Nash Equilibrium and Party Polarization in an Electoral Model with Mixed Motivations*

Shino Takayama† Yuki Tamura‡ Terence Yeo§

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Abstract

We study the existence problem of Nash equilibrium as well as party polarization in an electoral competition model. Each party maximizes a sum of party members' expected utility and office rent. In other words, each party has mixed motivations. A class of models with an uncertainty about the median voter position has been increasingly important and Drouvelis, Saporiti and Vriend (2014) present an experimental study to support a model with mixed motivations. But the inclusion of office rent renders the payoff of each party discontinuous. This makes it difficult to apply a usual fixed point argument to prove the existence of Nash equilibrium. By using a recently developed concept, *C-security* in McLennan, Monteiro and Tourky (2011), we provide conditions under which a pure strategy Nash equilibrium (PSE) or a mixed strategy Nash equilibrium (MSE) exists within a fairly general setting, and further the analysis by presenting conditions under which various types of policy choices, including polarization, arise in equilibrium.

Key Words: Noncooperative games, electoral competition, existence of equilibrium.

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†Takayama; School of Economics, University of Queensland, St Lucia, QLD 4072, Australia; email: s.takayama@economics.uq.edu.au; tel: +61-7-3346-7379; fax: +61- 7-3365-7299.

‡Tamura; Department of Economics, University of Rochester

§Yeo (corresponding author); School of Economics, University of Queensland; email: t.yeo@uq.edu.au; tel: +61-7-3365-6570; fax: +61- 7-3365-7299.

1 Introduction

This paper studies an electoral model with two parties. Voters have single-peaked utility functions. In the model, there is an uncertainty about the median voter's bliss point. Each party maximizes a sum of party members' expected utility and office rent, which is a value of holding office. In other words, each party has mixed motivations.

Ball (1999) initiated the study of existence of a pure strategy Nash equilibria (PSE) under the mixed-motivation assumptions and pointed out that the game with mixed motivations may fail to possess a PSE. Saporiti (2008) furthers the analysis by showing the existence of a PSE when office rents are equal. A class of models with a stochastic winning chance has become important. Roemer (1997) provides a micro-foundation for this setting by using an income distribution, and in the model without office rents, provides the existence result and analyzes policy outcomes. Recently by using a linear utility function and a uniform distribution function for the median voter's bliss point, Drouvelis, Saporiti and Vriend (2014) provides the equilibrium analysis in the model with office rents.

This paper extends these results to a more general setting. Our equilibrium analysis is useful for constructing a tractable model in the presence of an uncertainly about the median voter and with mixed motivations. This paper first provides the extended version of the Saporiti (2008) model by using a more general class of utility functions and distribution functions for the median voter's bliss point, which leads to stochastic winning chances, and then provides the existence result in the case where office rents can be heterogenous. We also further the analysis by presenting conditions under which various types of policy choices arise in equilibrium.

Methodologically, Saporiti (2008) applies the concepts of payoff security and reciprocally upper semi-continuity proposed by Reny (1999) and show the existence of a PSE when office rents are the same. In this paper, by using the concept of *C-security* proposed by McLennan, Monteiro and Tourky (2011), we provide conditions for the existence of a PSE. C-security generalizes Reny's better-reply security to non-quasiconcave games. C-security allows us to study the case where office rents can be different and also investigate how the differences of office rents lead to different policy outcomes.

In our model, there is an uncertainty about the median voter's bliss point, and we call the median of the distribution of the median voter's bliss point the *center*. Convergence is a situation whereby both parties choose the same policy. Polarization is a situation whereby both parties choose policies on the different sides of the center. One-sided differentiation is a situation whereby both parties choose different policies but on the same side relative to the center. Our three main theorems present conditions under which several types of policy outcomes, including convergence, one-sided differentiation, and polarization, arise in equilibrium.

Our results about party polarization can be summarized as follows. If both parties' office rents are sufficiently high, there exists a PSE, and each party announces a policy located on the center. This is because each party values winning the office rather than maximizing their party's voter welfare. However, as the office rents decrease, a PSE may fail to exist and each party tends to choose somewhere between the party's bliss point and the center. Another interesting phenomenon can arise in equilibrium. When one party's office rent is higher than that of the other, the equilibrium policies are biased toward the preferred policy of the one party whose office rent is lower. Then, the other party chooses a policy between this opponent's policy choice and the bliss point for the center.

Because our method allows us to consider the case such that office rents are heterogenous, we can also analyze how the differences of office rents result in equilibrium policy positions. Our propositions characterizing an equilibrium policy outcome encompass other important works in the literature. The closest predecessor of our theorems is the one in Drouvelis, Saporiti and Vriend (2014). Drouvelis, Saporiti and Vriend (2014) find that both polarization as well as one-sided differentiation could occur in equilibrium by assuming that the median voter's bliss point is uniformly distributed. Their theoretical predictions are supported by the data collected from a laboratory experiment. Our analysis generalizes their theoretical result to a broader class of distribution functions for the median voter's position and utility functions. Drouvelis, Saporiti and Vriend (2014) provide characterizations of an equilibrium with different values on holding office.

Another model that our results can be applied to is the Roemer (1997) model. Our existence theorem implies the existence of a PSE in the original Roemer (1997) model. In Roemer (1994), it is shown that in a model with no uncertainty, the only equilibrium consists of both parties proposing the median voter's bliss point. On the other hand, in Roemer (1997), it is shown that each equilibrium involves parties putting forth different policies when there is uncertainty about the median voter's bliss point. However as the uncertainty becomes smaller, then the policies of the two parties converge just as in Roemer (1994). Our model captures these features of the two models. In our model, office rents are deterministic variables. The first and third theorems state that when office rents are small, the equilibrium is similar to the one in Roemer (1997), where both parties choose different policies, and when office rents are large, the equilibrium is similar to the one in Roemer (1994). Finally, we show that by using the Roemer (1997) framework, as the difference of the incomes between the two parties' supporters increases, the degree of polarization increases. In the last section, within the Roemer (1997) framework, by using examples of a particular utility function and a distribution function, which satisfy the assumptions for our theorems, we numerically illustrate how party polarization arises depending on the magnitude of office rents for the two parties.

Models with uncertainty about the median voter's bliss point have been increasingly important in the political science literature (Aragones and Palfrey, 2005, Takayama, 2007, Saporiti, 2008, Hummel, 2013, Takayama, 2014, Drouvelis, Saporiti and Vriend, 2014). Further, as Drouvelis, Saporiti and Vriend (2014) claims, the mixed motivations hypothesis is conceptually more realistic than the traditional hypotheses of candidates' motivations. However, electoral competition models with mixed motivations generally have discontinuous and non-quasiconcave payoffs, which makes it difficult to guarantee the existence of a PSE. Our analysis shows how the concept of C-security can be used in a political competition model to guarantee the PSE existence, even when parties are significantly heterogeneous.

Politics has been increasingly polarized across the world (see Benoit and Laver, 2006, McCarty, Poole and Rosenthal, 2016). The classical works including Wittman (1983), Hansson and Stuart (1984), Calvert (1985), and Roemer (1994) address this issue of party polarization and study whether electoral equilibrium is characterized by the median voter position. This theme continued to be an important one in the literature. Recent works including Alesina and Rosenthal (2000), McMurray (2015), Eyster and Kittsteiner (2007) and Esponda and Pouzo (2016) also address this issue.

Sometimes, polarization is viewed as a potential cause of dysfunctional politics. Meanwhile, Smidt (2015) illustrates that parties are likely to benefit from polarization by gaining reliable supporters among even nonpartisans. While we have observed political parties' polarization around the world, the median voter theorem is a core in the theoretic analysis of political competition. When the median voter theorem holds, there is a very strong incentive for parties to choose what the median voter prefers. How office-seeking parties balance the *centrifugal* incentive to appeal to their voters against the *centripetal* motivation to appeal to voters in the general population is a key theme in the theoretical literature of electoral competition. Our approach in this paper is to adopt a fairly general framework in the literature and by applying the notion of C-security, to explore the mechanism involved in various types of policy choices including polarization. By showing that polarized differentiation arises by using the concept of C-security, our analysis bridges the literature of political competition and the advances in equilibrium analysis of discontinuous games.

The paper is organized as follows. The second section describes the model and the three main theorems. The third section provides preliminary results and the fourth section provides the proofs of the main theorems. The fifth section presents the results of party polarization. The last section discusses our results with the literature and provides numerical illustrations on the results. By applying the theorem in Simon and Zame (1990), the sixth section presents the existence of a mixed strategy Nash equilibrium (MSE). We add that in the last section's numerical illustrations, we assume a particular functional forms for the distribution function as well as the utility function; however, our theoretical results prior to the numerical illustration do not rely on a particular functional form.

2 The Model

We describe an electoral competition game $\mathcal{G}$. Let $X \equiv [0,1]$ be the policy space. There are two political parties, *Party L* and *Party R* and a continuum of voters, indexed by a bliss point $\tau \in [0,1]$. Parties have mixed motivations in the sense that they are interested in winning the election not just to benefit their own party members, but also to capture office rents. Let $k^i \geq 0$ be the *office rent*, which is the intrinsic value that party $i = L, R$ places on holding office. The values of k^i are common knowledge.

The voter's utility is $v : X \to \mathbb{R}$ and given by $v(\tau, x)$ where τ is the voter's bliss point and x is an implemented policy.

Assumption 1. For each $\tau \in X$, $v(\tau, x)$ is differentiable with respect to x except at $x = \tau$, and weakly concave with respect to x with a single-peak $x = \tau$.

A simple example of utility functions that satisfy Assumption 1 is $v(\tau, x) = -|x - \tau|$ ($|\cdot|$ is the absolute value). Saporiti (2008) and Drouvelis, Saporiti and Vriend (2014) use this linear utility function (see Assumption 1 in Saporiti (2008)).

For every pair of policies $x_i, x_{-i} \in X$, define $\sigma(x_i, x_{-i})$ such that if $x_i \neq x_{-i}$,

$$v(\sigma(x_i, x_{-i}), x_i) = v(\sigma(x_i, x_{-i}), x_{-i})$$

and if $x_i = x_{-i}$, $\sigma(x_i, x_i) = x_i$. Then $\sigma(x_i, x_{-i})$ identifies the bliss point of the voter who is indifferent between two policy positions x_i, x_{-i}. When $x_i = x_{-i}$, all voters are indifferent between the two policies. In this case, we set $\sigma(x_i, x_i) = x_i$ so that for every x_i, x_{-i}, $\sigma(x_i, x_{-i})$ is uniquely defined and continuous everywhere.

Define $\pi : X^2 \to [0,1]$ to be Party L's winning probability. The two parties' objective functions are:

$$\begin{aligned} &\mathrm{E}\Pi_L(x_L, x_R) = \pi(x_L, x_R)(v(\tau_L, x_L) + k^L) + (1 - \pi(x_L, x_R))v(\tau_L, x_R) \text{ and} \\ &\mathrm{E}\Pi_R(x_R, x_L) = \pi(x_L, x_R)v(\tau_R, x_L) + (1 - \pi(x_L, x_R))(v(\tau_R, x_R) + k^R). \end{aligned} \tag{1}$$

There are two motivations for each party. In addition to capturing office rent, Party L is also motivated to maximize the utility of voters whose bliss point is τ_L, while Party R wants to do the same for voters with a bliss point of τ_R. Let τ_m denote the bliss point of the median voter. Assume τ_m is distributed according with a continuous distribution function G in the interval $[\tau_L, \tau_R]$. When $\tau_L = \tau_R$, then the median voter's bliss point denoted by τ_m satisfies $\tau_m = \tau_L = \tau_R$. In this situation, $x_L^* = x_R^* = \tau_m$ is a PSE, because choosing a different policy decreases their voter's utility as well as the winning probability. As our focus is to study a situation under which a PSE may not exist, following Saporiti (2008) (see Assumption 5) and Roemer (1997), we make the following assumption.

Assumption 2.

$$\tau_L < \tau_m < \tau_R.$$

Because

$$\pi(x_L, x_R) = \Pr[v(\tau_m, x_L) \geq v(\tau_m, x_R)],$$

for every $x_L, x_R \in [\tau_L, \tau_R]$,

$$\pi(x_L, x_R) = \begin{cases} G(\sigma(x_L, x_R)) & \text{if } x_L < x_R, \\ \frac{1}{2} & \text{if } x_L = x_R = x, \\ 1 - G(\sigma(x_L, x_R)) & \text{if } x_L > x_R. \end{cases} \tag{2}$$

Following Saporiti (2008), Drouvelis, Saporiti and Vriend (2014) and Roemer (1997), we assume that in the event of a tie each party wins with an equal probability. Formally, we define an equilibrium in our model.

Definition 1. A strategy profile (x_L^*, x_R^*) is a PSE if for each $i = L, R$, $x_i^* \in X$ is a best response to $x_{-i}^* \in X$ such that $\mathrm{E}\Pi_i(x_i^*, x_{-i}^*) \geq \mathrm{E}\Pi_i(x_i, x_{-i}^*)$ for every $x_i \in X$.

To clearly state the PSE existence conditions in the following theorems, we first define for each $i = L, R$ and $x_i, x_{-i} \in X$,

$$\begin{array}{rcl} U_0^i(x_i, x_{-i}) & = & G(\sigma(x_L, x_R))(v(\tau_i, x_i) + k^i) + (1 - G(\sigma(x_L, x_R)))v(\tau_i, x_{-i}) \\ U_1^i(x_i, x_{-i}) & = & (1 - G(\sigma(x_L, x_R)))(v(\tau_i, x_i) + k^i) + G(\sigma(x_L, x_R))v(\tau_i, x_{-i}) \end{array}.$$

Notice that each $U_0^i(x_i, x_{-i})$ and $U_1^i(x_i, x_{-i})$ are continuous with respect to each $x_i, x_{-i} \in X$, because G and v are continuous in these variables. Then, for each $i = L, R$ and $x_i, x_{-i} \in X$,

$$\mathrm{E}\Pi_i(x_i, x_{-i}) = \begin{cases} U_0^i(x_i, x_{-i}) & \text{if } x_i < x_{-i} \\ v(h_i, x_i) + \frac{1}{2}k^i & \text{if } x_i = x_{-i} \\ U_1^i(x_i, x_{-i}) & \text{if } x_i > x_{-i} \end{cases}. \tag{3}$$

For a fixed $\bar{x}$, we denote a maximizer x of function $U_0^L(x, \bar{x})$ in the interval $[\tau_L, \bar{x}]$ by $\mathbf{x}_0^L(\bar{x})$, and a maximizer x of function $U_1^R(x, \bar{x})$ given $\bar{x}$ in the interval $[\bar{x}, \tau_R]$ by $\mathbf{x}_1^R(\bar{x})$. Because $U_0^L(x, \bar{x})$ and $U_1^R(x, \bar{x})$ are both continuous in x, these maximizers are well-defined in these compact intervals.

The following assumption corresponds to Assumption 6 in Saporiti (2008).

Assumption 3. Assume that $\ln(G(\sigma(x_L, x_R)))$ and $\ln(1 - G(\sigma(x_L, x_R)))$ are concave in x_L and x_R when $x_L \leq x_R$, respectively.

