Nash Equilibrium and Party Polarization in an Electoral Competition Model*

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Abstract

We study the existence problem of Nash equilibrium as well as party polarization in an electoral competition model. In our model, political parties also value holding office (office rent) in addition to maximizing their party members' utility. A class of models with an uncertainty about the median voter position has been increasingly important and Drouvelis, Saporiti and Vriend (2014) present an experimental study to support a model with office rent. But the inclusion of office rent renders the payoff of each party discontinuous. This makes it difficult to apply a usual fixed point argument to prove the existence of Nash equilibrium. By using a recently developed concept, *C-security* in McLennan, Monteiro and Tourky (2011), we provide conditions under which a pure strategy Nash equilibrium (PSE) or a mixed strategy Nash equilibrium (MSE) exists within a fairly general setting, and further the analysis by presenting conditions under which various types of policy choices, including polarization, arise in equilibrium.

Key Words: Noncooperative games, electoral competition, existence of equilibrium.

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1 Introduction

This paper makes several contributions to the literature of political competition. We provide an existence proof of Nash equilibrium within a fairly general framework in the literature. Further, we present conditions under which several types of policy outcomes, including convergence, one-sided differentiation, and polarization, arise in equilibrium. In our model, there is an uncertainty about the median voter's bliss point, and we call the median of the distribution of the median voter's bliss point the *center*. Convergence is a situation whereby both parties choose the same policy. Polarization is a situation whereby both parties choose policies on the different sides of the center. One-sided differentiation is a situation whereby both parties choose different policies but on the same side relative to the center.

A model with uncertainty about the median voter's bliss point has been increasingly important in the political science literature (Aragones and Palfrey, 2005, Takayama, 2007, Saporiti, 2008, Hummel, 2013, Takayama, 2014, Drouvelis, Saporiti and Vriend, 2014). In this paper, we study the model with uncertainty about the median voter's bliss point, which also includes an additional feature of mixed motivation where political parties value holding office, in addition to maximizing their party members' utility. As Drouvelis, Saporiti and Vriend (2014) claims, the mixed motivations hypothesis is conceptually more realistic than the traditional hypotheses of candidates' motivations. However, electoral competition models with mixed motivations generally have discontinuous and non-quasiconcave payoffs, which makes it difficult to guarantee the existence of a pure strategy Nash equilibrium (PSE).

In our model, there are two political parties and two goods: a *private good* and a *public good*. Public goods are financed by a tax on income, and voters' income levels are heterogeneous. Voters care about tax rates, and there is an uncertainty about the median income. Our model adopts a mixed motivation framework, which extends the Roemer (1997) model by incorporating an *office rent* for a party, which is a positive payoff from being in power. Our model is also viewed as a general version of the Drouvelis, Saporiti and Vriend (2014) model, which uses a mixed motivation framework under the assumption of uniform distribution about the median voter's bliss point. In our model, we do not assume any particular functional form for the distribution, and the voters' bliss points are not necessarily symmetrically distributed.

The first contribution of this paper is the existence proof, which provides a connection with recent developments of game theory literature. To the best of our knowledge, this is the first to apply the concept of *C-security* by McLennan, Monteiro and Tourky (2011) to the existence result in this framework. By using C-security, we provide a simple proof of the existence of a PSE. C-security generalizes Reny's better-reply security to non-quasiconcave games. Our existence theorem implies the existence of a PSE in the original Roemer (1997) model. Finally, we also show that when each candidate wins with equal probability in an electoral tie, there is always an equilibrium, either a mixed strategy Nash equilibrium (MSE) or a PSE, by using the theorem in Simon and Zame (1990). We also show that when a tie is not broken with equal probability, an equilibrium does not exist.

The second contribution is to provide the conditions under which party polarization arises in PSE. Our results about party polarization can be summarized as follows. If the degree of parties' office rent is sufficiently high, there exists a PSE, and each party announces a policy located on the center. This is because each party values winning the office rather than maximizing their party's voter welfare. However, as the degree of the office rents decreases, an equilibrium in pure strategies may fail to exist and each party tends to choose somewhere between the party's bliss point and the center. Another interesting phenomenon can arise in equilibrium. When one party's office rent is higher than that of the other, the equilibrium policies are biased toward the preferred policy of the one party whose office rent is lower. Then, the other party chooses a policy between this opponent's policy choice and the bliss point for the center.

In Roemer (1994), it is shown that in a model with no uncertainty, the only equilibrium consists of both parties proposing the median voter's bliss point. On the other hand, in Roemer (1997), it is shown that each equilibrium involves parties putting forth different policies when there is uncertainty about the median voter's bliss point. However as the uncertainty becomes smaller, then the policies of the two parties converge just as in Roemer (1994). Our model captures these features of the two models. In our model, office rents are deterministic variables. The first two theorems state that when office rents are small, the equilibrium is similar to the one in Roemer (1997), where both parties choose different policies, and when office rents are large, the equilibrium is similar to the one in Roemer (1994).

The closest predecessor of our third theorem is the one in Drouvelis, Saporiti and Vriend (2014). Drouvelis, Saporiti and Vriend (2014) find that both polarization as well as one-sided differentiation could occur in equilibrium by assuming that the median voter's bliss point is uniformly distributed. Their theoretical predictions are supported by the data collected from a laboratory experiment. Our analysis generalizes their theoretical result to a broader class of distribution functions for the median voter's position. Another aspect is that having a public good in the model, our analysis implies that when one-sided differentiation arises, either more or less of the public good than the level associated with the center is provided with certainty, and this distortion is toward the party which is less motivated by office rents.

