Dietzfelbinger et al. ieq:**

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**Theorem (Friesen & T. 2012+).**

For each prime number $p$, there exists a family $M(t)$ of fooling-set matrices over $\mathbb{k} = \mathbb{F}_p$ of size $n = n(t)$ and rank $r = r(t)$, with $n(t) \to \infty$ as $t \to \infty$, and such that

$$\frac{n(t)}{r(t)^2} \to 1$$

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**Dietzfelbinger et al. ieq:**

$n \leq (\mathrm{rk}_{\mathbb{k}} M)^2$

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- In short: the inequality $n \leq \left(\mathrm{rk}_{\mathbb{F}_p} M\right)^2$ is best possible
- . . . including the constant (one) inf front of $(\mathrm{rk}\, M)^2$

# Outline

# How to prove this

Easy approach to lower bounds on exponent:

- Find *one* good matrix $M$, e.g.,

$$M := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

- It is a fooling-set matrix of size $a = 6$ with rank $b = 3$
- Hence $n \leq \mathrm{rk}^{\log_b a} = \mathrm{rk}^{\log_3 6} = \mathrm{rk}^{1.63\ldots}$
- For an infinite family of matrices: take $M^{\otimes k}$ ($k$-fold tensor product)
  - is a fooling set matrix of size $a^k$
  - with rank $b^k$

# Bounds achieved in this way

| Year | Authors | Exponent |
|---|---|---|
| 1996 | Dietzfelbinger et al. | $\log_3 4 = 1.26\ldots$ |
| 1996 | Hühne | $\log_4 6 = 1.29\ldots$ |
| 2012 | Klauck & de Wolf | $\log_3 6 = 1.63\ldots$ |
| 2013 | Lee | $\log_5 15 = 1.68\ldots$ |

# Outline

# How we proved this

- $p$ prime, $\Bbbk := \mathbb{F}_p$
- $r$ integer $r \geq 2$

We define $f\colon \mathbb{Z} \to \Bbbk$ by linear recurrence relation

- $f(k+r) = -f(k) - f(k+1)$ for all $k \in \mathbb{Z}$
- $f(0) = 1$, and $f(1) = \cdots = f(r-1) = 0$.

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- We don't yet know how to choose $r$ and $n$.

# The rank

- $p, r, n$ fixed.
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**Lemma (easy).**

$$\operatorname{rk} M \leq r$$

(In fact, $\operatorname{rk} M = r$ if $n \geq r$.)

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**Trivial:**

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**Not trivial:**

$$M_{k,\ell} M_{\ell,k} \stackrel{!}{=} 0 \text{ whenever } k \neq \ell.$$

# Fooling-set property in terms of $f$

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$M$ is fooling-set matrix, if and only if,

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Mini proof.

$M_{k,\ell}M_{\ell,k} = f(k-\ell)f(\ell-k)$

# Main lemma

**Lemma.**

For all integers $t \geq 1$, if we let

- $r := p^t + 1$ and
- $n := r(r-1) + 1$,

then $f(k)f(-k) = 0$ for all $k \in \mathbb{Z} \setminus n\mathbb{Z}$.

This lemma proves the main theorem.

# How to prove the main lemma

**Lemma 1 (Combinatorial part).**

"Something about blocks of zero."

**Lemma 2 (Algebraic part).**

If $r = p^t + 1$ for some integer $t \geq 1$,
then $n := r(r-1) + 1$ is a period of the function $f$.

Finally: Putting together part.

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Proof.

...
Prove that values of $f$ are (more or less) binomials $\binom{r}{j}$. ...

Finally: Putting together part.

# Outlook

| Field | Entries | Result |
|---|---|---|
| $\chi(\mathbb{k}) = 2$ | $0/1$ | ieq asymptotically best possible |
| $\chi(\mathbb{k}) > 2$ | general | ieq asymptotically best possible |
| $\chi(\mathbb{k}) > 2$ | $0/1$ | ?!?? |
| $\chi(\mathbb{k}) = 0$ | general | up to constant<br>(very recently—will be in full version of paper) |
| $\chi(\mathbb{k}) = 0$ | $0/1$ | ?!?? |
| $\chi(\mathbb{k}) = 0$ | polytope | ?!?? |

**Open problems.**

1. Prove tightness with constant 1 in characteristic 0!
2. Prove tightness for $0/1$ matrices!
3. Prove tightness for matrices defined by polytopes!