We will show that the relevant interval for our existence analysis is $[\tau_L, \tau_R]$ in Proposition 1. By Assumption 1, $v(\tau_L, x)$ and $v(\tau_R, x)$ is differentiable in (τ_L, τ_R). So, we use $v'(\tau_L, x)$ and $v'(\tau_R, x)$ to denote the first derivatives of these two functions with respect to x in (τ_L, τ_R). Because at $x = \tau_L$, $v(\tau_L, x)$ is not differentiable while at $x = \tau_R$, $v(\tau_R, x)$ is not differentiable, particularly for these two points, we define

$$v'(\tau_L, \tau_L) \equiv \lim_{\epsilon \to 0} \frac{v(\tau_L, \tau_L) - v(\tau_L, \tau_L + \epsilon)}{\epsilon}, v'(\tau_R, \tau_R) \equiv \lim_{\epsilon \to 0} \frac{v(\tau_R, \tau_R) - v(\tau_R, \tau_R - \epsilon)}{\epsilon}.$$

Because v is differentiable with respect to x except at $x = \tau$ as in Assumption 1, $\sigma(x_L, x_R)$ satisfies the same property by the implicit function theorem. Further, because $v(\tau, x)$ is strictly decreasing in $|x - \tau|$, $\sigma(x_L, x_R)$ is also increasing with respect to x_L and x_R, respectively. These observations are summarized in the following lemma.

Lemma 1. For every $x_L, x_R \in X$, $\sigma(x_L, x_R)$ is differentiable with respect to x_L and x_R except for $x_L = x_R$, and monotonically increasing with respect to x_L and x_R, respectively.

Because G is continuous, the first derivative exists except for countably many points. Denote

$$G'_{-}(\sigma) \equiv \lim_{\epsilon \to 0} \frac{G(\sigma) - G(\sigma - \epsilon)}{\epsilon}, G'_{+}(\sigma) \equiv \lim_{\epsilon \to 0} \frac{G(\sigma + \epsilon) - G(\sigma)}{\epsilon}.$$

Similarly, denote

$$\sigma'_{1-}(x_L, x_R) \equiv \lim_{\epsilon \to 0} \frac{\sigma(x_L, x_R) - \sigma(x_L - \epsilon, x_R)}{\epsilon}, \sigma'_{2+}(x_L, x_R) \equiv \lim_{\epsilon \to 0} \frac{\sigma(x_L, x_R + \epsilon) - \sigma(x_L, x_R)}{\epsilon}.$$

Then the following lemma is a slightly more general version of Assumption A4* in Roemer (1997), and is quite common in the literature on electoral competition. Our generalization is only technical, as in our model, σ does not need to be differentiable everywhere.

In Roemer (1997), v and G are twice differentiable, and thus one-sided differentiation for G and σ as above is not necessary.

In Saporiti (2008), the utility function is symmetric with respect to a single peak τ, that is, $v(\tau, \tau - x) = v(\tau, \tau + x)$ for every $x < \tau$. Because v is single-peaked by Assumption 1, $v(\tau, x)$ is strictly decreasing in $|\tau - x|$. Thus, if $v(\sigma(x_L, x_R), x_L) = v(\sigma(x_L, x_R), x_R)$ for some $x_L < x_R$, then the symmetry and single-peakedness of v would require $\sigma(x_L, x_R) - x_L = x_R - \sigma(x_L, x_R)$ and so $\sigma(x_L, x_R) = \frac{x_L + x_R}{2}$. Hence, if v is symmetric around each bliss point, σ is differentiable everywhere and the first derivative with respect to either variable x_1 or x_2 is $\frac{1}{2}$. In general, this holds for any symmetric utility function (which may be non-linear).

As our argument for the existence proof does not rely on any assumption on symmetry for the utility function, we use one-sided differentiation as above.

Lemma 2. When $x_L < x_R$, $\frac{\sigma'_{1-}(x_L, x_R) G'_{-}(\sigma(x_L, x_R))}{G(\sigma(x_L, x_R))}$ is decreasing in x_L and $\frac{\sigma'_{2+}(x_L, x_R) G'_{+}(\sigma(x_L, x_R))}{1 - G(\sigma(x_L, x_R))}$ is increasing in x_R.

Proof. Since the argument is symmetric, we only prove the result for x_R. Because G is continuous, the first derivative exists except for countably many points. Then by Assumption 3, the change in $\ln(G(\sigma(x_L, x_R)))$, which is $\lim_{\epsilon \to 0} \frac{\frac{G(\sigma(x_L, x_R)) - G(\sigma(x_L - \epsilon, x_R))}{\sigma(x_L, x_R) - \sigma(x_L - \epsilon, x_R)} \cdot \frac{\sigma(x_L, x_R) - \sigma(x_L - \epsilon, x_R)}{\epsilon}}{G(\sigma(x_L, x_R))}$, is decreasing in x_L. □

To state our main theorems, for every $x_L, x_R \in X$ with $x_L \leq x_R$, define $u_0^L(x_L, x_R)$ to be

$$\sigma'_1(x_L, x_R) G'_{-}(\sigma(x_L, x_R))(v(\tau_L, x_L) - v(\tau_L, x_R) + k^L) + v'(\tau_L, x_L) G(\sigma(x_L, x_R)). \quad (4)$$

Similarly, for every $x_L, x_R \in X$ with $x_L \leq x_R$, define $u_1^R(x_R, x_L)$ to be

$$\sigma_2'(x_L, x_R)G_+'(\sigma(x_L, x_R))(v(\tau_R, x_R) - v(\tau_R, x_L) + k^R) + v'(\tau_R, x_R)(1 - G(\sigma(x_R, x_L))). \quad (5)$$

Then define

$$Y_L = \{x \in X : u_0^L(x, x) < 0\}; \text{ and } Y_R = \{x \in X : u_1^R(x, x) > 0\}.$$

Y_L is the set of strategies such that when both parties choose these policies, party L's payoff is decreasing while Y_R is the set of strategies such that when both parties choose these policies, party R's payoff is increasing.

When $G(\sigma(x, x)) = 0$, $x = \tau_L$ and $\tau_L \notin Y_L$, and when $G(\sigma(x, x)) = 1$, $x = \tau_R$ and $\tau_R \notin Y_R$. So, we modify the conditions Y_L and Y_R, as these points are not included in the sets, which allows us to apply Assumption 3 in a more convenient way:

$$Y_L = \{x \in X : \frac{\sigma_{1-}'(x, x)G_-'(\sigma(x, x))}{G(\sigma(x, x))} < -\frac{v'(\tau_L, x)}{k^L}\}; \text{ and}$$
$$Y_R = \{x \in X : \frac{\sigma_{2+}'(x, x)G_+'(\sigma(x, x))}{1 - G(\sigma(x, x))} < \frac{v'(\tau_R, x)}{k^R}\}.$$

Now, we state our main theorems.

Theorem 1. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. If $[\tau_L, \hat{\tau}] \subseteq Y_R$ and $[\hat{\tau}, \tau_R] \subseteq Y_L$, then the game $\mathcal{G}$ has a PSE where two equilibrium policies are different at $x_L^* = \mathbf{x}_1^L(x_R^*)$ and $x_R^* = \mathbf{x}_0^R(x_L^*)$.

Theorem 2. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3.

1. Suppose that $k^L < \frac{G(\sigma(\tau_L,\hat{\tau}))(v(\tau_L,\tau_L)-v(\tau_L,\hat{\tau}))}{\frac{1}{2}-G(\sigma(\tau_L,\hat{\tau}))}$ such that there is a $\bar{y}_L < \hat{\tau}$ that solves $U_1^L(\bar{y}_L, \bar{y}_L) = U_0^L(\tau_L, \bar{y}_L)$. If $[\tau_L, \bar{y}_L] \subseteq Y_R$ and $[\bar{y}_L, \tau_R] \subseteq Y_L$, then the game $\mathcal{G}$ has a PSE where two equilibrium policies are different at $x_L^* = \mathbf{x}_0^L(x_R^*)$ and $x_R^* = \mathbf{x}_1^R(x_L^*)$.
2. Alternatively, suppose that $k^R < \frac{(1-G(\sigma(\tau_R,\hat{\tau})))(v(h_R,\tau_R)-v(h_R,\hat{\tau}))}{G(\sigma(\tau_R,\hat{\tau}))-\frac{1}{2}}$ such that there is a $\bar{y}_R > \hat{\tau}$ that solves $U_1^R(\tau_R, \bar{y}_R) = U_0^R(\bar{y}_R, \bar{y}_R)$. If $[\tau_L, \bar{y}_R] \subseteq Y_R$ and $[\bar{y}_R, \tau_R] \subseteq Y_L$, then the game $\mathcal{G}$ has a PSE where two equilibrium policies are different at $x_L^* = \mathbf{x}_0^L(x_R^*)$ and $x_R^* = \mathbf{x}_1^R(x_L^*)$.

Theorems 1 and 2 are obtained by using a variant of *C-security* proposed in McLennan, Monteiro and Tourky (2011). For each $i = L, R$, define a function, $\underline{u}_i : X \to \mathbb{R}$ as follows:

$$\underline{u}_L(d_L, x_R) = \liminf_{\tilde{x}_R \to x_R} \mathrm{E}\Pi_L(d_L, \tilde{x}_R) \text{ and } \underline{u}_R(d_R, x_L) = \liminf_{\tilde{x}_L \to x_L} \mathrm{E}\Pi_R(d_R, \tilde{x}_L).$$

This is the payoff that strategy d_i can almost guarantee to player i if his opponents play any strategies close enough to x_{-i}. For $\alpha \in \mathbb{R}$, define:

$$\begin{aligned} B_i^\alpha(x_{-i}) &= \{y_i \in X_i : \underline{u}_i(y_i, x_{-i}) \geq \alpha\}; \text{ and} \\ C_i^\alpha(x_{-i}) &= \text{con } B_i^\alpha(x_{-i}), \end{aligned}$$

where $\text{con } Z$ is the convex hull of the set Z.

Definition (C-security)**.** The game is C-secure on $Z \subset X \times X$ if there is an $\alpha = (\alpha_L, \alpha_R) \in \mathbb{R}^2$ such that

(1) for each $i = L, R$ and any $z \in Z$, $B_i^{\alpha_i}(z_{-i})$ is nonempty; and

(2) for any $z \in Z$, there is some player $i \in \{L, R\}$ such that $z_i \notin C_i^{\alpha_i}(z_{-i})$.

The game $\mathcal{G}$ is *C-secure* at $(x_L, x_R) \in X^2$ if it is *C-secure* in some neighborhood of (x_L, x_R).

The following theorem restates the main theorem in McLennan, Monteiro and Tourky (2011).

Theorem (Theorem MMT)**.** If the game $\mathcal{G}$ is C-secure at each $(x_L, x_R) \in X^2$ that is not a Nash equilibrium, then $\mathcal{G}$ has a PSE.

Loosely speaking, C-security concerns a non-PSE strategy profile and considers the situation where one party slightly changes the non-PSE strategy. Then it requires that by this deviation, at least one party has an incentive to deviate from the neighborhood of the non-PSE strategy profile. The difficulty of applying this concept is that we have to show that there is such a profitable deviation for every non-PSE strategy profile. To verify this property, we will show that for each opponent's strategy, there is a unique maximizer of the expected payoff which is different from the same strategy with the opponent's strategy. Later we will prove Lemma 5, which indicates that the relevant parts of the expected payoff are U_0^L and U_1^R. Theorem 1 and Theorem 2 state that when at the cut-off $\bar{y}_L$, or $\bar{y}_R$, U_0^L is decreasing and U_1^R is increasing in response to the opponent's policy choice, then both U_0^L and U_1^R are either bell-shaped or monotone (decreasing and increasing respectively), and for any non-PSE strategy, there is an incentive to deviate to the well-defined maximizers. In this way, we show that the game is C-secure. The key idea is that we will find a cut-off $\bar{y}_L$ (or $\bar{y}_R$) such that when $\bar{y}_L \in Y_L \cap Y_R$ (or $\bar{y}_R$), Party R takes a strategy $x_R > \bar{y}_L$ (or Party L takes a strategy $x_L < \bar{y}_L$). In Theorem 1, $\bar{y}_L = \bar{y}_R = \hat{\tau}$ and in Theorem 2, $\bar{y}_L < \hat{\tau}$ or $\bar{y}_R > \hat{\tau}$.

Thus, Theorem 2 is a variation of Theorem 1. A special care is necessary for Theorem 2. As our proof will show, when $\bar{y}_L = \bar{y}_R = \hat{\tau}$ as in Theorem 1, a maximal payoff of each party is greater than the payoff of choosing the same strategy as the opponent. However, when $\hat{\tau}$ may not be contained in both sets as in Theorem 2, we need another condition to guarantee that a maximal payoff is greater than the payoff of choosing the same strategy. The conditions on k^L and k^R in Theorem 2 indeed guarantee this. This is crucial to apply C-security. The condition imposed in the sets Y_L and Y_R sets the limit on how much each party prefers making a marginal deviation toward their median voter's bliss point rather than choosing a tie. The condition imposed in Theorem 2 sets a further condition on how much Party L prefers the choice by their median voter (τ_L, or τ_R) over winning the election ($\hat{\tau}$).

The third theorem concerns the situation where both parties choose the same policy $\hat{\tau}$ in a PSE. When k^L and k^R are both significantly large, both parties essentially try to maximize their winning probability. In this situation, by the median voter theorem, both parties choose the median of the median voter's bliss point. Theorem 3 states this result and gives the cut-off point for the office rents above which this situation arises in equilibrium When conditions (L) and (R) hold, the equilibrium such that both parties choose $\hat{\tau}$ occurs. Because condition (L) and (R) respectively guarantees that $U_0^L(x_L, \hat{\tau})$ is increasing in x_L up to $\hat{\tau}$, and that $U_1^R(x_R, \hat{\tau})$ is decreasing in x_R up to $\hat{\tau}$, both parties choose $\hat{\tau}$ in equilibrium. The expected payoffs are continuous at $\hat{\tau}$, which maximizes the expected payoff for each party in response to the opponent choice of $\hat{\tau}$.

Theorem 3. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Then $(x_L^*, x_R^*) = (\hat{\tau}, \hat{\tau})$ is a PSE if and only if conditions (L) and (R) hold:

- (L) $\frac{\sigma'_{1-}(\hat{\tau},\hat{\tau})G'_{-}(\sigma(\hat{\tau},\hat{\tau}))}{G(\sigma(\hat{\tau},\hat{\tau}))} \geq \frac{-v'(\tau_L,\hat{\tau})}{k^L}$;
- (R) $\frac{\sigma'_{2+}(\hat{\tau},\hat{\tau})G'_{+}(\sigma(\hat{\tau},\hat{\tau}))}{1-G(\sigma(\hat{\tau},\hat{\tau}))} \geq \frac{v'(\tau_R,\hat{\tau})}{k^R}$.

Theorems 1 and 3 generalize the first theorem in Saporiti (2008). To see the connection more clearly, we propose the next corollary. In the Saporiti (2008) model, the median voter's bliss point τ_m is known with certainty. Voters' bliss points are distributed according to a continuous distribution function F. Parties perceive the fraction of the electorate that will vote for them with a certain noise. This noise is modelled as an error term, distributed according to a continuously differentiable distribution function H on $[-\beta, \beta]$ for some $\beta > 0$ and $H(0) = \frac{1}{2}$. Given F, there must exist a theoretical voter that is indifferent between two parties' policies. But the bliss point of the actual indifferent voter is a random variable determined by H centered on the theoretical indifferent voter. As in our case, we simply assume that there is an uncertainty about the median voter's bliss point, which is distributed according to G, to obtain a comparison to the main theorem Saporiti (2008), in the next corollary, we assume that $\tau_R - \hat{\tau} = \hat{\tau} - \tau_L$.