Finally, we show that as the difference of the incomes between the two parties' supporters increases, the degree of polarization increases. We also provide numerical illustrations on how party polarization arises depending on the magnitude of office rents for the two parties.

Politics has been increasingly polarized across the world (see McCarty, Poole and Rosenthal, 2016, Benoit and Laver, 2006). The classical works including Wittman (1983), Hansson and Stuart (1984), Calvert (1985), and Roemer (1994) address this issue of party polarization and study whether electoral equilibrium is characterized by the median voter position. This theme continued to be an important one in the literature. Recent works including Alesina and Rosenthal (2000), McMurray (2015), Eyster and Kittsteiner (2007) and Esponda and Pouzo (2016) also address this issue.

Sometimes, polarization is viewed as a potential cause of dysfunctional politics. Meanwhile, Smidt (2015) illustrates that parties are likely to benefit from polarization by gaining reliable supporters among even nonpartisans. While we have observed political parties' polarization around the world, the median voter theorem is a core in the theoretic analysis of political competition. When the median voter theorem holds, there is a very strong incentive for parties to choose what the median voter prefers. How office-seeking parties balance the *centrifugal* incentive to appeal to their voters against the *centripetal* motivation to appeal to voters in the general population is a key theme in the theoretical literature of electoral competition. Our approach in this paper is to adopt a fairly general framework in the literature and by applying the notion of C-security, to explore the mechanism involved in various types of policy choices including poralization. By doing so, our paper provides fundamental contributions to the issue.

The paper is organized as follows. The second section describes the model and the three main theorems. The third section provides preliminary results and the fourth section provides the proofs of the main theorems. The fifth section presents the existence of an MSE when a tie breaks at an equal probability. The sixth section presents the results of party polarization, and provides numerical illustrations on the results. The last section concludes.

2 The Model

We describe an electoral competition game $\mathcal{G}$. Consider an economy with two goods: a *private good* and a *public good*. There is a continuum of voters with a *direct utility function*, $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$u(b, g) = b + \varphi(g),$$

where φ is increasing and strictly concave, b is the individual voter's consumption level of the private good, and g is the per capita value of the public good.

We denote income by h and its mean by $\bar{h}$. As the public good is financed entirely by tax revenue, the political issue is a tax rate t. Denote the set of possible rates $[0, 1]$ by X. Define a voter's *indirect utility function*, $v : \mathbb{R} \times X \to \mathbb{R}$, by

$$v(h, x) = (1 - x)h + \varphi(x\bar{h}). \tag{1}$$

Then, since φ is strictly concave, $v(h, x)$ can be shown to be single-peaked in x. By using (1), we define the monotonic bliss point of a voter whose income is h, $\tau_h \in [0, 1]$, as

$$\tau_h = \operatorname*{argmax}_{x \in X} v(h, x). \tag{2}$$

There are two political parties, *Party L* and *Party R*. Party L wants to maximize the utility of voters whose income is h_L, while Party R wants to do the same for voters with an income of h_R. We assume $h_L < h_R$. Parties have mixed motivations in the sense that they are interested in winning the election not just to benefit their own party members, but also to capture office rents. Let $k^i \geq 0$ be the *office rent*, which is the intrinsic value that party $i = L, R$ places on holding office. The values of k^i are common knowledge.

The randomly distributed median voter's income has a continuous cumulative distribution function G and its support is $[h_L, h_R]$. Because $v(h, x)$ is single-peaked in t, for each party $i = L, R$, there is a single ideal policy τ_{h_i}. For simplicity, denote $\tau_{h_R} = \tau_R$, and $\tau_{h_L} = \tau_L$, respectively. We can think of these as each party's bliss point. The voter with the median income is the median voter. Let τ_m denote the median voter's bliss point. Then, $\tau_R < \tau_m < \tau_L$. Let $\hat{h}$ denote $G(\hat{h}) = \frac{1}{2}$. Then $\hat{h}$ is the median of a median voter's income. Because $G(h_L) = 0$, $G(h_R) = 1$ and G is continuous, $\hat{h}$ exists in (h_L, h_R). Let $\hat{\tau}$ denote voter $\hat{h}$'s bliss point and then $\hat{\tau} = \operatorname*{argmax}_{x \in X} v(\hat{h}, x)$. We call $\hat{h}$ the *center*.

Each party i independently and simultaneously announces a policy $x_i \in X$. After observing the two announced policies, each voter votes for a party whose announced policy is closer to his ideal policy. Finally, parties implement their announced policies if they win the election.

Given a pair of announced policies (x_L, x_R), let $\pi(x_L, x_R)$ be the probability that L wins the election. When both parties choose the same policy $x \in X$, we set $\pi(x, x) = p \in [0, 1]$ as the tie-breaking rule.[1] Because (1) is single-peaked in x, the median voter theorem holds and Party L wins if and only if the median voter votes for Party L. Thus, Party L wins if and only if $v(h_m, x_L) \geq v(h_m, x_R)$, equivalently $h_m \geq \frac{\varphi(x_R \bar{h}) - \varphi(x_L \bar{h})}{x_R - x_L}$.