# Discrete Math in Tartu

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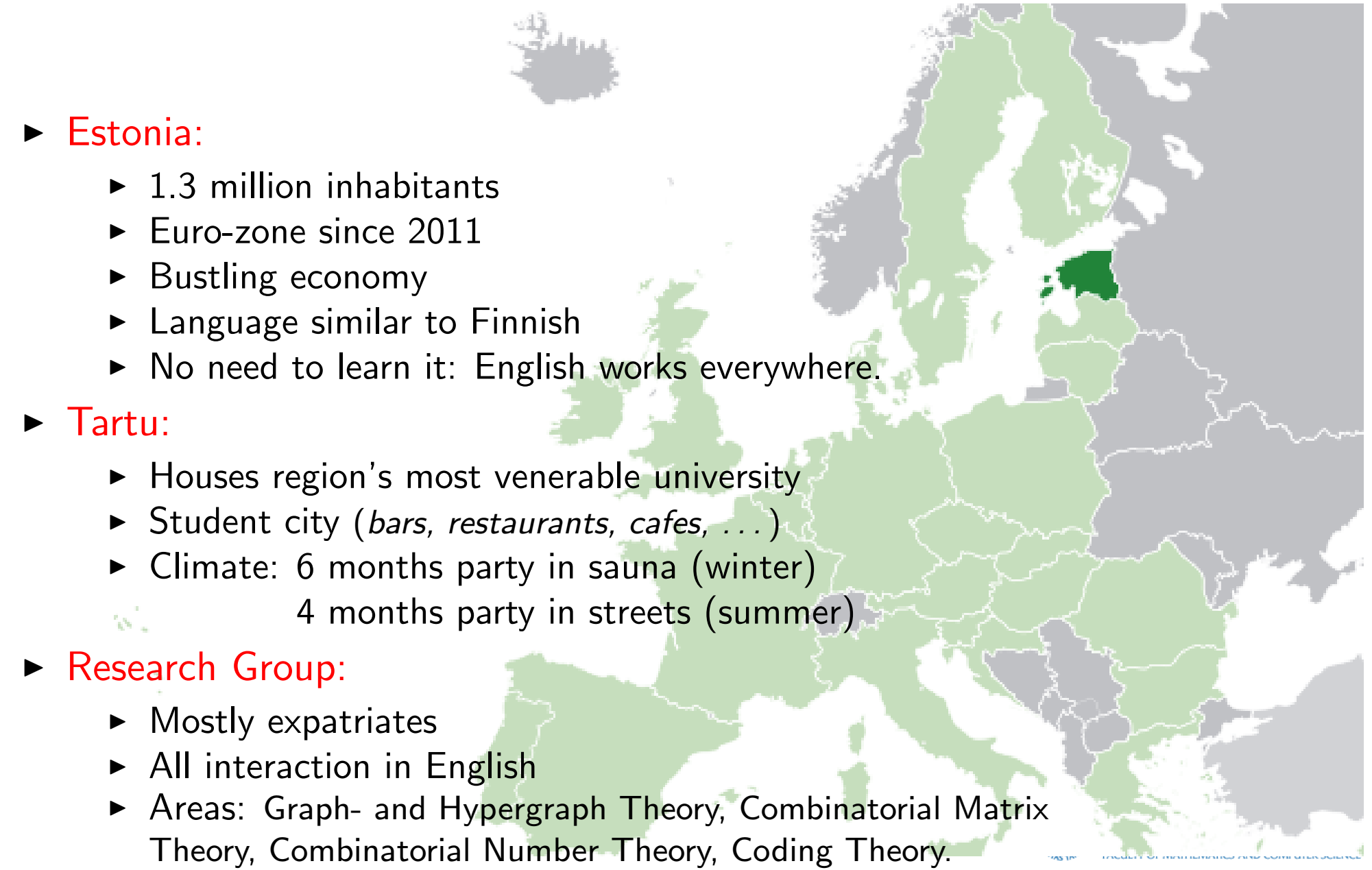

- Estonia:
  - 1.3 million inhabitants
  - Euro-zone since 2011
  - Bustling economy
  - Language similar to Finnish
  - No need to learn it: English works everywhere.
- Tartu:
  - Houses region's most venerable university
  - Student city (*bars*, *restaurants*, *cafes*, ...)
  - Climate: 6 months party in sauna (winter)
    4 months party in streets (summer)
- Research Group:
  - Mostly expatriates
  - All interaction in English
  - Areas: Graph- and Hypergraph Theory, Combinatorial Matrix Theory, Combinatorial Number Theory, Coding Theory.