Corollary 1. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Suppose that G is differentiable at $\hat{\tau}$, and $v(\tau, x)$ is symmetric with respect to $\tau \in X$. If $\hat{\tau} = \frac{\tau_L+\tau_R}{2}$, and $k^L = k^R = \bar{k}$, then there exists a PSE. Further,

(1) if $2\sigma'_{1-}(\hat{\tau},\hat{\tau})G'_{-}(\hat{\tau}) \geq \frac{-v'(\tau_L,\hat{\tau})}{\bar{k}}$, $(x_L^*, x_R^*) = (\hat{\tau}, \hat{\tau})$ is a PSE;

(2) otherwise, there is a PSE where two equilibrium policies are different at $x_L^* = \mathbf{x}_1^L(x_R^*)$ and $x_R^* = \mathbf{x}_0^R(x_L^*)$.

Proof. Note that because $v(\tau, x)$ is symmetric with respect to τ and $\hat{\tau} = \frac{\tau_L+\tau_R}{2}$, $v'(\tau_L, \hat{\tau}) = v'(\tau_R, \hat{\tau})$. Because G is differentiable at $\frac{1}{2}$, $G'_{-}(\frac{1}{2}) = G'_{+}(\frac{1}{2})$. Because $v(\hat{\tau}, x)$ is symmetric with respect to $\hat{\tau}$, $\sigma'_{1-}(\hat{\tau},\hat{\tau}) = \sigma'_{2+}(\hat{\tau},\hat{\tau})$. Thus, (L) holds if and only if (R) holds. When (L) and (R) hold, then Theorem 3 is applicable. When (L) and (R) do not hold, then Theorem 1 holds. As a result, we obtain the corollary. □

In Saporiti (2008), the differentiability of the winning probability is assumed and the first order condition is used to study a location of possible equilibria. The differentiability of the winning probability is equivalent to the differentiability of G in our model. Case 1 in Theorem 1 of Saporiti (2008) corresponds to item (1) while case 2 corresponds to item (2) in our model. By using C-security, Corollary 1 gives more detailed characterization of a PSE to the theorem in Saporiti (2008).

Drouvelis, Saporiti and Vriend (2014) use a uniform distribution for G and solve for a PSE. They show that one-sided differentiation can arise in equilibrium. In Section 5, we present a condition under which both parties choose different policies but on the same side relative to $\hat{\tau}$ under a general form of distribution functions (Proposition 5 and 4). In the last section, we will discuss the relationship between Theorems 1 and 3, and the results in Roemer (1994) and Roemer (1997), which consider the model in which there is no office rents, namely $k^L = k^R = 0$.

3 The Two Propositions

The goal in this subsection is to prove two propositions. Let $\mathcal{G}'$ be a restricted game of $\mathcal{G}$ in the sense that a policy space in $\mathcal{G}'$ is $[\tau_L, \tau_R]$. In the first proposition, we show that an equilibrium of $\mathcal{G}'$ is also an equilibrium of $\mathcal{G}$. In the second proposition, we provide the conditions under which $\mathbf{x}_0^L(\bar{x}_R)$ and $\mathbf{x}_1^R(\bar{x}_L)$ are the unique maximizers of the expected payoffs in the game $\mathcal{G}'$.

As the statements of the three theorems show, in equilibrium, voter $\hat{\tau}$ is a key. Suppose voter $\hat{\tau}$ is indifferent between some policy $\mathcal{X}(\bar{x})$ and $\bar{x}$. When $\bar{x} = \hat{\tau}$, the only policy about which voter $\hat{\tau}$ is indifferent with $\hat{\tau}$ is $\hat{\tau}$ itself. Thus, define $\mathcal{X}(\bar{x})$ to satisfy

$$\begin{cases} \sigma(\mathcal{X}(\bar{x}), \bar{x}) = \hat{\tau} & \text{if } \bar{x} \neq \hat{\tau} \\ \mathcal{X}(\bar{x}) = \hat{\tau} & \text{if } \bar{x} = \hat{\tau}. \end{cases} \tag{6}$$

The next lemma shows the locations of the two policies $\bar{x}$ and $\mathcal{X}(x)$.

Lemma 3. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, and 2. When $\bar{x} \leq \hat{\tau}$, $\hat{\tau} \leq \mathcal{X}(\bar{x})$. When $\bar{x} > \hat{\tau}$, $\hat{\tau} > \mathcal{X}(\bar{x})$.

Proof. First, when $\bar{x} = \hat{\tau}$, we have $\hat{\tau} = \mathcal{X}(\bar{x})$. Without loss of generality, suppose $\bar{x} < \hat{\tau}$. Because $v(\hat{\tau}, x)$ is increasing up to $\hat{\tau}$ and decreasing, $v(\hat{\tau}, \bar{x}) = v(\hat{\tau}, \mathcal{X}(\bar{x}))$ holds only if $\hat{\tau} < \mathcal{X}(\bar{x})$. □

In what follows, we show that an equilibrium of $\mathcal{G}'$ is also an equilibrium of $\mathcal{G}$. This will allow us to focus on $[\tau_L, \tau_R]$. In equilibrium, there are three possible cases for Party R's policy x_R: 1. $\tau_R \geq x_R \geq \tau_L$, 2. $x_R > \tau_R$, 3. $x_R < \tau_L$. In case 2, $x_L = \tau_R$ yields a winning probability 1 for Party L, while in case 3, $x_L = \tau_L$ also guarantees a win for Party L. Then, Party R can increase the payoff by choosing $x_R \in [\tau_L, \tau_R]$. Therefore, there is no strategy to support cases 2 and 3 as a PSE. On the other hand, if there is a PSE, the strategy for each party must belong to the interval $[\tau_L, \tau_R]$.

Lemma 4. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, and 2.

- For every $x_L \in X$ and $x_R > \tau_R$, there is an $\hat{x}_R \leq \tau_R$ such that $\mathrm{E}\Pi_R(\hat{x}_R, x_L) > \mathrm{E}\Pi_R(x_R, x_L)$.
- For every $x_R \in X$ and $x_L < \tau_L$, there is an $\hat{x}_L \geq \tau_L$ such that $\mathrm{E}\Pi_L(\hat{x}_L, x_R) > \mathrm{E}\Pi_L(x_L, x_R)$.

Proof. By symmetry, we only prove the first statement. If $x_L \geq \tau_R$, then $\hat{x}_R = \tau_R$ satisfies $\mathrm{E}\Pi_R(\hat{x}_R, x_L) > \mathrm{E}\Pi_R(x_R, x_L)$ for every $x_R > \tau_R$. Now, fix $x_L = \bar{x}$ with $\bar{x} < \tau_R$. Due to the single-peakedness of $v(\tau_R, x)$ with respect to x, there is an $\tilde{x}_R$ such that $\sigma(\bar{x}, \tilde{x}_R) = \tau_R$. The proof shows that there is an $\hat{x}_R \leq \tau_R$ such that

$$\mathrm{E}\Pi_R(\hat{x}_R, \bar{x}) > \mathrm{E}\Pi_R(x_R, \bar{x}), \tag{7}$$

for (Part I) $x_R \geq \tilde{x}_R$; and (Part II) $x_R \in (\tau_R, \tilde{x}_R)$.

(Part I) For $x_R \geq \tilde{x}_R$. Take $\hat{x}_R = \bar{x} + \epsilon < \tau_R$ for a sufficiently small ϵ. Because $v(\tau_R, \tilde{x}_R) = v(\tau_R, \bar{x}) < v(\tau_R, \hat{x}_R)$ and $G(\sigma(\bar{x}, \hat{x}_R)) < 1$,

$$\begin{aligned} \mathrm{E}\Pi_R(\hat{x}_R, \bar{x}) &= (1 - G(\sigma(\bar{x}, \hat{x}_R)))(v(\tau_R, \hat{x}_R) + k^R) + G(\sigma(\bar{x}, \hat{x}_R))v(\tau_R, \bar{x}) \\ &> (1 - G(\tau_R))(v(\tau_R, \tilde{x}_R) + k^R) + G(\tau_R)v(\tau_R, \bar{x}) \\ &= v(\tau_R, \bar{x}) = \mathrm{E}\Pi_R(x_R, \bar{x}). \end{aligned}$$

(Part II) $x_R \in (\tau_R, \tilde{x}_R)$. Let $\hat{x}_R = \tau_R$, and then $\sigma(\bar{x}, \tau_R) < \sigma(\bar{x}, x_R)$ for every $x_R \in (\tau_R, \tilde{x}_R)$. Because $v(\tau_R, \tau_R) > v(\tau_R, x_R)$,

$$\begin{aligned} \mathrm{E}\Pi_R(\hat{x}_R, \bar{x}) &= (1 - G(\sigma(\bar{x}, \tau_R)))(v(\tau_R, \tau_R) + k^R) + G(\sigma(\bar{x}, \tau_R))v(\tau_R, \bar{x}) \\ &> (1 - G(\sigma(\bar{x}, x_R)))(v(\tau_R, x_R) + k^R) + G(\sigma(\bar{x}, x_R))v(\tau_R, \bar{x}) \\ &= \mathrm{E}\Pi_R(x_R, \bar{x}). \end{aligned}$$

□

In equilibrium, in response to x_L, Party R tends to choose a policy toward τ_R than x_L, and in response to x_R, Party L tends to choose a policy toward τ_L. The next lemma shows this result.

Lemma 5. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, and 2.

- For every $x_L \in [\tau_L, \tau_R]$ and $x_R \in [\tau_L, x_L)$, $\hat{x}_R = \max\{x_L, \hat{\tau}\}$ satisfies

$$\mathrm{E}\Pi_R(x_R^*, \bar{x}) > \mathrm{E}\Pi_R(x_R, \bar{x}).$$

- For every $x_R \in [\tau_L, \tau_R]$ and $x_L \in (x_R, \tau_R]$, $\hat{x}_L = \min\{x_R, \hat{\tau}\}$ satisfies

$$\mathrm{E}\Pi_L(x_L^*, \bar{x}) > \mathrm{E}\Pi_L(x_L, \bar{x}).$$

Proof. Since the proof is symmetrical, we only prove the first statement. Let $x_L \in [\tau_L, \tau_R]$ and $x_R \in [\tau_L, x_L)$. First, we show the result holds for $\hat{x}_R = \hat{\tau}$ when $x_L \leq \hat{\tau}$. The difference between $\mathrm{E}\Pi_R(\hat{\tau}, x_L)$ and $\mathrm{E}\Pi_R(x_R, x_L)$ is

$$\pi(x_L, \hat{\tau})(v(\tau_R, \hat{\tau}) - v(\tau_R, x_L)) + \pi(x_L, x_R)(v(\tau_R, x_L) - v(\tau_R, x_R)) \\ + (\pi(x_L, \hat{\tau}) - \pi(x_L, x_R))k^R > 0.$$

The inequality holds because $x_R < x_L \leq \hat{\tau} < \tau_R$, $\pi(\hat{\tau}, x_L) > \pi(x_R, x_L)$ and $v(\tau_R, \hat{\tau}) \geq v(\tau_R, x_L) > v(\tau_R, x_R)$.

Second, if $x_L > \hat{\tau}$, we show the result holds for $\hat{x}_R = x_L$. If $v(\tau_R, x_R) + k^R \geq v(\tau_R, x_L)$, $1 - \pi(x_L, x_R + \epsilon) > 1 - \pi(x_L, x_R)$ and then we have

$$\begin{aligned} \mathrm{E}\Pi_R(x_R, x_L) &= (1 - \pi(x_L, x_R))(v(\tau_R, x_R) + k^R) + \pi(x_L, x_R)v(\tau_R, x_L) \\ &< (1 - \pi(x_L, x_R + \epsilon))(v(\tau_R, x_R) + k^R) + \pi(x_L, x_R + \epsilon)v(\tau_R, x_L) \\ &< (1 - \pi(x_L, x_R + \epsilon))(v(\tau_R, x_R + \epsilon) + k^R) + \pi(x_L, x_R + \epsilon)v(\tau_R, x_L) \\ &= \mathrm{E}\Pi_R(x_R + \epsilon, x_L). \end{aligned}$$

Because $x_R < x_L$ is arbitrary, by applying the same logic, $\mathrm{E}\Pi_R(x_R, x_L) < \mathrm{E}\Pi_R(x_L, x_L)$.

On the other hand, suppose $v(\tau_R, x_R) + k^R < v(\tau_R, x_L)$. Because $\sigma(x_R, x_L) \in (\tau_L, \tau_R)$, $\pi(x_R, x_L) > 0$, which implies

$$\begin{aligned} \mathrm{E}\Pi_R(x_L, x_L) - \mathrm{E}\Pi_R(x_R, x_L) &= \tfrac{1}{2}k^R + \pi(x_R, x_L)(v(\tau_R, x_L) - v(\tau_R, x_R) - k^R) \\ &> \tfrac{1}{2}k^R \geq 0. \end{aligned}$$

□

Lemma 6. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, and 2. If there is a PSE (x_L^*, x_R^*) in the game $\mathcal{G}$, then x_L^* and x_R^* belong to $[\tau_L, \tau_R]$ and each party obtains a strictly positive winning probability in equilibrium.

Proof. Suppose that (x_L^*, x_R^*) is a PSE in the game $\mathcal{G}$. By Lemma 4, $x_R^* \leq \tau_R$ and $x_L^* \geq \tau_L$. By Lemma 5, $x_L^* \leq x_R^*$. Thus, we obtain $\tau_L \leq x_L^* \leq x_R^* \leq \tau_R$. This completes the first part of our proof. To show the second part without loss of generality, on the contrary, suppose that Party R wins with certainty in a PSE (x_L^*, x_R^*) and then by (2), $G(\sigma(x_L^*, x_R^*)) = 0$. Note that $\sigma(\mathcal{X}(x_R^*), x_R^*) = \hat{\tau}$. Then, $G(\sigma(\mathcal{X}(x_R^*), x_R^*)) = \frac{1}{2}$ and thus because $G(\sigma(x_L^*, x_R^*)) = 0$,

$$\begin{aligned} \mathrm{E}\Pi_L(\mathcal{X}(x_R^*), x_R^*) &= \frac{1}{2}(v(\tau_L, \mathcal{X}(x_R^*)) + k^L) + \frac{1}{2}v(\tau_L, x_R^*) \\ &> v(\tau_L, x_R^*) = \mathrm{E}\Pi_L(x_L^*, x_R^*), \end{aligned}$$

because $\mathcal{X}(x_R^*) < \hat{\tau}$ by Lemma 3, and $v(\tau_L, \mathcal{X}(x_R^*)) > v(\tau_L, x_R^*)$. However, this is a contradiction with the assumption that x_L^* is a PSE strategy. □

Proposition 1. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, and 2. A pair of strategies (x_L^*, x_R^*) is a PSE in the game $\mathcal{G}$ if and only if it is a PSE in the restricted game $\mathcal{G}'$ with the restricted strategy space $[\tau_L, \tau_R]$.