From (1),

$$\pi(x_L, x_R) = \Pr[v(h_m, x_L) \geq v(h_m, x_R)].$$

The two parties' objective functions are:

$$\begin{aligned} \mathrm{E}\Pi_L(x_L, x_R) &= \pi(x_L, x_R)(v(h_L, x_L) + k^L) + (1 - \pi(x_L, x_R))v(h_L, x_R) \text{ and} \\ \mathrm{E}\Pi_R(x_R, x_L) &= \pi(x_L, x_R)v(h_R, x_L) + (1 - \pi(x_L, x_R))(v(h_R, x_R) + k^R). \end{aligned} \tag{3}$$

Note that $\frac{\varphi(x_R \bar{h}) - \varphi(x_L \bar{h})}{x_R - x_L}$ is $\bar{h}$ times a slope of $\varphi(x \bar{h})$ between x_L and x_R. For every $x \in X$, let

$$l(x) \equiv \lim_{\epsilon \to 0} \frac{\varphi((x + \epsilon) \bar{h}) - \varphi(x \bar{h})}{\epsilon}.$$

Then,

$$l(x) = \bar{h}\varphi'(x\bar{h}).$$

Because φ is continuous and strictly concave, this limit is well defined for every $x \in X$.

[1] As we discuss later, Roemer (1997) assumed $p = \frac{1}{2}$.

Given player i, we denote player i's opponent by $-i$. Then, for every $x_i, x_{-i} \in X$, define a cut-off between two policy positions to be:

$$\sigma(x_i, x_{-i}) = \begin{cases} \frac{\varphi(x_i\,\bar{h}) - \varphi(x_{-i}\,\bar{h})}{x_i - x_{-i}} & \text{if } x_i \neq x_{-i}, \\ l(x_i) & \text{if } x_i = x_{-i}. \end{cases}$$

The following assumptions correspond to Assumption A4* in Roemer (1997), and Assumption 6 in Saporiti (2008), and they are quite common in the literature on electoral competition. As described by Saporiti (2008), these assumptions guarantee that $\ln(G(\sigma(x_L, x_R)))$ and $\ln(1 - G(\sigma(x_L, x_R)))$ are concave in x_L and x_R when $x_L \geq x_R$, respectively.

Assumption 1. Assume that when $x_R < x_L$, $\frac{\sigma_1'(x_L, x_R)G'(\sigma(x_L, x_R))}{G(\sigma(x_L, x_R))}$ is decreasing in x_L and $\frac{\sigma_2'(x_L, x_R)G'(\sigma(x_L, x_R))}{1 - G(\sigma(x_L, x_R))}$ is increasing in x_R.

Now, we state our main theorems.

Theorem 1. A strategy $(x_L^*, x_R^*) = (\hat{\tau}, \hat{\tau})$ is a PSE if and only if $p = \frac{1}{2}$, and the following conditions (NL) and (NR) hold:

(NL) $2\sigma_1'(\hat{\tau}, \hat{\tau})G'(\sigma(\hat{\tau}, \hat{\tau})) \leq \frac{-v'(h_L, \hat{\tau})}{k^L}$; and

(NR) $2\sigma_2'(\hat{\tau}, \hat{\tau})G'(\sigma(\hat{\tau}, \hat{\tau})) \leq \frac{v'(h_R, \hat{\tau})}{k^R}$.

To show the existence conditions when conditions (NL) and (NR) do not hold, we define the following sets:

$$Y_L = \{x \in X : k^L \sigma_1'(x, x)G'(\sigma(x, x)) > -v'(h_L, x)G(\sigma(x, x))\},$$

and

$$Y_R = \{x \in X : k^R \sigma_2'(x, x)G'(\sigma(x, x)) > v'(h_R, x)(1 - G(\sigma(x, x)))\}.$$

As we explain later, Y_L is the set of strategies such that when both parties choose these policies, party L's payoff is increasing while Y_R is the set of strategies such that when both parties choose these policies, party R's payoff is decreasing.

Notice that when $G'(\sigma(x, x)) = 0$ if and only if $G(\sigma(x, x)) = 0, 1$. When $G(\sigma(x, x)) = 0$, $x \notin Y_L$ and when $G(\sigma(x, x)) = 1$, $x \notin Y_R$. So, we modify the conditions Y_L and Y_R, as these points are not included in the sets, which allows us to apply Assumption 1 in a more convenient way.

$$Y_L = \{x \in X : \frac{\sigma_1'(x, x)G'(\sigma(x, x))}{G(\sigma(x, x))} > \frac{-v'(h_L, x)}{k^L}\}$$

and

$$Y_R = \{x \in X : \frac{\sigma_2'(x, x)G'(\sigma(x, x))}{1 - G(\sigma(x, x))} > \frac{v'(h_R, x)(1 - G(\sigma(x, x)))}{k^R}\}.$$

Theorem 2. Suppose that $[\tau_R, \hat{\tau}] \subseteq Y_L$ and $[\hat{\tau}, \tau_L] \subseteq Y_R$. Then the game $\mathcal{G}$ has a PSE where two equilibrium policies are different at $x_R^* = \mathbf{x}_0^R(x_L^*)$ and $x_L^* = \mathbf{x}_1^L(x_R^*)$.

Theorem 3.

1. Suppose that $k^L < \frac{G(\sigma(\tau_L,\hat{\tau}))(v(h_L,\tau_L)-v(h_L,\hat{\tau}))}{\frac{1}{2}-G(\sigma(\tau_L,\hat{\tau}))}$ so that there is a $\bar{y} \in (\hat{\tau}, \tau_L)$ that solves

$$k^L = \frac{G(\sigma(\tau_L, \bar{y}))(v(h_L, \tau_L) - v(h_L, \bar{y}))}{1 - G(\sigma(\bar{y}, \bar{y})) - G(\sigma(\tau_L, \bar{y}))}.$$

   If $[\tau_R, \bar{y}] \subseteq Y_L$ and $[\bar{y}, \tau_L] \subseteq Y_R$, then the game $\mathcal{G}$ has a PSE where two equilibrium policies are different at $x_R^* = \mathbf{x}_0^R(x_L^*)$ and $x_L^* = \mathbf{x}_1^L(x_R^*)$.

2. Suppose that $k^R < \frac{(1-G(\sigma(\tau_R,\hat{\tau})))(v(h_R,\tau_R)-v(h_R,\hat{\tau}))}{G(\sigma(\tau_R,\hat{\tau}))-\frac{1}{2}}$ so that there is a $\bar{y} \in (\tau_R, \hat{\tau})$ that solves

$$k^R = \frac{(1 - G(\sigma(\tau_R, \bar{y})))(v(h_R, \tau_R) - v(h_R, \bar{y}))}{G(\sigma(\bar{y}, \bar{y})) + G(\sigma(\tau_R, \bar{y})) - 1}.$$

   If $[\tau_R, \bar{y}] \subseteq Y_L$ and $[\bar{y}, \tau_L] \subseteq Y_R$, then the game $\mathcal{G}$ has a PSE where two equilibrium policies are different at $x_R^* = \mathbf{x}_0^R(x_L^*)$ and $x_L^* = \mathbf{x}_1^L(x_R^*)$.

There are two points that we should note here. The first point is that in Theorem 2, we will show that for any strategy $x_R \in Y_L$, Party L's best response is well defined, and for any strategy $x_L \in Y_R$, Party R's best response is well defined. Further, we will show that a maximal payoff of each party is greater than the payoff of choosing the same strategy as the opponent when $\hat{\tau}$ is contained in both sets. However, when $\hat{\tau}$ may not be contained in both sets, we need another condition to guarantee that a maximal payoff is greater than the payoff of choosing the same strategy. The conditions on k^L and k^R indeed guarantee this. For example, when $\bar{y} = \tau_L$, then the LHS is zero, and thus the RHS, k^L, is greater than the LHS. When $\bar{y}$ decreases from τ_L to $\hat{\tau}$, the LHS decreases. Thus, by the intermediate value theorem, we can guarantee such a $\bar{y}$ to solve the equations.

The second point is the difference in the two distinct conditions imposed on k^L and k^R, in the sets Y_L and Y_R, and the ones imposed in Theorem 3.[2] The condition imposed in the sets Y_L and Y_R sets the limit on how much each party prefers making a marginal deviation toward their median voter's bliss point rather than choosing a tie. The condition imposed in Theorem 3 sets a further condition on how much Party L prefers the choice by their median voter (τ_L, or τ_R) over winning the election ($\hat{\tau}$).

The closest comparison to the first two theorems is the results in Roemer (1994) and Roemer (1997). These works consider the model in which there is no office rents, namely $k^L = k^R = 0$

[2]We cannot draw a general conclusion on which limit of the conditions on k^L (e.g., for Party L, $\frac{G(\sigma(\tau_L,\hat{\tau}))(v(h_L,\tau_L)-v(h_L,\hat{\tau}))}{\frac{1}{2}-G(\sigma(\tau_L,\hat{\tau}))}$ or $\frac{-v'(h_L,x)G(\sigma(x,x))}{\sigma_1'(x,x)G'(\sigma(x,x))}$), is larger as it requires a further condition on the maximal and minimal of the hazard rate.

and define *single crossing property* (SCP) in the following way. Let H denote a set of possible incomes. Then, a family of functions $\{v(h,x)|h \in H\}$ satisfies the *single crossing property* if for all h and $x_1 \neq x_2$, $v(h,x_1) = v(h,x_2)$ implies that for $h' \neq h$, $v(h',x_1) \neq v(h',x_2)$.

In Roemer (1994), it is shown that in a model with no uncertainty, the only equilibrium consists of both parties proposing the median voter's bliss point when the SCP holds. On the other hand, in Roemer (1997), it is shown that a PSE exists and each equilibrium involves parties putting forth different policies when there is uncertainty about the median voter's bliss point, while as the uncertainty becomes smaller, then the equilibrium converges to the one in Roemer (1994) such that both parties choose the same policy.

First of all, when $k^L = 0$ and $k^R = 0$, $[\tau_R, \hat{\tau}] \subseteq Y_L$ and $[\hat{\tau}, \tau_L] \subseteq Y_R$ hold. Thus Theorem 2 implies the existence of a PSE in the Roemer (1997) model.

Second, Theorems 1 and 2 show that our model captures the interesting features of the Roemer (1994) and Roemer (1997) models. In our model, there is an uncertainty about the median voter's bliss point, while office rents k^L and k^R are deterministic variables. The first two theorems state that when office rents are small, the equilibrium is similar to the one in Roemer (1997) where both parties choose different policies, and when they are large, the equilibrium is similar to the one in Roemer (1994).

Theorem 3 is a variation of Theorem 2. In Theorem 2, $\bar{y} = \hat{\tau}$ and in Theorem 3, $\bar{y} \neq \hat{\tau}$. We will show that when $[\tau_R, \bar{y}] \subseteq Y_L$ and $[\bar{y}, \tau_L] \subseteq Y_R$, then in response to $x_R \in [\tau_R, \bar{y}]$, there is a best response of Party L such that $x_L > x_R$. Because $\bar{y} \in Y_L \cap Y_R$, $(\bar{y}, \bar{y})$ is not a PSE and we will show that $\bar{y}$ locates between x_R and x_L in equilibrium. When $\bar{y} = \hat{\tau}$, Theorem 2 shows the existence of a PSE and later in Proposition 3, we will show the condition under which two parties choose policies on the different sides of $\hat{\tau}$. Similarly, when $\bar{y} \neq \hat{\tau}$, we will show that both parties could choose different policies on the same side relative to $\hat{\tau}$.