Proof.
(Only if part) Suppose that (x_L^*, x_R^*) is a PSE in the game $\mathcal{G}$. Then,

$$\mathrm{E}\Pi_L(x_L^*, x_R^*) \geq \mathrm{E}\Pi_L(x, x_R^*) \text{ for all } x \in X; \text{ and} \tag{8}$$
$$\mathrm{E}\Pi_R(x_R^*, x_L^*) \geq \mathrm{E}\Pi_R(x_R^*, x) \text{ for all } x \in X. \tag{9}$$

Because $[\tau_L, \tau_R] \subset X$, we obtain:

$$\mathrm{E}\Pi_L(x_L^*, x_R^*) \geq \mathrm{E}\Pi_L(x, x_R^*) \text{ for all } x \in [\tau_L, \tau_R]; \text{ and} \tag{10}$$
$$\mathrm{E}\Pi_R(x_R^*, x_L^*) \geq \mathrm{E}\Pi_R(x_R^*, x) \text{ for all } x \in [\tau_L, \tau_R]. \tag{11}$$

By Lemma 6, x_L^* and x_R^* are in the interval $[\tau_L, \tau_R]$ and thus feasible in the game $\mathcal{G}'$. Thus, they are also a PSE in the game $\mathcal{G}'$.
(If part) Suppose that (x_L^*, x_R^*) is a PSE in the game $\mathcal{G}'$. Then, (10) and (11) hold. By way of contradiction, suppose that it is not a PSE in the game $\mathcal{G}$, and without loss of generality, assume that there is an $\tilde{x}_R \in X \setminus [\tau_L, \tau_R]$ which is a better response than x_R^* to x_L^*. By Lemma 5, because a possible best response is greater than x_L^*, we can assume that $\tilde{x}_R > \tau_L$ and it satisfies:

$$\mathrm{E}\Pi_R(\tilde{x}_R, x_L^*) > \mathrm{E}\Pi_R(x_R^*, x_L^*). \tag{12}$$

By Lemma 4, there is an $x_R \in [\tau_L, \tau_R]$ such that

$$\mathrm{E}\Pi_R(x_R, x_L^*) > \mathrm{E}\Pi_R(\tilde{x}_R, x_L^*). \tag{13}$$

By (12) and (13), we obtain

$$\mathrm{E}\Pi_R(x_R, x_L^*) > \mathrm{E}\Pi_R(x_R^*, x_L^*). \tag{14}$$

However, because $x_R \in [\tau_L, \tau_R]$, (14) is a contradiction with (10). □

Now we study the expected payoffs by using two continuous functions. Note that a discontinuity of $\mathrm{E}\Pi_i(x_L, x_R)$ arises only at $x_L = x_R$, and $\mathrm{E}\Pi_i(x_L, x_R)$ is continuous everywhere else. Thus, when $x_L < x_R$, the expected payoff for each party is a continuous function and similarly, when $x_L > x_R$, the expected payoff for each party is another continuous function.

For every $i = L, R$ and $x_{-i} \in X$, let

$$BR_i(x_{-i}) = \operatorname*{argmax}_{x \in X} \mathrm{E}\Pi_i(x, x_{-i}).$$

Then, BR_i is Party i's best response to Party $-i$'s strategy x_{-i}.

By Lemmas 5 and 15, we know that if equilibrium policies of two parties are not equal to $\hat{\tau}$, then we must have $x_L^* < x_R^*$. By (3), when $x_L^* < x_R^*$,

$$\mathrm{E}\Pi_L(x_L^*, x_R^*) = U_0^L(x_L^*, x_R^*) \text{ and } \mathrm{E}\Pi_R(x_R^*, x_L^*) = U_1^R(x_R^*, x_L^*).$$

Since $\mathbf{x}_1^R(x_L^*)$ and $\mathbf{x}_0^L(x_R^*)$ are respectively the maximizers of $U_1^R(x, x_L^*)$ and $U_0^L(x, x_R^*)$, when $\mathbf{x}_1^R(x_L^*) > \mathbf{x}_0^L(x_R^*)$, $(\mathbf{x}_0^L(x_R^*), \mathbf{x}_1^R(x_L^*))$ is a PSE, which leads us to $BR_L(x_R^*) = \mathbf{x}_0^L(x_R^*)$ and $BR_R(x_L^*) = \mathbf{x}_1^R(x_L^*)$. Theorem 1 and Theorem 2 present the conditions under which $\mathbf{x}_1^R(x_L^*) > \mathbf{x}_0^L(x_R^*)$ holds.

On the other hand, even if $\mathbf{x}_1^R(x_L^*) > \mathbf{x}_0^L(x_R^*)$ does not hold, there could still be a PSE. Theorem 3 provides the necessary and sufficient conditions under which at $\hat{\tau}$, U_1^R is decreasing and U_0^L is increasing in response to the opponent's strategy of $\hat{\tau}$. As we see in Lemma 8 and 9, Y_L and Y_R are the set of strategies $(x_L, x_R) = (x, x)$ at which $\mathrm{E}\Pi_L(x, x)$ and $\mathrm{E}\Pi_R(x, x)$ are increasing and decreasing, respectively. Thus, choosing the same policy does not constitute a PSE when a strategy x is in these sets. Note that when $\hat{\tau} \notin Y_L$ or Y_R, condition (L) or (R) holds.

Fix $\bar{x} \in X$ arbitrarily. We consider the relative values of the two continuous functions U_0^i and U_1^i for each $i = L, R$. Voter $\hat{\tau}$'s bliss point is a key factor in the positioning. For the purpose of proving the next lemma, by symmetry, we focus on Party L, and thus our interest here is U_0^L and U_1^L. Take $x, \bar{x} \in X$. By Lemma 5, we consider $x \leq \bar{x}$. Notice that the difference between $U_0^L(x, \bar{x})$ and $U_1^L(x, \bar{x})$ is

$$U_0^L(x, \bar{x}) - U_1^L(x, \bar{x}) = (2G(\sigma(x, \bar{x})) - 1)(v(\tau_L, x) - v(\tau_L, \bar{x}) + k^L). \tag{15}$$

The two functions, U_0^L and U_1^L, intersect at $\mathcal{X}(\bar{x})$. Further, because $\sigma(x, \bar{x}) < \sigma(\mathcal{X}(\bar{x}), \bar{x})$, if $x < \mathcal{X}(\bar{x})$, the fact that G is strictly increasing and $v(\tau_L, x) > v(\tau_L, \bar{x})$ for $x < \bar{x}$ implies that:

$$\begin{cases} U_0^L(x, \bar{x}) > U_1^L(x, \bar{x}) & \text{if } x > \mathcal{X}(\bar{x}) \\ U_0^L(x, \bar{x}) = U_1^L(x, \bar{x}) & \text{if } x = \mathcal{X}(\bar{x}). \end{cases} \tag{16}$$

Now that we know the relative values of the two functions U_0^L and U_1^L, we study the shapes of these functions.

Lemma 7. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. For all $x_L, x_R \in (\tau_L, \tau_R)$ with $x_L \leq x_R$, $u_0^L(x_L, x_R) > 0$ implies $u_0^L(x, x_R) > 0$ for all $x \in [\tau_L, x_L]$ and $u_1^R(x_R, x_L) < 0$ implies $u_1^R(x, x_L) < 0$ for all $x \in [x_R, \tau_R]$.

Proof. Suppose $u_0^L(x_L, x_R) > 0$. Then

$$\frac{\sigma_1'(x_L, x_R) G_-'(\sigma(x_L, x_R))}{G(\sigma(x_L, x_R))} > \frac{-v'(\tau_L, x_L)}{v(\tau_L, x_L) - v(\tau_L, x_L) + k^L}. \tag{17}$$

By Lemma 3, the LHS of (17) is decreasing in x_L. Since $v(\tau_L, x_L)$ is weakly concave in $x_L \in (\tau_L, \tau_R)$, and $-v'(\tau_L, x_L) \geq 0$ for all $x_L > \tau_L$, the RHS of (17) is increasing in $x \in [\tau_L, x_L]$. Hence, for all $x \in [\tau_L, x_L]$, we have $u_0^L(x, x_R) > 0$. The proof for the second statement is similar by using $u_1^R(x_R, x_L)$. □

Lemma 7 assures that there is a unique maximizer of $U_0^L(x, \bar{x})$ or $U_1^R(x, \bar{x})$. Even when G or v is not differentiable at one point, the piece-wise differentiability of these functions guarantees the existence of maximizers in the closed intervals. When there is a point at which $U_0^L(x, \bar{x})$ is decreasing, then either $U_0^L(x, \bar{x})$ is decreasing up to that point or U-shaped with a single-peak as up to one point, $u_0^L(x, x_R) > 0$ and after this point, $u_0^L(x, x_R) < 0$. If G and v satisfy differentiability, then the maximizer must satisfy the first order condition for each party.

To see this, suppose that $Y_L \subset X$ is non-empty. Let $\bar{x} \in Y_L$. Because $\bar{x} \in Y_L$, we have

$$\frac{\sigma_{1-}'(\bar{x}, \bar{x}) G_-'(\sigma(\bar{x}, \bar{x}))}{G(\sigma(\bar{x}, \bar{x}))} < \frac{-v'(\tau_L, \bar{x})}{k^L}. \tag{18}$$

Note that (18) guarantees that the function $U_0^L(x, \bar{x})$ is strictly decreasing at $x = \bar{x}$. In the interval of $[\tau_L, \bar{x}]$, $U_0^L(x, \bar{x})$ is continuous. Every continuous function achieves the maximum in the compact set. Thus, $\mathbf{x}_0^L(\bar{x}) = x_L \in (\tau_L, \bar{x}]$ exists.

By Assumption 1, $v(\tau, \bar{x})$ is linear or strictly concave with respect to x. First, if $v(\tau, \bar{x})$ is linear, then $\mathbf{x}_0^L(\bar{x}) = \tau_L$ because $U_0^L(x, \bar{x})$ is strictly decreasing. Second, we consider the case when $v(\tau, \bar{x})$ is strictly concave with respect to x. Then the single-peakedness of v gives $v'(\tau_L, \tau_L) = 0$, and the following holds:

$$\frac{\sigma_{1-}'(\tau_L, \bar{x}) G_-'(\sigma(\tau_L, \bar{x}))}{G(\sigma(\tau_L, \bar{x}))} > \frac{-v'(\tau_L, \tau_L)}{v(\tau_L, \tau_L) - v(\tau_L, \bar{x}) + k^L}. \tag{19}$$

On the other hand, (18) states

$$\frac{\sigma_{1-}'(\bar{x}, \bar{x}) G_-'(\sigma(\bar{x}, \bar{x}))}{G(\sigma(\bar{x}, \bar{x}))} < \frac{-v'(\tau_L, \bar{x})}{k^L} = \frac{-v'(\tau_L, \bar{x})}{v(\tau_L, \bar{x}) - v(\tau_L, \bar{x}) + \tau_L}. \tag{20}$$

Because by (18) and (19), the maximal x_L to satisfy $u_0^L(x, x_R) > 0$ in $(\tau_L, \bar{x})$ exists, and by Lemma 7, maximizes $U_0^L(x, \bar{x})$, the existence of $\mathbf{x}_0^L(\bar{x})$ is unique. When G is differentiable, because both sides of (18) and (19) are continuous in $x_L \in X$, there is a unique $\mathbf{x}_0^L(\bar{x}) = x_L \in (\tau_L, \bar{x})$ to satisfy the first order condition given in the next Lemma as (21).

It should be worth noting that v is differentiable in (τ_L, τ_R) and Lemma 7 holds without the differentiability of G. Lemma 7 states that given x_R, if $u_0^L(x_L, x_R) < 0$, then there is no $x \in (x_L, x_R)$ such that $u_0^L(x_L, x_R) > 0$. If there is x_L such that $u_0^L(x, x_R) > 0$ for any $x < x_L$ and $u_0^L(x_L, x_R) < 0$, then x_L is the maximizer of $U_0^L(x_L, x_R)$. Overall, we obtain the following lemma.

Lemma 8. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Let $\bar{x} \in Y_L$. Then there is an $\mathbf{x}_0^L(\bar{x}) \in (\tau_L, \bar{x})$ such that

- $U_0^L(x, \bar{x})$ is U-shaped with a single peak $\mathbf{x}_0^L(\bar{x})$;
- if $v(\tau, x)$ is linear, then $\mathbf{x}_0^L(\bar{x}) = \tau_L$;
- if $v(\tau, x)$ is strictly concave and G is differentiable, then $\mathbf{x}_0^L(\bar{x}) \in (\tau_L, \bar{x})$ satisfies the first order condition:

$$\frac{\sigma'_{1-}(\mathbf{x}_0^L(\bar{x}), \bar{x}) G'_{-}(\sigma(\mathbf{x}_0^L(\bar{x}), \bar{x}))}{G(\sigma(\mathbf{x}_0^L(\bar{x}), \bar{x}))} = \frac{-v'(\tau_L, \mathbf{x}_0^L(\bar{x}))}{v(\tau_L, \mathbf{x}_0^L(\bar{x})) - v(\tau_L, \bar{x}) + k^L}. \tag{21}$$

A similar argument works for Party R.

Lemma 9. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Let $\bar{x} \in Y_R$. Then there is an $\mathbf{x}_1^R(\bar{x}) \in (\bar{x}, \tau_R)$ such that

- $U_1^R(x, \bar{x})$ is U-shaped with a single peak $\mathbf{x}_1^R(\bar{x})$;
- if $v(\tau, x)$ is linear, then $\mathbf{x}_1^R(\bar{x}) = \tau_R$;
- if $v(\tau, x)$ is strictly concave and G is differentiable, then $\mathbf{x}_1^R(\bar{x}) \in (\bar{x}, \tau_R)$ satisfies the first order condition:

$$\frac{\sigma'_{2+}(\mathbf{x}_1^R(\bar{x}), \bar{x}) G'_{+}(\sigma(\bar{x}, \mathbf{x}_1^R(\bar{x})))}{1 - G(\sigma(\bar{x}, \mathbf{x}_1^R(\bar{x})))} = \frac{v'(\tau_R, \mathbf{x}_1^R(\bar{x}))}{v(\tau_R, \mathbf{x}_1^R(\bar{x})) - v(\tau_R, \bar{x}) + k^R}. \tag{22}$$

By Proposition 1, we can focus on the restricted game $\mathcal{G}'$. To verify that the game $\mathcal{G}'$ is C-secure, the next lemma is essential. By Lemmas 5, 8 and 9, we obtain the following result, which guarantees that for any non-PSE strategy profile, deviation to the best response is profitable. The strategy of proving the result is as follows. Consider a non-PSE $(\bar{x}_L, \bar{x}_R)$. When $\bar{x}_L > \bar{x}_R$, by Lemma 5, we know that there is a profitable deviation for both parties. When $\bar{x}_L \leq \bar{x}_R$, then at least one of $\bar{x}_L$ and $\bar{x}_R$ satisfies $\bar{x}_L \in Y_R$ and $\bar{x}_R \in Y_L$. Then if $\mathbf{x}_0^L(\bar{x}_R)$ or $\mathbf{x}_1^R(\bar{x}_L)$ is the unique maximizer, the game satisfies C-security. In the next subsections, we show that this requirement for the unique maximizers is satisfied.

Proposition 2. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3.

- In response to $\bar{x}_R \in Y_L$ with $\bar{x}_R \geq \hat{\tau}$, $BR_L(\bar{x}_R) = \mathbf{x}_0^L(\bar{x}_R)$.
- In response to $\bar{x}_L \in Y_R$ with $\bar{x}_L \leq \hat{\tau}$, $BR_R(\bar{x}_L) = \mathbf{x}_1^R(\bar{x}_L)$.