Drouvelis, Saporiti and Vriend (2014) use a uniform distribution for G and solve for a PSE. They show that one-sided differentiation can arise in equilibrium. In Section 6, we present a condition under which both parties choose different policies but on the same side relative to $\hat{\tau}$ under a general form of distribution functions (Proposition 4 and 5). To provide these characterization results, we first present Theorem 3, which shows the existence of a PSE when $\bar{y} \neq \hat{\tau}$.

3 Preliminary Results

3.1 Basic Results

This subsection consists of two parts. First, we present some useful features about the function σ. Second, we show that a PSE in the game $\mathcal{G}$ with the strategy space X is a PSE in the game $\mathcal{G}'$ with the restricted strategy space $[\tau_R, \tau_L]$.

We begin with the first part, presenting useful features of σ. Because σ is $\bar{h}$ times a slope between two points in the function φ, and φ is strictly concave, the following two lemmas hold.

Lemma 1. For every $x, \bar{x} \in X$,

(1) $\sigma(x, \bar{x})$ is decreasing in x;

(2) $\sigma(x, \bar{x}) = \sigma(\bar{x}, x)$.

By (2) of Lemma 1, $\frac{\mathrm{d}\sigma(x,\bar{x})}{\mathrm{d}x} = \frac{\mathrm{d}\sigma(\bar{x},x)}{\mathrm{d}x}$, for every $x, \bar{x} \in X$, denote

$$\sigma_1'(x, \bar{x}) \equiv \frac{\mathrm{d}\sigma(x, \bar{x})}{\mathrm{d}x} = \frac{\mathrm{d}\sigma(\bar{x}, x)}{\mathrm{d}x} \equiv \sigma_2'(\bar{x}, x),$$

where the subscript 1 and 2 respectively denote the first derivative with respect to the first and second variable.

Lemma 2. For every $x, \bar{x} \in X$,

(1) $\sigma_1'(x, \bar{x}) = \sigma_2'(\bar{x}, x)$;

(2) $\sigma_1'(x, \bar{x})$ and $\sigma_2'(\bar{x}, x)$ are strictly negative in x;

(3) $\sigma_1'(\bar{x}, \bar{x}) = l'(\bar{x})$;

(4) $G(\sigma(\hat{\tau}, \hat{\tau})) = \frac{1}{2}$.

Proof. For every $x \in X$,

$$\sigma_1'(x, \bar{x}) = \lim_{\delta \to 0} \frac{\sigma(x + \delta, \bar{x}) - \sigma(x, \bar{x})}{\delta} = \lim_{\delta \to 0} \frac{\frac{\varphi((x+\delta)\,\bar{h}) - \varphi(\bar{x}\,\bar{h})}{x+\delta-\bar{x}} - \frac{\varphi(x\,\bar{h}) - \varphi(\bar{x}\,\bar{h})}{x-\bar{x}}}{\delta}.$$

Because φ is strictly concave, $\frac{\varphi((x+\delta)\,\bar{h}) - \varphi(\bar{x}\,\bar{h})}{x+\delta-\bar{x}} < \frac{\varphi(x\,\bar{h}) - \varphi(\bar{x}\,\bar{h})}{x-\bar{x}}$ and thus we obtain (2) together with (1). Particularly, when $x = \bar{x}$, the continuity and strict concavity of φ guarantees (2), because denoting $\bar{y} = \bar{x} + \delta$,

$$\begin{aligned} \sigma_1'(\bar{x}, \bar{x}) &= \lim_{x \to \bar{x}} \sigma_1'(x, \bar{x}) = \lim_{\delta \to 0} \frac{\frac{\varphi((\bar{y}-\delta)\,\bar{h}) - \varphi(\bar{y}\,\bar{h})}{(-\delta)} - l(\bar{x})}{\delta} \\ &= \lim_{\delta \to 0} \frac{l(\bar{y}) - l(\bar{x})}{\delta} = \lim_{\delta \to 0} \frac{l(\bar{x} + \delta) - l(\bar{x})}{\delta} = l'(\bar{x}), \end{aligned}$$

which gives us (3).

Finally, note that

$$\frac{\mathrm{d}\, v(\hat{h}, \hat{\tau})}{\mathrm{d}\, x} = -\hat{h} + \varphi'(\hat{\tau}\bar{h})\bar{h} = 0,$$

and we have $l(\hat{\tau}) = \bar{h}\varphi'(\hat{\tau}\bar{h}) = \hat{h}$. Then, recalling that $l(\hat{\tau}) = \lim_{x\to\hat{\tau}} \sigma(x,\hat{\tau})$ and $\hat{h}$ is the median of the distribution of the median voter's income, we have

$$G(\sigma(\hat{\tau},\hat{\tau})) = \lim_{x\to\hat{\tau}} G\big(\sigma(x,\hat{\tau})\big) = G\big(l(\hat{\tau})\big) = G\big(\hat{h}\big) = \frac{1}{2}.$$

□

As we will see later, in equilibrium, voter $\hat{h}$ is a key. Suppose voter $\hat{h}$ is indifferent between some policy $\mathcal{X}(\bar{x})$ and $\bar{x}$. Then,

$$(1-\bar{x})\hat{h} + \varphi(\bar{x}\bar{h}) = (1-\mathcal{X}(\bar{x}))\hat{h} + \varphi(\mathcal{X}(\bar{x})\bar{h}). \tag{4}$$

When $\bar{x} = \hat{\tau}$, the only policy about which voter $\hat{h}$ is indifferent with $\hat{\tau}$ is $\hat{\tau}$ itself. Thus, define $\mathcal{X}(\bar{x})$ to satisfy

$$\begin{cases} \sigma(\mathcal{X}(\bar{x}),\bar{x}) = \hat{h} & \text{if } \bar{x} \neq \hat{\tau} \\ \mathcal{X}(\bar{x}) = \hat{\tau} & \text{if } \bar{x} = \hat{\tau}. \end{cases} \tag{5}$$

Note that $\mathcal{X}(\bar{x})$ always exists because v is U-shaped and peaked at $\hat{\tau}$ for voter $\hat{h}$, the horizontal line going through $(\bar{x}, v(\bar{x}))$ always intersects with a different point of v, unless $\bar{x} = \hat{\tau}$, and $\mathcal{X}(\bar{x})$ is uniquely determined. By using this property, the next lemma shows the locations of the two policies $\bar{x}$ and $\mathcal{X}(x)$.

Lemma 3. When $\bar{x} \leq \hat{\tau}$, $\hat{\tau} \leq \mathcal{X}(\bar{x})$. When $\bar{x} > \hat{\tau}$, $\hat{\tau} > \mathcal{X}(\bar{x})$.

Proof. First, when $\bar{x} = \hat{\tau}$, we have $\hat{\tau} = \mathcal{X}(\bar{x})$. Without loss of generality, suppose $\bar{x} < \hat{\tau}$. Because $v(\hat{h}, x)$ is increasing up to $\hat{\tau}$ and decreasing, the equality as in (4) holds only if $\hat{\tau} < \mathcal{X}(\bar{x})$. □

In the second part, we show several basic features of equilibrium, particularly that a PSE in the game $\mathcal{G}$ with the strategy space X is a PSE in the game $\mathcal{G}'$ with the restricted strategy space $[\tau_R, \tau_L]$. This will allow us to focus on $[\tau_R, \tau_L]$. In equilibrium, there are three possible cases for Party R's policy x_R: 1. $\tau_L \geq x_R \geq \tau_R$, 2. $x_R > \tau_L$, 3. $x_R < \tau_R$. In case 2, $x_L = \tau_L$ yields a winning probability 1 for Party L, while in case 3, $x_L = \tau_R$ also guarantees a win for Party L. Then, Party R can increase the payoff by choosing $x_R \in [\tau_R, \tau_L]$. Therefore, there is no strategy to support cases 1 and 2 as a PSE. On the other hand, if there is a PSE, the strategy for each party must belong to the interval $[\tau_R, \tau_L]$.

Note that for every $x_L, x_R \in [\tau_R, \tau_L]$, we obtain:

$$\pi(x_L, x_R) = \begin{cases} 1 - G(\sigma(x_L, x_R)) & \text{if } x_L < x_R, \\ p & \text{if } x_L = x_R = x, \\ G(\sigma(x_L, x_R)) & \text{if } x_L > x_R. \end{cases} \tag{6}$$

In equilibrium, in response to x_L, Party R tends to choose a policy toward τ_R than x_L, and in response to x_R, Party L tends to choose a policy toward τ_L. The next lemma shows this result.

Lemma 4.

- Fix $x_R = \bar{x}$ in X. A best response of Party L to $\bar{x}$ denoted by x_L^* satisfies $\bar{x} \leq x_L^*$.
- Fix $x_L = \bar{x}$ in X. A best response of Party R to $\bar{x}$ denoted by x_R^* satisfies $\bar{x} \geq x_R^*$.

Proof. Since the proof is symmetrical, we only prove the first statement. Suppose, by way of contradiction, $\bar{x} > x_L^*$ holds. First, we show that if $\bar{x} \leq \hat{\tau}$, Party L profitably deviates to $\hat{\tau}$. The difference between $\mathrm{E}\Pi_L(\hat{\tau}, \bar{x})$ and $\mathrm{E}\Pi_L(x_L^*, \bar{x})$ is

$$\pi(\hat{\tau}, \bar{x})(v(h_L, \hat{\tau}) - v(h_L, \bar{x})) + \pi(x_L^*, \bar{x})(v(h_L, \bar{x}) - v(h_L, x_L^*)) \\ + (\pi(\hat{\tau}, \bar{x}) - \pi(x_L^*, \bar{x}))k^L > 0.$$

The inequality holds because $x_L^* < \bar{x} \leq \hat{\tau} < \tau_L$, so that $v(h_L, \hat{\tau}) - v(h_L, \bar{x}) > 0$, $v(h_L, \bar{x}) - v(h_L, x_L^*) > 0$ and $\pi(\hat{\tau}, \bar{x}) - \pi(x_L^*, \bar{x}) > 0$.