Proof. First, suppose $x_R = \bar{x} \geq \hat{\tau}$. By Lemma 8, $\mathbf{x}_0^L(\bar{x})$ exists uniquely and $\mathbf{x}_0^L(\bar{x}) < \bar{x}$ holds, which implies $\mathrm{E}\Pi_L(\mathbf{x}_0^L(\bar{x}), \bar{x}) = U_0^L(\mathbf{x}_0^L(\bar{x}), \bar{x}) > U_0^L(x, \bar{x}) = \mathrm{E}\Pi_L(x, \bar{x})$ for every $x \leq \bar{x}$. By Lemma 3, since $\bar{x} \geq \hat{\tau}$, $\mathcal{X}(\bar{x}) \leq \bar{x}$, and then by (16), $U_0^L(\bar{x}, \bar{x}) \geq U_1^L(\bar{x}, \bar{x})$. Thus,

$$\mathrm{E}\Pi_L(BR_L(\bar{x}), \bar{x}) = U_0^L(\mathbf{x}_0^L(\bar{x}), \bar{x}) > U_0^L(\bar{x}, \bar{x}) \geq U_1^L(\bar{x}, \bar{x}).$$

Further, since $\bar{x} \geq \hat{\tau}$, by Lemma 5, $x_L^* = \bar{x}$ satisfies that for every $x_L > \bar{x}$,

$$\mathrm{E}\Pi_L(x_L^*, \bar{x}) > \mathrm{E}\Pi_L(x_L, \bar{x}) = U_1^L(x_L, \bar{x}).$$

Because $\bar{x} \geq \hat{\tau}$ implies $G(\sigma(\bar{x}, \bar{x})) \geq \frac{1}{2}$, $\mathrm{E}\Pi_L(x_L^*, \bar{x}) \leq U_0^L(\bar{x}, \bar{x}) < U_0^L(\mathbf{x}_0^L(\bar{x}), \bar{x})$, this completes the proof for the first statement. By symmetry, we can prove the second statement. □

4 Proofs of the Main Theorems

4.1 Proofs of Theorem 1 and Theorem 2

The next lemma shows that when there is a profitable deviation to a unique maximizer, the game is C-secure. A special care is needed to show condition (2), because if there are two strategies $x_1, x_2 \in B_i^{\alpha_i}(z_{-i})$ for some i and $x_1 < x_i < x_2$ for a non-PSE strategy x_i, $x_i \in C_i^{\alpha_i}(z_{-i})$ because $[x_1, x_2] \subseteq C_i^{\alpha_i}(z_{-i})$. Lemmas 2 and 5 assure that this does not happen.

Lemma 10. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Suppose that there is $\bar{y}$ such that $[\tau_L, \bar{y}] \subseteq Y_R$ and $[\bar{y}, \tau_R] \subseteq Y_L$. If

- $U_0^L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R) > U_1^L(x_L, \bar{x}_R)$ for every $\bar{x}_R \in [\bar{y}, \tau_R]$ and $x_L \geq \bar{x}$; and
- $U_1^R(\mathbf{x}_1^R(\bar{x}_L), \bar{x}_L) > U_0^R(x_R, \bar{x}_L)$ for every $\bar{x}_L \in [\tau_L, \bar{y}]$ and $x_R \geq \bar{x}$,

the game $\mathcal{G}'$ is C-secure at each $(x_L, x_R) \in [\tau_L, \tau_R]^2$ that is not a Nash equilibrium.

Proof. Let $(\bar{x}_L, \bar{x}_R) \in [\tau_L, \tau_R]^2$ that is not a PSE. We start by deriving $\underline{u}_L$ and $\underline{u}_R$. Since $\mathrm{E}\Pi_i(x_i, x_{-i})$ is continuous at all strategy profiles (x_L, x_R) for which $x_L \neq x_R$, and discontinuous at (x_L, x_R) where $x_L = x_R$ (with the exception of $x_L = x_R = \hat{\tau}$), it is easily shown that for every $i = L, R$ and $x_i, x_{-i} \in [\tau_R, \tau_L]$,

$$\underline{u}_i(x_i, x_{-i}) = \begin{cases} \mathrm{E}\Pi_i(x_i, x_{-i}) & \text{if } x_i \neq x_{-i} \\ \min\{U_0^i(x_{-i}, x_{-i}), U_1^i(x_{-i}, x_{-i})\} & \text{otherwise}. \end{cases} \tag{23}$$

By symmetry, suppose that $\bar{y} \leq \hat{\tau}$. Because $(\bar{x}_L, \bar{x}_R)$ is not a PSE, $\bar{x}_L = BR_L(\bar{x}_R)$ and $\bar{x}_R = BR_R(\bar{x}_L)$ do not simultaneously hold. Then, suppose $\bar{x}_L \neq BR_L(\bar{x}_R)$. Take (z_L, z_R) in the ϵ-neighborhood of $(\bar{x}_L, \bar{x}_R)$. Take $\delta > 0$ sufficiently small and let

$$\alpha^L = \mathrm{E}\Pi_L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R) - \delta; \text{ and } \alpha^R = \mathrm{E}\Pi_R(\mathbf{x}_1^R(\bar{x}_L), \bar{x}_L) - \delta.$$

First, suppose $\bar{x}_L \leq \bar{x}_R$. Notice that because $[\tau_L, \bar{y}] \subseteq Y_R$ and $[\bar{y}, \tau_R] \subseteq Y_L$, either $\bar{x}_L \in Y_R$ or $\bar{x}_R \in Y_L$ (or both) holds. We first consider the case of $\bar{x}_L \in Y_R$. By the continuity of $U_1^L(x_L, x_R)$ with respect to both variables in the ϵ-neighborhood of $(\bar{x}_L, \bar{x}_R)$,

$$U_0^L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R) = \mathrm{E}\Pi_L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R) > \mathrm{E}\Pi_L(\bar{x}_L, \bar{x}_R) = U_1^L(\bar{x}_L, \bar{x}_R) \approx U_1^L(z_L, z_R),$$

Then by the continuity of $U_0^L(\mathbf{x}_0^L(\bar{x}_R), x_R)$ with respect to x_R, we have

$$U_0^L(\mathbf{x}_0^L(\bar{x}_R), z_R) \geq U_0^L(x_L, z_R) \;=\; \mathrm{E}\Pi_L(x_L, z_R) > U_1^L(z_L, z_R).$$

Then by the continuity of $U_0^L(\mathbf{x}_0^L(\bar{x}_R), x_R)$, $|U_0^L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R) - U_0^L(\mathbf{x}_0^L(\bar{x}_R), z_R)| < \delta$. Thus $\mathbf{x}_0^L(\bar{x}_R) \in B_L^{\alpha_L}(z_R)$. Also, by Lemma 2, for every $x > \bar{x}_R$,

$$U_1^L(x, \bar{x}_R) = \mathrm{E}\Pi_L(x, \bar{x}_R) < U_0^L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R) = \mathrm{E}\Pi_L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R).$$

Thus $[\bar{x}_R, \tau_R] \cap C_L^{\alpha_L}(z_R) = \varnothing$, because by the continuity of $U_0^L(x, \bar{x}_R)$, $C_L^{\alpha_L}(z_R)$ is a closed interval with its size less than ϵ_δ for a sufficiently small ϵ_δ. Then $z_L \notin C_L^{\alpha_L}(z_R)$.

Because $z_L \notin C_L^{\alpha_L}(z_R)$, we are only required to show that $B_R^{\alpha_R}(z_L)$ is not empty. Because $U_1^R(\mathbf{x}_1^R(\bar{x}_L), \bar{x}_L) - \delta$ is well-defined, by the continuity of $U_1^R(\mathbf{x}_1^R(\bar{x}_L), x_L)$ with respect to x_L, $\mathbf{x}_1^R(\bar{x}_L) \in B_R^{\alpha_R}(z_L)$ and we obtain the result.

Second we consider the case of $\bar{x}_L > \bar{x}_R$. By Lemma 5, when $\hat{x}_R = \max\{\bar{x}_L, \hat{\tau}\}$ and $\hat{x}_L = \min\{\bar{x}_R, \hat{\tau}\}$, for every $x_L > \bar{x}_R$ and $x_R < \bar{x}_L$, $\mathrm{E}\Pi_L(\hat{x}_L, \bar{x}_R) > \mathrm{E}\Pi_L(x_L, \bar{x}_R)$ and $\mathrm{E}\Pi_R(\hat{x}_R, \bar{x}_L) > \mathrm{E}\Pi_R(x_R, \bar{x}_L)$. Thus for every $x_L > \bar{x}_R$ and $x_R < \bar{x}_L$,

$$\mathrm{E}\Pi_L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R) > U_1^L(x_L, \bar{x}_R) \approx U_1^L(x_L, z_R);$$
$$\mathrm{E}\Pi_R(\mathbf{x}_1^R(\bar{x}_L), \bar{x}_L) > U_0^R(x_R, \bar{x}_L) \approx U_0^R(x_R, z_R).$$

Then by the continuity of $\mathrm{E}\Pi_L(\mathbf{x}_0^L(\bar{x}_R), \bar{x}_R)$ and $\mathrm{E}\Pi_R(\mathbf{x}_1^R(\bar{x}_L), \bar{x}_L)$ in the ϵ-neighborhood of $(\bar{x}_L, \bar{x}_R)$, $B_L^{\alpha_L}(z_R)$ and $B_R^{\alpha_R}(z_L)$ are not empty. Also, $z_L \notin C_L^{\alpha_L}(z_R)$ and $z_R \notin C_R^{\alpha_R}(z_L)$. This completes the proof. □

4.2 Proof of Theorem 1

Proof of Theorem 1. Note that $\bar{y} = \hat{\tau}$ and both statements of Proposition 2 hold. Then Lemma 10 is applicable. Thus, by Lemma 10, the game $\mathcal{G}'$ is C-secure. Thus, by Theorem MMT, there exists a PSE in the restricted game $\mathcal{G}'$. By Proposition 1, this PSE is also a PSE in the game $\mathcal{G}$. □

4.3 Proof of Theorem 2

The proof of Theorem 2 is slightly more complicated than that of Theorem 1. To apply C-security, in Theorem 1, we are required to show that for every $x > \hat{\tau}$,

$$\mathrm{E}\Pi_L(BR_L(x), x) = U_0^L(\mathbf{x}_0^L(x), x) > U_1^L(x, x).$$

Because $x > \hat{\tau}$ and thus $BR_L(x) \leq x$, (16) implies that $U_0^L(x,x) \geq U_1^L(x,x)$. However, when x can be smaller than $\hat{\tau}$ in the case of Theorem 2, we cannot directly obtain $U_0^L(x,x) \geq U_1^L(x,x)$ to apply Lemma 2, because when $x < \hat{\tau}$, similarly to the relationship in (16), it is possible to have $U_0^L(x,x) < U_1^L(x,x)$. Then C-security could be violated as Party L can be better off by choosing the same policy with Party R.

Thus to have a well-defined best response in this case, we need some $x_L < x_R$ for Party L such that it is a better response than choosing the same policy x_R. The next lemma gives this condition. When there is $\bar{y}_L$ as specified in the next lemma, at $\bar{y}_L$, $U_0^L(x, \bar{y}_L)$ is decreasing and there is a better response x_L than $\bar{y}_L$ for Party L, which is smaller than $\bar{y}_L$. Note that because $\bar{y}_L < \hat{\tau}$, the second statement of Lemma 2 is applicable for Party R and then we can guarantee that

$$\mathrm{E}\Pi_R(BR_R(x), x) = U_1^R(\mathbf{x}_1^R(x), x) > U_0^R(x, x).$$

Lemma 11. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Suppose that $k^L < \frac{G(\sigma(\tau_L,\hat{\tau}))(v(\tau_L,\tau_L)-v(\tau_L,\hat{\tau}))}{\frac{1}{2}-G(\sigma(\tau_L,\hat{\tau}))}$ so that there is a $\bar{y}_L < \hat{\tau}$ that solves $U_1^L(\bar{y}_L, \bar{y}_L) = U_0^L(\tau_L, \bar{y}_L)$. Then, for every $x \geq \bar{y}_L$, $U_0^L(\mathbf{x}_0^L(x), x) > U_1^L(x, x)$.

Proof. When $U_1^L(\bar{y}_L, \bar{y}_L) = U_0^L(\tau_L, \bar{y}_L)$, because

$$\begin{aligned} U_1^L(\bar{y}_L, \bar{y}_L) &= v(\tau_L, \bar{y}_L) + (1 - G(\sigma(\bar{y}_L, \bar{y}_L)))k^L \\ &= G(\sigma(\tau_L, \bar{y}_L))(v(\tau_L, \tau_L) + k^L) + (1 - G(\sigma(\tau_L, \bar{y}_L)))v(\tau_L, \bar{y}_L) \\ &= U_0^L(\tau_L, \bar{y}_L), \end{aligned}$$

we obtain

$$k^L = \frac{G(\sigma(\tau_L, \bar{y}_L))(v(\tau_L, \tau_L) - v(\tau_L, \bar{y}_L))}{1 - G(\sigma(\bar{y}_L, \bar{y}_L)) - G(\sigma(\tau_L, \bar{y}_L))}.$$

When $x = \tau_L$,

$$k^L \geq 0 = \frac{G(\sigma(\tau_L, \tau_L))(v(\tau_L, \tau_L) - v(\tau_L, \tau_L))}{1 - G(\sigma(\tau_L, \tau_L)) - G(\sigma(\tau_L, \tau_L))}.$$

When x increases from τ_L to $\hat{\tau}$, by the assumption for this lemma,

$$k^L < \frac{G(\sigma(\tau_L, \hat{\tau}))(v(\tau_L, \tau_L) - v(\tau_L, \hat{\tau}))}{1 - G(\sigma(\hat{\tau}, \hat{\tau})) - G(\sigma(\tau_L, \hat{\tau}))}.$$

Because $\frac{G(\sigma(\tau_L,x))(v(\tau_L,\tau_L)-v(\tau_L,x))}{1-G(\sigma(x,x))-G(\sigma(\tau_L,x))}$ is continuous with respect to x, by the intermediate value theorem, we can guarantee the existence of $\bar{y}_L$.