Second, if $\bar{x} > \hat{\tau}$, we show that Party L profitably deviates to $\bar{x}$. Note that the assumption of $\bar{x} > x_L^*$ requires that x_L^* is the interior solution satisfying the following first order condition:

$$\frac{v'(h_L, x_L^*)}{v(h_L, x_L^*) - v(h_L, \bar{x}) + k^L} = \frac{\sigma_1'(x_L^*, \bar{x})G'(\sigma(x_L^*, \bar{x}))}{1 - G(\sigma(x_L^*, \bar{x}))}. \tag{7}$$

Since $x_L^* < \tau_L$, $v'(h_L, x_L^*) > 0$. Because $\sigma(x_L, \bar{x})$ is decreasing in x_L, $\sigma_1'(x_L^*, \bar{x}) < 0$. For (7) to hold, we must have $v(h_L, x_L^*) - v(h_L, \bar{x}) + k^L < 0$. This implies that

$$\begin{aligned} \mathrm{E}\Pi_L(\bar{x}, \bar{x}) - \mathrm{E}\Pi_L(x_L^*, \bar{x}) &= pk^L + \pi(x_L^*, \bar{x})(v(h_L, \bar{x}) - v(h_L, x_L^*) - k^L) \\ &> pk^L \\ &> 0. \end{aligned}$$

Thus, Party L profitably deviates to x_R^*. Hence, we must have $\bar{x} \leq x_L^*$. □

Lemma 5.

- For every $x_R \in X$ and $x_L > \tau_L$, there is an $x_L^* \leq \tau_L$ such that $\mathrm{E}\Pi_L(x_L^*, x_R) > \mathrm{E}\Pi_L(x_L, x_R)$.
- For every $x_L \in X$ and $x_R < \tau_R$, there is an $x_R^* \geq \tau_R$ such that $\mathrm{E}\Pi_R(x_R^*, x_L) > \mathrm{E}\Pi_R(x_R, x_L)$.

Proof. By symmetry, we only prove the first statement. If $x_R \geq \tau_L$, then $x_L^* = \tau_L$ satisfies $\mathrm{E}\Pi_L(x_L^*, x_R) > \mathrm{E}\Pi_L(x_L, x_R)$ for every $x_L > \tau_L$. Now, fix x_R with $x_R < \tau_L$. Due to the single-peakedness of $v(h_L, x)$ with respect to x, there is an $\tilde{x}_L > \tau_L$ such that $v(h_L, x_R) = v(h_L, \tilde{x}_L)$. Because v is strictly concave, for every $x > \tilde{x}_L$,

$$G(\sigma(x_R, x)) < G(\sigma(x_R, \tilde{x}_L)) < G(\sigma(x_R, x_R)), \text{ and} \tag{8}$$

$$v(h_L, x_R) = v(h_L, \tilde{x}_L) > v(h_L, x). \tag{9}$$

Then,

$$\begin{aligned} U_1^L(x_R, x_R) &= G(\sigma(x_R, x_R))(v(h_L, x_R) + k^L) + (1 - G(\sigma(x_R, x_R)))v(h_L, x_R) \\ &\geq G(\sigma(\tilde{x}_L, x_R))(v(h_L, \tilde{x}_L) + k^L) + (1 - G(\sigma(\tilde{x}_L, x_R)))v(h_L, x_R) \\ &= U_1^L(\tilde{x}_L, x_R) = \mathrm{E}\Pi_L(\tilde{x}_L, x_R), \end{aligned} \tag{10}$$

where $U_1^L(x_R, x_R) = \mathrm{E}\Pi_L(\tilde{x}_L, x_R)$ if and only if $k^L = 0$.

Take $x_L^* = x_R + \epsilon$ for ϵ sufficiently small. The proof consists of three parts, and shows that $\mathrm{E}\Pi_L(x_L^*, x_R) > \mathrm{E}\Pi_L(x, x_R)$ for $x = \tilde{x}_L$ (Part I), $x < \tilde{x}_L$ (Part II) and $x \in (\tau_L, \tilde{x}_L)$ (Part III).

(Part I) Because $x_R < \tau_L$, $v(h_L, x_R) < v(h_L, x_L^*)$. When $k^L > 0$, since $U_1^L(x_R, x_R) > \mathrm{E}\Pi_L(\tilde{x}_L, x_R)$ by (10), due to the continuity of $U_1^L(x, x_R)$, we obtain

$$\mathrm{E}\Pi_L(x_L^*, x_R) = U_1^L(x_L^*, x_R) > \mathrm{E}\Pi_L(\tilde{x}_L, x_R). \tag{11}$$

When $k^L = 0$, we also have (11) because $v(h_L, x_R) < v(h_L, x_L^*)$ and thus:

$$\begin{aligned} \mathrm{E}\Pi_L(x_L^*, x_R) &= U_1^L(x_L^*, x_R) \\ &= G(\sigma(x_L^*, x_R))v(h_L, x_L^*) + (1 - G(\sigma(x_L^*, x_R)))v(h_L, x_R) \\ &> G(\sigma(x_L^*, x_R))v(h_L, x_R) + (1 - G(\sigma(x_L^*, x_R)))v(h_L, x_R) \\ &= G(\sigma(x_R, x_R))v(h_L, x_R) + (1 - G(\sigma(x_R, x_R)))v(h_L, x_R) \\ &= U_1^L(x_R, x_R) = \mathrm{E}\Pi_L(\tilde{x}_L, x_R), \end{aligned}$$

where the last equality holds by (10).