Let $x > \bar{y}_L$ and then

$$\begin{aligned} U_1^L(\bar{y}_L, \bar{y}_L) - U_1^L(x, x) &= v(\tau_L, \bar{y}_L) - v(\tau_L, x) + k^L(G(\sigma(x, x)) - G(\sigma(\bar{y}_L, \bar{y}_L))); \\ U_0^L(\tau_L, \bar{y}_L) - U_0^L(\tau_L, x) &= (G(\sigma(\tau_L, \bar{y}_L)) - G(\sigma(\tau_L, x)))(v(\tau_L, \tau_L) + k^L) \\ &\quad + (1 - G(\sigma(\tau_L, \bar{y}_L)))v(\tau_L, \bar{y}_L) - (1 - G(\sigma(\tau_L, x)))v(\tau_L, x). \end{aligned}$$

Suppose $U_1^L(x, x) \geq U_0^L(\tau_L, x)$. Then

$$\begin{aligned} &k^L(G(\sigma(x, x)) + G(\sigma(\tau_L, x)) - G(\sigma(\bar{y}_L, \bar{y}_L)) - G(\sigma(\tau_L, \bar{y}_L))) \\ &\leq G(\sigma(\tau_L, \bar{y}_L))(v(\tau_L, \tau_L) - v(\tau_L, \bar{y}_L)) - G(\sigma(\tau_L, x))(v(\tau_L, \tau_L) - v(\tau_L, x)). \end{aligned}$$

However, $G(\sigma(\tau_L, \bar{y}_L))(v(\tau_L, \tau_L) - v(\tau_L, \bar{y}_L)) < G(\sigma(\tau_L, x))(v(\tau_L, \tau_L) - v(\tau_L, x))$ while $G(\sigma(x, x)) + G(\sigma(\tau_L, x)) > G(\sigma(\bar{y}_L, \bar{y}_L)) + G(\sigma(\tau_L, \bar{y}_L))$. Thus

$$0 \leq k^L(G(\sigma(x, x)) + G(\sigma(\tau_L, x)) - G(\sigma(\bar{y}_L, \bar{y}_L)) - G(\sigma(\tau_L, \bar{y}_L))) < 0.$$

This is a contradiction. Thus $U_1^L(x, x) < U_0^L(\tau_L, x)$. Because $U_0^L(\tau_L, x) \leq U_0^L(\mathbf{x}_0^L(x), x)$, we obtain the desired result. □

In an exactly symmetrical way, we can obtain the following result.

Lemma 12. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Suppose that $k^R < \frac{(1-G(\sigma(\tau_R,\hat{\tau})))(v(h_R,\tau_R)-v(h_R,\hat{\tau}))}{G(\sigma(\tau_R,\hat{\tau}))-\frac{1}{2}}$ so that there is a $\bar{y}_R > \hat{\tau}$ that solves $U_1^R(\tau_R, \bar{y}_R) = U_0^R(\bar{y}_R, \bar{y}_R)$. Then, for every $x \leq \bar{y}_R$, $U_1^R(\mathbf{x}_1^R(x), x) > U_0^R(x, x)$.

Now as the combination of Lemma 2 and the second statement of Lemma 11, we obtain the following lemma, which makes Lemma 10 applicable in this case.

Lemma 13. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Let $\bar{y}_L$ be as defined in Lemma 11. Suppose $[\tau_L, \bar{y}_L] \subseteq Y_R$ and $[\bar{y}_L, \tau_R] \subseteq Y_L$. Then

- $BR_L(\bar{x})$ uniquely exists in response to $\bar{x} \in [\bar{y}_L, \tau_R]$ and for every $x \geq \bar{x}$,

$$\mathrm{E}\Pi_L(BR_L(\bar{x}), \bar{x}) = U_0^L(\mathbf{x}_0^L(\bar{x}), \bar{x}) > U_1^L(x, \bar{x}).$$

- $BR_R(x)$ uniquely exists in response to $x \in [\tau_L, \bar{y}_L]$ and for every $x \leq \bar{x}$,

$$\mathrm{E}\Pi_R(BR_R(\bar{x}), \bar{x}) = U_1^R(\mathbf{x}_1^R(\bar{x}), \bar{x}) > U_0^R(x, \bar{x}).$$

Proof. Suppose $\bar{y}_L < \hat{\tau}$. First, let $\bar{x} \in [\bar{y}_L, \tau_R]$. By Lemma 5, for every $x < \bar{x}$, there is $x^* \leq \bar{x}$ such that $U_0^L(x^*, \bar{x}) > U_1^L(x, \bar{x})$. By Lemma 8, $\mathbf{x}_0^L(\bar{x})$ exists uniquely and $\mathbf{x}_0^L(\bar{x}) < \bar{x}$. Thus $U_0^L(\mathbf{x}_0^L(\bar{x}), \bar{x}) \geq U_0^L(x^*, \bar{x}) > U_1^L(x, \bar{x})$. Together with Lemma 11 for the case $x = \bar{x}$, we obtain the first part. Second, let $\bar{x} \in [\tau_L, \bar{y}_L]$. Then $\bar{x} \in Y_R$. Because $\bar{y}_L < \hat{\tau}$, $\bar{x} < \hat{\tau}$. By the second statement of Lemma 2, we obtain the second part. □

Lemma 14. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, 2, and 3. Let $\bar{y}_R$ be as defined in Lemma 12. Suppose $[\tau_L, \bar{y}_R] \subseteq Y_R$ and $[\bar{y}_R, \tau_R] \subseteq Y_L$. Then

- $BR_L(x)$ exists in response to $x \in [\bar{y}_R, \tau_R]$ and for every $x \geq \bar{x}$,

$$\mathrm{E}\Pi_L(BR_L(\bar{x}), \bar{x}) = U_0^L(\mathbf{x}_0^L(\bar{x}), \bar{x}) > U_1^L(x, \bar{x}).$$

- $BR_R(x)$ exists in response to $x \in [\tau_L, \bar{y}_R]$ and for every $x \leq \bar{x}$,

$$\mathrm{E}\Pi_R(BR_R(\bar{x}), \bar{x}) = U_1^R(\mathbf{x}_1^R(\bar{x}), \bar{x}) > U_0^R(x, \bar{x}).$$

Proof of Theorem 2. Similarly with Theorem 1, by Lemma 13, Lemma 10 is applicable. Finally, by Lemma 10 and Theorem MMT, there exists a PSE. □

4.4 Proof of Theorem 3

In this section, we study the condition under which both parties choose the same policy $\hat{\tau}$. Because the next theorem concerns the case where both parties choose the same policy, the value of p matters to the result. In the next lemma, we specify a condition under which a tie is *not* a PSE.

Lemma 15. Suppose that the game $\mathcal{G}$ satisfies Assumptions 1, and 2. Let $\bar{x} \neq \hat{\tau}$. Then, $(x_L, x_R) = (\bar{x}, \bar{x})$ is not a PSE.

Proof. Let $\bar{x} \neq \hat{\tau}$ be the policy chosen by parties, such that $x_L = x_R = \bar{x}$. Party L has no incentive to deviate from $x_L = \bar{x}$ only if

$$\tfrac{1}{2} \geq \max\left\{G(\sigma(\bar{x}, \bar{x})), 1 - G(\sigma(\bar{x}, \bar{x}))\right\},$$

because

$$\mathrm{E}\Pi_L(\bar{x}, \bar{x}) \geq \max\left\{\lim_{\varepsilon\to 0} \mathrm{E}\Pi_L(\bar{x} - \varepsilon, \bar{x}), \lim_{\varepsilon\to 0} \mathrm{E}\Pi_L(\bar{x} + \varepsilon, \bar{x})\right\}.$$

But $\frac{1}{2} \geq \max\left\{G(\sigma(\bar{x}, \bar{x})), 1 - G(\sigma(\bar{x}, \bar{x}))\right\}$ immediately implies that

$$\tfrac{1}{2} \leq \min\left\{G(\sigma(\bar{x}, \bar{x})), 1 - G(\sigma(\bar{x}, \bar{x}))\right\} < \max\left\{G(\sigma(\bar{x}, \bar{x})), 1 - G(\sigma(\bar{x}, \bar{x}))\right\}.$$

Note that the last inequality is strict because $\bar{x} \neq \hat{\tau}$. Then, for Party R,

$$\mathrm{E}\Pi_R(\bar{x}, \bar{x}) < \max\left\{\lim_{\varepsilon\to 0} \mathrm{E}\Pi_R(\bar{x} - \varepsilon, \bar{x}), \lim_{\varepsilon\to 0} \mathrm{E}\Pi_R(\bar{x} + \varepsilon, \bar{x})\right\},$$

and Party R has an incentive to deviate from $x_R = \bar{x}$ whenever Party L has no incentive to deviate from $x_L = \bar{x}$. By symmetry, Party L has an incentive to deviate from the strategy profile $(\bar{x}, \bar{x})$ if Party R does not. Hence, $(\bar{x}, \bar{x})$ cannot be a PNE as at least one party can be better off by deviating from $(\bar{x}, \bar{x})$. □

Proof of Theorem 3.
(If part) Suppose both (L) and (R) hold. Then, we have (L) $\frac{\sigma_1'(\hat{\tau},\hat{\tau})G_-'(\sigma(\hat{\tau},\hat{\tau}))}{G(\sigma(\hat{\tau},\hat{\tau}))} \geq \frac{-v'(\tau_L,\hat{\tau})}{k^L}$. We will show that $(\hat{\tau},\hat{\tau})$ is a PSE, or equivalently, that $BR_L(\hat{\tau}) = BR_R(\hat{\tau}) = \hat{\tau}$. By symmetry, we only prove $BR_L(\hat{\tau}) = \hat{\tau}$.

The first part of Lemma 5 shows that for $x_L > \hat{\tau}$,

$$\mathrm{E}\Pi_L(x_L,\hat{\tau}) < \mathrm{E}\Pi_L(\hat{\tau},\hat{\tau}). \tag{24}$$

Because $u_0^L(\hat{\tau},\hat{\tau}) > 0$ (see (4) for the definition of u_0^L), (24) also holds for all $x_L \in (\tau_L,\hat{\tau}]$ by Lemma 7. Finally, since condition (L) holds, (24) also holds for all $x_L < \tau_L$.

We have thus shown that $\hat{\tau}$ is maximizer of $\mathrm{E}\Pi_L(x_L,\hat{\tau})$ in X and thus, $BR_L(\hat{\tau}) = \hat{\tau}$. The proof that $BR_R(\hat{\tau}) = \hat{\tau}$ is identical.
(Only if part) Suppose $(\hat{\tau},\hat{\tau})$ is a PSE. By the same logic with (24), there is no better-response than $\hat{\tau}$ for Party L in response to Party R's strategy $\hat{\tau}$. Suppose to the contrary that condition (L) is false, and then $u_0^L(\hat{\tau},\hat{\tau}) < 0$ holds. Then, there must exist some $\varepsilon > 0$ sufficiently small such that $\mathrm{E}\Pi_L(\hat{\tau}-\varepsilon,\hat{\tau}) > \mathrm{E}\Pi_L(\hat{\tau},\hat{\tau})$, and $BR_L(\hat{\tau}) \neq \hat{\tau}$. Hence $(\hat{\tau},\hat{\tau})$ cannot be a PSE if condition (L) does not hold. The proof for the necessity of condition (R) for $(\hat{\tau},\hat{\tau})$ to be a PSE is identical. □

5 The Results of Party Polarization

In this section, we present three propositions about equilibrium policy choices. They demonstrate the conditions about office rent under which polarization, right-sided differentiation, and left-sided differentiation arise in equilibrium, where right-sided differentiation is a situation whereby both parties choose policies greater than the median of the distribution of the median voter's bliss point.

The intuition of the propositions can be summarized as follows. If the degree of parties' office motives is sufficiently high, there exists a PSE, and each party announces a policy located on the center. However, as the degree of the office rent decreases, an equilibrium in pure strategies may fail to exist. When one party's office rent is higher than the other, there is a situation such that equilibrium policies are both biased toward the bliss point of one party whose office rent is relatively lower. Then, the other party chooses a policy between the opponent's policy and the center.

For each $x_L \in X$, define

$$Z_L(x_L) = \{x_R \in X : u_0^L(x_L,x_R) < 0 \text{ and } x_R \geq x_L\}.$$

and for each $x_R \in X$, define

$$Z_R(x_R) = \{x_L \in X : u_1^R(x_R,x_L) > 0 \text{ and } x_L \leq x_R\},$$

Then, $Z_R(x_R)$ is the set of Party L's strategies x_L for which $\mathrm{E}\Pi_R(x, x_L)$ is increasing at $x = x_R$, while $Z_L(x_L)$ is the set of Party R's strategies x_R for which $\mathrm{E}\Pi_L(x, x_L)$ is decreasing at $x = x_L$. Notice that $Z_L(x_L)$ and $Z_R(x_R)$ are respectively the generalizations of Y_L and Y_R because $x \in Y_R$ if and only if $x \in Z_R(x)$ and $x \in Y_L$ if and only if $x \in Z_L(x)$.

Proposition 3.

- If a PSE (x_L^*, x_R^*) satisfies $x_L^* < \hat{\tau} < x_R^*$, then $x_L^* \in Z_R(\hat{\tau})$ and $x_R^* \in Z_L(\hat{\tau})$.
- Conversely, suppose $[\tau_L, \hat{\tau}] \subseteq Y_R \cap Z_R(\hat{\tau})$ and $[\hat{\tau}, \tau_R] \subseteq Y_L \cap Z_L(\hat{\tau})$. Then, a PSE (x_L^*, x_R^*) satisfies $x_L^* < \hat{\tau} < x_R^*$.

Proof. Suppose that the parties choose policies on different sides of the center such that $x_L^* < \hat{\tau} < x_R^*$. Then, by Lemmas 8 and 9, it must be the case that $U_0^L(x, x_R^*)$ is decreasing at $x = \hat{\tau}$ and $U_1^R(x, x_L^*)$ is increasing at $x = \hat{\tau}$. Thus, we obtain the first statement.

On the other hand, suppose $[\tau_L, \hat{\tau}] \subseteq Y_R$ and $[\hat{\tau}, \tau_R] \subseteq Y_L$. By Theorem 1, there exists a PSE (x_L^*, x_R^*). On the contrary, suppose that $x_R^* < \hat{\tau}$. By Lemmas 5 and 15, $x_L^* < x_R^* < \hat{\tau}$. However, because $x_L^* \in Z_R(\hat{\tau})$, $\mathrm{E}\Pi_R(x, x_L^*)$ is increasing at $x = \hat{\tau}$ and $\mathbf{x}_1^R(x_L^*) > \hat{\tau}$, which contradicts $x_R^* < \hat{\tau}$. □

Even when equilibrium policies of two parties are different at $x_L^* = \mathbf{x}_0^L(x_R^*)$ and $x_R^* = \mathbf{x}_1^R(x_L^*)$, there are two possible situations in the sense that equilibrium policies are located on different sides or on the same side of the center. In the next two propositions, we provide the conditions under which one-sided differentiation arises, in which both parties choose policies on the same side with respect to $\hat{\tau}$. By symmetry, we only prove Proposition 5.

Proposition 4.

- If a PSE (x_L^*, x_R^*) satisfies $x_L^* < x_R^* < \hat{\tau}$, then $x_L^* \notin Z_R(\hat{\tau})$.
- Conversely, suppose that the conditions of Theorem 2-1 hold. When $[\hat{\tau}, \tau_R] \subseteq Z_L(\hat{\tau})$ and $(\tau_L, \hat{\tau}) \cap Z_R(\hat{\tau}) = \varnothing$, a PSE (x_L^*, x_R^*) satisfies $x_L^* < x_R^* < \hat{\tau}$.

Proposition 5.

- If a PSE (x_L^*, x_L^*) satisfies $\hat{\tau} < x_L^* < x_R^*$, then $x_R^* \notin Z_L(\hat{\tau})$.
- Conversely, suppose that the conditions of Theorem 2-2 hold. When $[\tau_L, \hat{\tau}] \subseteq Z_R(\hat{\tau})$ and $(\hat{\tau}, \tau_R) \cap Z_L(\hat{\tau}) = \varnothing$, a PSE (x_R^*, x_R^*) satisfies $\hat{\tau} < x_L^* < x_R^*$.

Proof. Suppose that a PSE (x_L^*, x_R^*) satisfies $\hat{\tau} < x_L^* < x_R^*$. Then, by Lemma 8, it must be the case that $U_0^L(x, x_R^*)$ is increasing at $x = \hat{\tau} + \epsilon$ for sufficiently small ϵ. By Lemma 7, we obtain the first statement.

To show the second statement, note that by Theorem 2, there is a PSE (x_L^*, x_R^*). On the contrary, suppose $\hat{\tau} \geq x_L^*$. Because $[\tau_L, \hat{\tau}] \subseteq Z_R(\hat{\tau})$, $x_L^* \in Z_R(\hat{\tau})$ and then $U_1^L(x, x_L^*)$ is increasing at $x = \hat{\tau}$. Thus, $x_R^* > \hat{\tau}$ as it maximizes $U_1^L(x, x_R^*)$. However, as $(\hat{\tau}, \tau_R) \cap Z_L(\hat{\tau}) = \varnothing$ indicates $x_R^* \notin Z_L(\hat{\tau})$, $\mathrm{E}\Pi_L(x, x_R^*)$ is increasing at $x = \hat{\tau}$ and this contradicts the assumption that x_L^* constitutes a PSE. □

6 The Existence of Mixed Strategy Equilibrium

We have given conditions under which a PSE exists. The next proposition is the existence result including an MSE. We borrow the argument from Simon and Zame (1990). Simon and Zame (1990) introduce an endogenous sharing rule which guarantees the existence of an MSE. Our strategy then is to show that an MSE in the game with an endogenous sharing rule is an MSE in the original game and a tie happens with a probability zero, so that a sharing rule indeed does not matter in the original game.

Proposition 6. An equilibrium (a PSE or/and an MSE) exists.

For every $(x_L, x_R) \in X \times X$, define a *payoff correspondence* $Q : X \times X \to \mathbb{R}^2$ to be

$$Q(x_L, x_R) = (\mathrm{E}\Pi_L(x_L, x_R), \mathrm{E}\Pi_L(x_L, x_R)).$$

A *sharing rule* is a Borel measurable function $q : X \times X \to \mathbb{R}^2$ such that $q(x_L, x_R) \in Q(x_L, x_R)$ for every $(x_L, x_R) \in X \times X$. Notice that the payoff correspondence Q is discontinuous when $x_L = x_R$.

A mixed strategy for party $i = L, R$ is a probability measure on X and a mixed strategy profile is a pair (α_L, α_R) of mixed strategies. A *solution* for the game $\mathcal{G}$ is a sharing rule q and a mixed strategy profile (α_L, α_R) such that q is a Borel measurable selection from the payoff correspondence Q, and (α_L, α_R) is a profile of mixed strategies such that for each $i = L, R$ and each probability measure β_i on X,

$$\int q_i(x_L, x_R)\mathrm{d}(\alpha_i \times \alpha_{-i}) \geq \int q_i(x_L, x_R)\mathrm{d}(\beta_i \times \alpha_{-i}). \tag{25}$$

Each sharing rule q defines a game $\mathcal{G}_q$, while a solution for $\mathcal{G}$ is a sharing rule q and a profile of mixed strategies (α_L, α_R) which constitutes a Nash equilibrium in the game $\mathcal{G}_q$. By the main theorem of Simon and Zame (1990), every game with an endogenous sharing rule has a solution.

Proof of Proposition 6. When both conditions (L) and (R) hold, by Theorem 3, a PSE exists. Fan-Glicksberg's fixed point theorem[1] is an extension of Kakutani's fixed point theorem to

[1]For details, see page 108 of McLennan (2014).

correspondences with infinite dimensional domains, stating that if V is a locally convex topological vector space, $X \times X \subset V$ is nonempty, convex, and compact, and $F : X \times X \to X \times X$ is an upper semicontinuous convex valued correspondence, then F has a fixed point.

Let X^* be a dense subset of $X \times X$ and a bounded continuous function $\psi : X^* \to \mathbb{R}^2$. Let $C_\psi : X \times X \to \mathbb{R}^2$ be the correspondence whose graph is the closure of the graph of ψ and define $Q_\psi(x_L, x_R)$ to be the convex hull of $C_\psi(x_L, x_R)$ for each $(x_L, x_R) \in X \times X$. We claim that Q_ψ defined above is bounded, upper semi-continuous, has nonempty convex, compact values and $Q_\psi(x_L, x_R) = \psi(x_L, x_R)$ for each $(x_L, x_R) \in X^*$. Any selection q from the correspondence Q_ψ agrees with ψ on X^*, and thus, every sharing rule is an extension of the given payoff function ψ on X^* to the entire space $X \times X$. Therefore, for each $i = L, R$, and α_{-i}, a best response to α_{-i} satisfying (25) exists and by Fan-Glicksberg's fixed point theorem, a fixed point exists. Then, its fixed point is a Nash equilibrium in the game $\mathcal{G}$ with a sharing rule q.

Now, we consider the case such that either condition (L) or (R) does not hold. By symmetry, suppose that condition (L) does not hold. We show that a tie happens in equilibrium with probability zero. On the contrary, suppose that for some $\bar{x} \in X$, a probability measure β_i on X satisfies $\beta_i(\bar{x}) > 0$ for each $i = L, R$ and $\beta = (\beta_L, \beta_R)$ constitutes an MSE. Then, construct α_L such that $\alpha_L(\bar{x} - \epsilon) = \beta_L(\bar{x})$ and $\alpha_L(\bar{x}) = 0$. Then, when $\bar{x} \neq \hat{\tau}$,

$$\int q_L(x_L, x_R) \mathrm{d}(\alpha_L \times \beta_R) > \int q_L(x_L, x_R) \mathrm{d}(\beta_L \times \beta_R). \tag{26}$$

This is a contradiction with the assumption that $\beta = (\beta_L, \beta_R)$ constitutes an MSE in this case. Further when $\bar{x} = \hat{\tau}$, the same contradiction with (26) arises because since condition (L) does not hold, Party L is better off by deviating from $\hat{\tau}$. □

7 Discussion with the Literature

So far, by using C-security, we have given conditions under which various types of equilibria arise. Our results relating to party polarizations generalize the preceding works such as Roemer (1994, 1997) and Saporiti (2008), Drouvelis, Saporiti and Vriend (2014). In this section, we take the Roemer (1997) model and show how our results can be applied in the Roemer (1997) framework. Roemer (1997) is useful in the sense that this model provides a micro-foundation of the model we have considered in the previous section. Here, we will demonstrate the conditions about office rent under which each type of polarization arises in equilibrium. Finally, we show that when the income of the voter that each party tries to maximizes goes further away from the median, then the party's choice also goes further away from the middle point as a best response to the opponent's party choice. This result is in line with the empirical observation in Smidt (2015).

7.1 The Roemer (1997) Model

We describe an electoral competition game $\mathcal{G}$ in the Roemer (1997) model. In the model, there are two political parties and two goods: a *private good* and a *public good*. Public goods are financed by a tax on income, and voters' income levels are heterogeneous. Voters care about tax rates, and there is an uncertainty about the median income.

There is a continuum of voters with a *direct utility function*, $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$u(b, g) = b + \varphi(g),$$

where φ is increasing and strictly concave, b is the individual voter's consumption level of the private good, and g is the per capita value of the public good.

As the public good is financed entirely by tax revenue, the political issue is a tax rate x. Define a voter's *indirect utility function*, $v : \mathbb{R} \times X \to \mathbb{R}$, by

$$v(\tau, x) = (1 - x)h(\tau) + \varphi(x\bar{h}),$$

where $h(\tau)$ is an income of voter τ and $\bar{h}$ is the mean income in the economy. Since φ is strictly concave, $v(\tau, x)$ can be shown to be single-peaked in x. For each $\tau \in X$, define $h(\tau) = \bar{h}\varphi'(\tau\bar{h})$ and then τ maximizes $v(\tau, x)$ for a voter with income $h(\tau)$. Note that τ_R increases if and only if $h(\tau_R)$ decreases by the relationship $h(\tau) = \bar{h}\varphi'(\tau\bar{h})$ and the strict concavity of φ.

Now suppose that bliss points of party L's core electorates are represented by the income level w_L and party R's ones are represented by the income level w_R with $w_L > w_R$.[2] Then Assumptions 1 and 2 hold. After making Assumption 3 implying Lemma 3, which corresponds to Assumption A4* in Roemer (1997), our three main theorems and propositions are applicable to this model.

Roemer (1994) and Roemer (1997) consider the model in which there is no office rents, namely $k^L = k^R = 0$ and define *single crossing property* (SCP) in the following way. Let H denote a set of possible incomes. Then, a family of functions $\{v(\tau, x) | \tau \in X\}$ satisfies the *single crossing property* if for all $\tau \in X$ and $x_1 \neq x_2$, $v(\tau, x_1) = v(\tau, x_2)$ implies that for $\tau' \neq \tau$, $v(\tau', x_1) \neq v(\tau', x_2)$.

In Roemer (1994), it is shown that in a model with no uncertainty, the only equilibrium consists of both parties proposing the median voter's bliss point when the SCP holds. On the other hand, in Roemer (1997), it is shown that a PSE exists and each equilibrium involves parties putting forth different policies when there is uncertainty about the median voter's bliss point, while as the uncertainty becomes smaller, then the equilibrium converges to the one in Roemer (1994) such that both parties choose the same policy.

[2]This assumption is symmetrically opposite to the one in the original Roemer (1997) model. We assume $w_L > w_R$ so that $\tau_L < \tau_R$ holds as in our model of the second section.

First of all, when $k^L = 0$ and $k^R = 0$, $[\tau_L, \hat{\tau}] \subseteq Y_R$ and $[\hat{\tau}, \tau_R] \subseteq Y_L$ hold. Thus Theorem 1 implies the existence of a PSE in the Roemer (1997) model.

Second, Theorems 1 and 3 show that our model captures the interesting features of the Roemer (1994) and Roemer (1997) models. In our model, there is an uncertainty about the median voter's bliss point, while office rents k^L and k^R are deterministic variables. The first two theorems state that when office rents are small, the equilibrium is similar to the one in Roemer (1997) where both parties choose different policies, and when they are large, the equilibrium is similar to the one in Roemer (1994).

Finally, we show that when the income of the voter that each party tries to maximizes goes further away from the median, then the party's choice also goes further away from the middle point as a best response to the opponent's party choice. This result is in line with the empirical observation in Smidt (2015).

Proposition 7. Suppose that (x_L^*, x_R^*) is a PSE and $x_L^* \neq x_R^*$. Holding all others constant, as τ_R increases, $h(\tau_R)$ decreases and x_R^* increases. Similarly, holding all others constant, as τ_L decreases, $h(\tau_L)$ increases and x_L^* decreases.

Proof. For every $\tau_R \in X$ and $x \in X$,

$$\begin{aligned} -v'(\tau_L, x) &= h(\tau_L) - \bar{h}\varphi'(x\bar{h}) = \bar{h}\left(\varphi'(\tau_L\bar{h}) - \varphi'(x\bar{h})\right) \text{ and} \\ v(\tau_R, x) - v(\tau_R, \bar{x}) &= (\bar{x} - x)\bar{h}\varphi'(\tau_L\bar{h}) + \varphi(x\bar{h}) - \varphi(\bar{x}\bar{h}). \end{aligned}$$

Then

$$\begin{aligned} \frac{\mathrm{d}\left(-v'(\tau_L, x)\right)}{\mathrm{d}\tau_L} &= (\bar{h})^2\varphi''(\tau_L\bar{h}) \\ \frac{\mathrm{d}\left(v(\tau_L, x) - v(\tau_L, \bar{x})\right)}{\mathrm{d}\tau_L} &= (\bar{x} - x)(\bar{h})^2\varphi''(\tau_L\bar{h}). \end{aligned}$$

As τ_L decreases, $h(\tau_L)$ increases. Also, the denominator increases by $|(\bar{x} - x)(\bar{h})^2\varphi''(\tau_L\bar{h})|$, while the numerator increases by $|(\bar{h})^2\varphi''(\tau_L\bar{h})|$. Because $0 < \bar{x} - x < 1$ and the numerator increases more than the denominator, the RHS of (21) increases. Thus, the LHS becomes smaller than the RHS. When x increases, the LHS decreases by Assumption 3, while the RHS increases. Thus, the difference becomes even larger. To equalize both sides, x must decrease. □

7.2 Numerical Examples

To elaborate on C-security and the three propositions in the previous section, we present some numerical examples by using the Roemer (1997) model. In the examples of this subsection, we use a uniform distribution for incomes and let $h(\tau_R) = 0.2$, $h(\tau_L) = 0.8$, $\bar{h} = 0.5$ and

$\varphi(x\bar{h}) = 0.5\sqrt{x\bar{h}}$. Then, $\tau_L = 0.049$, $\hat{\tau} = 0.125$, and $\tau_R = 0.781$. Then Assumptions 1, 2, and 3 are satisfied by the settings for the examples.

The first two panels of the first Figure (1a) and (1b) show the payoffs of Party L and Party R, when the opponent's strategy equals $\bar{x} = 0.1$. In this example, C-security holds at $\bar{x}$, as clearly the maximizers $BR_L(\bar{x})$ and $BR_R(\bar{x})$ yield higher payoffs. However, when k^L increases to 2.00, as we see in the remaining two panels of the Figure (1c) and (1d), Party L's best response is not well defined in the sense that Party L would be better off by choosing as close as possible to $\bar{x}$, but once $\bar{x}$ is chosen, the payoff drops.

Next, Figure 2 provides the illustrative examples for party polarization. The first two panels, (2a) and (2b) present the payoffs in PSE when $(k^L, k^R) = (0.0005, 0.08)$. In the mathematical calculation, we have obtained $\bar{y}_L = 0.074$, and left-sided PSE where $x_R^* = 0.076$ and $x_L^* = 0.058$. In this example, Party R's best response x_R^* yields higher payoffs than $\hat{\tau}$ or x_L^* such that

$$\mathrm{E}\Pi_R(x_R^*, x_L^*) = 0.3453 > \mathrm{E}\Pi_R(\hat{\tau}, x_L^*) = 0.3449 > \mathrm{E}\Pi_R(\hat{\tau}, x_L^*) = 0.3437.$$

On the other hand, in the second two panels, (2c) and (2d), we set $(k^L, k^R) = (0.02, 0.01)$, where we obtain polarized PSE. In this example, we have obtained $x_R^* = 0.316$ and $x_L^* = 0.091$. In this situation, both parties choose different policies on the different sides with respect to $\hat{\tau}$.

Finally, we will demonstrate how the required conditions in the propositions are met in the above two examples of $(k^L, k^R) = (0.0005, 0.08)$ and $(k^L, k^R) = (0.02, 0.01)$. Recall that when $(k^L, k^R) = (0.0005, 0.08)$, we have $(x_L^*, x_R^*) = (0.058, 0.076)$ (left-sided PSE) and when $(k^L, k^R) = (0.02, 0.01)$, we have $(x_L^*, x_R^*) = (0.091, 0.316)$ (polarized PSE). Figure 3 illustrates the conditions in Proposition 4.

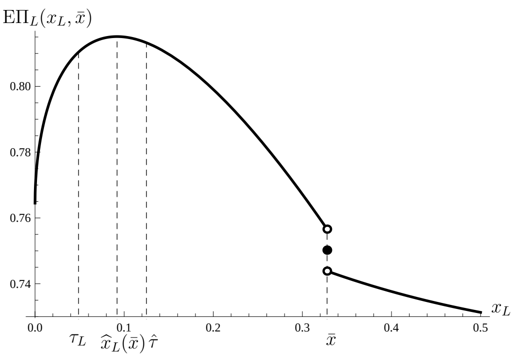
The first two panels, Figure (3a) and (3b) present the first derivatives, $u_1^R(x_R, x_R)$ and $u_0^L(x_L, x_L)$ on $[\tau_L, \tau_R]$ when $(k^L, k^R) = (0.0005, 0.08)$. As Theorem 2 requires, we can see that that $u_1^R(x_R, x_R)$ is strictly positive on $[\tau_L, \bar{y}_L]$, indicating that $[\tau_L, \bar{y}_L] \subset Y_R$, while $u_0^L(x_L, x_L)$ is strictly negative on $[\bar{y}_L, \tau_R]$, indicating that $[\bar{y}_L, \tau_R] \subset Y_L$. The second two panels (3c) and (3d) show $u_0^L(\hat{\tau}, x_R)$ on $[\hat{\tau}, \tau_R]$ and $u_1^R(\hat{\tau}, x_L)$ on $[\tau_L, \hat{\tau}]$. In this situation, both parties would not choose the same policy. Party R chooses a policy greater than the one that Party L chooses. Figure (3c) shows that $u_0^L(\hat{\tau}, x_R)$ is strictly negative on $[\hat{\tau}, \tau_R]$, which indicates that $[\hat{\tau}, \tau_R] \subset Z_L(\hat{\tau})$, as Proposition 4 requires. Further, Figure (3d) shows that $u_1^R(\hat{\tau}, x_L)$ is strictly negative on $[\tau_L, \bar{y}_L]$, which indicates that $[\tau_L, \bar{y}_L] \cap Z_R(\hat{\tau}) = \varnothing$, as Proposition 4 requires. Under this circumstance, because both parties' payoffs are decreasing at $\hat{\tau}$ in response to the opponent's strategy so that both choose policies smaller than $\hat{\tau}$, both $u_1^R(\hat{\tau}, x_L)$ and $u_0^L(\hat{\tau}, x_R)$ are decreasing on $[\tau_L, \hat{\tau}]$ and $[\hat{\tau}, \tau_R]$. Thus, in this situation, as we see from the first two panels of Figure (2a) and (2b), we obtain $x_R^* = 0.076$ and $x_L^* = 0.058$ as a PSE. As shown in Proposition 4, in this situation, both parties choose policies on the left side of $\hat{\tau}$.

Next, Figure 4 illustrates the conditions in Proposition 3. In this case, we set $(k^L, k^R) =$

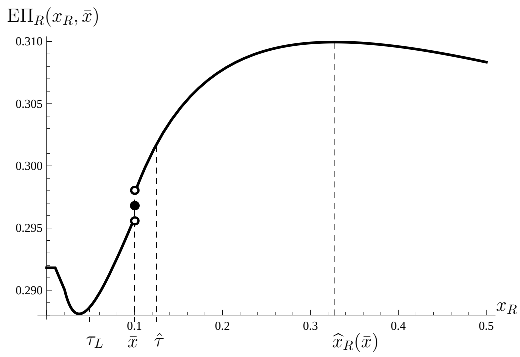
$(0.02, 0.01)$. The first two panels, Figure (4a) and (4b) present $u_0^L(x_L, x_L)$ and $u_1^R(x_R, x_R)$ on $[\tau_L, \tau_R]$, in which we can see that $[\tau_L, \hat{\tau}] \subset Y_R$ and $[\hat{\tau}, \tau_R] \subset Y_L$. By Lemmas 8 and 9, we can guarantee a best response to each strategy in Y_L or Y_R. The second two panels, (4c) and (4d) show that $u_0^L(\hat{\tau}, x_R)$ is strictly negative on $[\hat{\tau}, \tau_R]$ and $u_1^R(\hat{\tau}, x_L)$ is strictly positive on $[\tau_L, \hat{\tau}]$. This indicates that $[\hat{\tau}, \tau_R] \subset Y_L \cap Z_L(\hat{\tau})$ and $[\tau_L, \hat{\tau}] \subset Y_R \cap Z_R(\hat{\tau})$. Then, Party L's payoff is decreasing at $\hat{\tau}$, while Party R's payoff is increasing at $\hat{\tau}$, in response to the opponent's strategy in Y_L or Y_R. In this way, both parties choose policies on the different sides of $\hat{\tau}$, as proved in Proposition 3. More specifically, in this example, we obtain $x_R^* = 0.316$ and $x_L^* = 0.091$, as we have seen in the first two panels of Figure (2a) and (2b).

Figure 1: Payoffs ($\bar{x} = 0.1$)

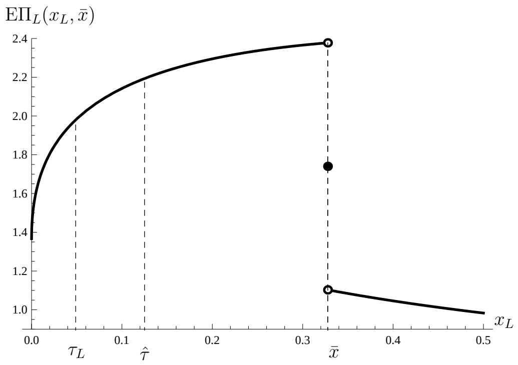
(a) L for $(k_L, k_R) = (0.02, 0.01)$



(b) R for $(k_L, k_R) = (0.02, 0.01)$



(c) L for $(k_L, k_R) = (2.00, 0.01)$



(d) R for $(k_L, k_R) = (2.00, 0.01)$

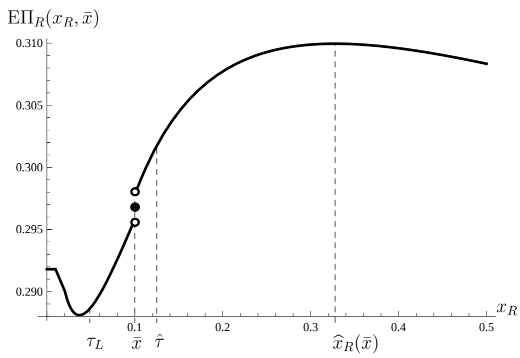
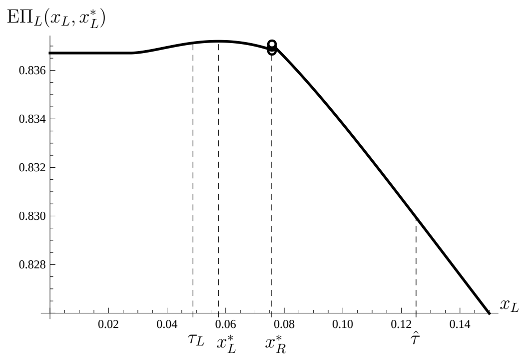
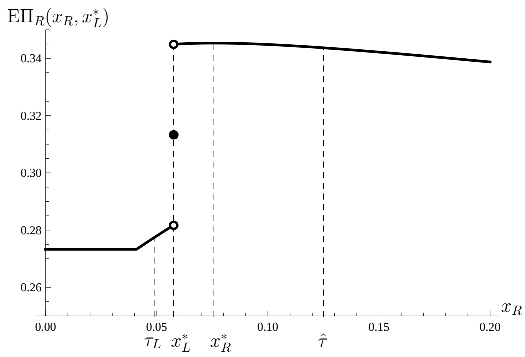


Figure 2: PSE

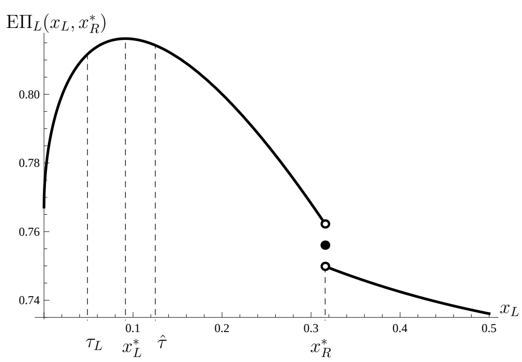
(a) L for $(k_L, k_R) = (0.0005, 0.08)$



(b) R for $(k_L, k_R) = (0.0005, 0.08)$



(c) L for $(k_L, k_R) = (0.02, 0.01)$



(d) R for $(k_L, k_R) = (0.02, 0.01)$

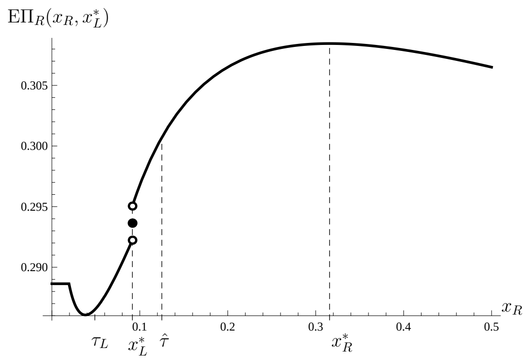
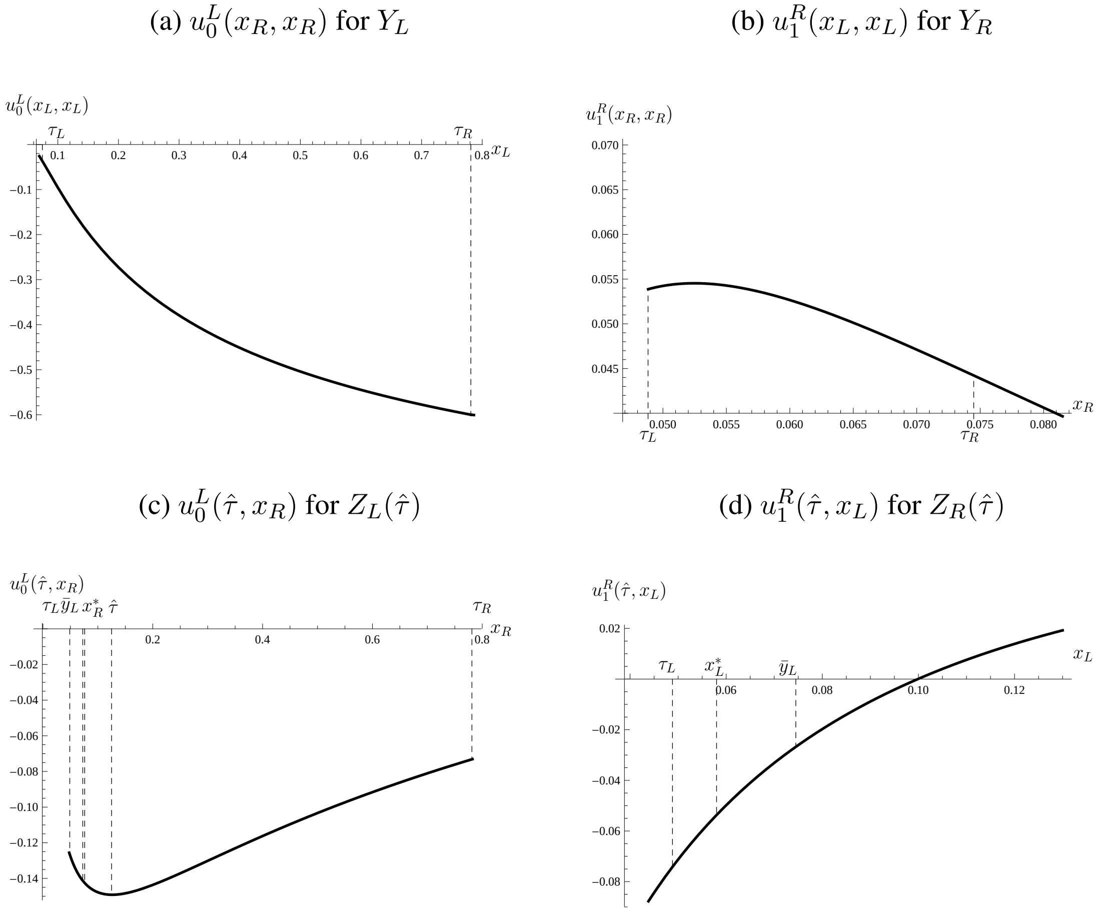


Figure 3: Payoffs in Left-Sided PSE



(a) $u_0^L(x_R, x_R)$ for Y_L

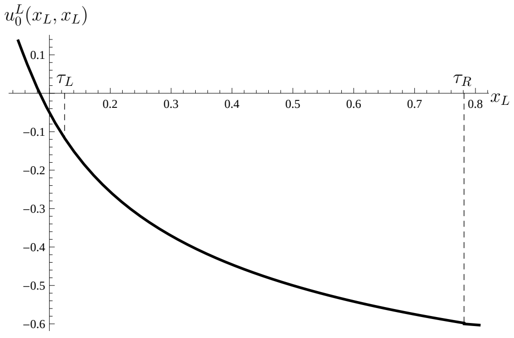
(b) $u_1^R(x_L, x_L)$ for Y_R

(c) $u_0^L(\hat{\tau}, x_R)$ for $Z_L(\hat{\tau})$

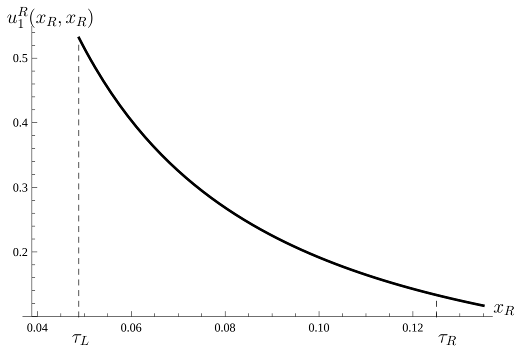
(d) $u_1^R(\hat{\tau}, x_L)$ for $Z_R(\hat{\tau})$

Figure 4: Payoffs in Poralized PSE

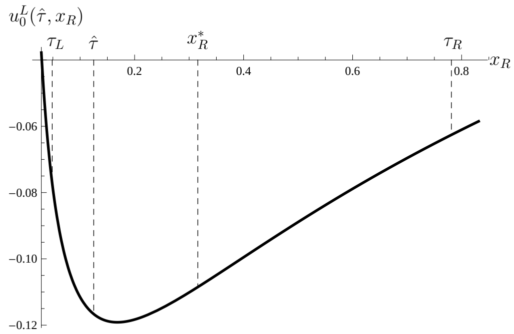
(a) $u_0^L(x_L, x_L)$ for Y_L



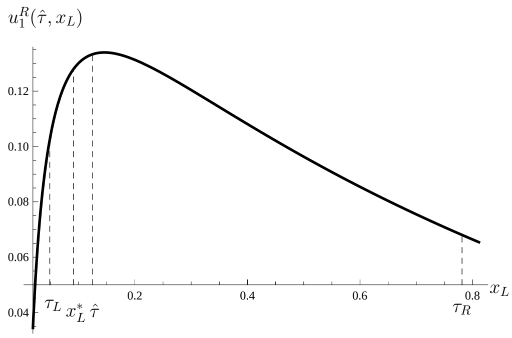
(b) $u_1^R(x_R, x_R)$ for Y_R



(c) $u_0^L(\hat{\tau}, x_R)$ for $Z_L(\hat{\tau})$



(d) $u_1^R(\hat{\tau}, x_L)$ for $Z_R(\hat{\tau})$



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