(Part II) For every $x > \tilde{x}_L$, by (8) and (9),

$$\begin{aligned} \mathrm{E}\Pi_L(\tilde{x}_L, x_R) &= G(\sigma(\tilde{x}_L, x_R))(v(h_L, \tilde{x}_L) + k^L) + (1 - G(\sigma(\tilde{x}_L, x_R)))v(h_L, x_R) \\ &= v(h_L, \tilde{x}_L) + G(\sigma(\tilde{x}_L, x_R))k^L \\ &> G(\sigma(x, x_R))(v(h_L, x) + k^L) + (1 - G(\sigma(x, x_R)))v(h_L, x_R) \\ &= \mathrm{E}\Pi_L(x, x_R). \end{aligned}$$

By (11), for every $x > \tilde{x}_L$,

$$\mathrm{E}\Pi_L(x_L^*, x_R) > \mathrm{E}\Pi_L(x, x_R). \tag{12}$$

(Part III) For each $x \in (\tau_L, \tilde{x}_L)$, due to the single-peakedness of $v(h_L, x)$ with respect to x, there is an $x_L^* \in (x_R, \tau_L)$ such that $v(h_L, x_L^*) = v(h_L, x)$. Because v is strictly concave,

$$G(\sigma(x, x_R)) < G(\sigma(x_L^*, x_R)).$$

Therefore, for every $x \in (\tau_L, \tilde{x}_L)$, because $v(h_L, x_R) < v(h_L, x) = v(h_L, x_L^*)$,

$$\begin{aligned} \mathrm{E}\Pi_L(x_L^*, x_R) &= G(\sigma(x_L^*, x_R))(v(h_L, x_L^*) + k^L) + (1 - G(\sigma(x_L^*, x_R)))v(h_L, x_R) \\ &> G(\sigma(x, x_R))(v(h_L, x) + k^L) + (1 - G(\sigma(x, x_R)))v(h_L, x_R) \\ &= \mathrm{E}\Pi_L(x, x_R). \end{aligned} \tag{13}$$

By (11), (12) and (13), we have shown that for every $x_R \leq \tau_L$ and $x_L > \tau_L$, there is some $x_L^* \leq \tau_L$ such that $\mathrm{E}\Pi_L(x_L^*, x_R) > \mathrm{E}\Pi_L(x_L, x_R)$. □

Lemma 6. If there is a PSE (x_L^*, x_R^*) in the game $\mathcal{G}$, then x_L^* and x_R^* belong to $[\tau_R, \tau_L]$ and each party obtains a strictly positive winning probability in equilibrium.

Proof. Suppose that (x_L^*, x_R^*) is a PSE in the game $\mathcal{G}$. By Lemma 5, $x_L^* \leq \tau_L$ and $x_R^* \geq \tau_R$. By Lemma 4, $x_R^* \leq x_L^*$. Thus, we obtain $\tau_R \leq x_R^* \leq x_L^* \leq \tau_L$. This completes the first part of our proof. To show the second part without loss of generality, on the contrary, suppose that Party L wins with certainty in a PSE (x_L^*, x_R^*) and then by (6), $G(\sigma(x_L^*, x_R^*)) = 0$. Because $G(\hat{h}) = \frac{1}{2}$, $\sigma(x_L^*, x_R^*) < \hat{h}$. Then, we claim $x_L^* > \hat{\tau}$, because otherwise, we would have $x_R^* \leq x_L^* \leq \hat{\tau}$ by Lemma 4 and $\sigma(x_L^*, x_R^*) \geq \sigma(\hat{\tau}, \hat{\tau})$ due to the strict concavity of φ, which implies that $G(\sigma(x_L^*, x_R^*)) \geq \frac{1}{2}$ by (4) of Lemma 2. Because $G(\sigma(x_L^*, x_R^*)) = 0$, this is not possible and we must have $x_L^* > \hat{\tau}$. Let $\sigma(x_L^*, \mathcal{X}(x_L^*)) = \hat{h}$. Then, $G(\sigma(x_L^*, \mathcal{X}(x_L^*))) = \frac{1}{2}$ and thus:

$$\begin{aligned}\mathrm{E}\Pi_R(\mathcal{X}(x_L^*), x_L^*) &= \frac{1}{2}(v(h_R, \mathcal{X}(x_L^*)) + k^R) + \frac{1}{2}v(h_R, x_L^*) \\ &> v(h_R, x_L^*) = \mathrm{E}\Pi_R(x_R^*, x_L^*),\end{aligned}$$

because $\mathcal{X}(x_L^*) < \hat{\tau}$ by Lemma 3, and $v(h_R, \bar{x}_R) > v(h_R, x_L^*)$. However, this is a contradiction with the assumption that x_R^* is a PSE strategy. □

Proposition 1. A pair of strategies (x_L^*, x_R^*) is a PSE in the game $\mathcal{G}$ if and only if it is a PSE in the restricted game $\mathcal{G}'$ with the restricted strategy space $[\tau_R, \tau_L]$.

Proof.
(Only if part) Suppose that (x_L^*, x_R^*) is a PSE in the game $\mathcal{G}$. Then,

$$\mathrm{E}\Pi_L(x_L^*, x_R^*) \geq \mathrm{E}\Pi_L(x, x_R^*) \text{ for all } x \in X; \text{ and} \tag{14}$$
$$\mathrm{E}\Pi_R(x_R^*, x_L^*) \geq \mathrm{E}\Pi_R(x_R^*, x) \text{ for all } x \in X. \tag{15}$$

Because $[\tau_R, \tau_L] \subset X$, we obtain:

$$\mathrm{E}\Pi_L(x_L^*, x_R^*) \geq \mathrm{E}\Pi_L(x, x_R^*) \text{ for all } x \in [\tau_R, \tau_L]; \text{ and} \tag{16}$$
$$\mathrm{E}\Pi_R(x_R^*, x_L^*) \geq \mathrm{E}\Pi_R(x_R^*, x) \text{ for all } x \in [\tau_R, \tau_L]. \tag{17}$$

By Lemma 6, x_L^* and x_R^* are in the interval $[\tau_R, \tau_L]$ and thus feasible in the game $\mathcal{G}'$. Thus, they are also a PSE in the game $\mathcal{G}'$.
(If part) Suppose that (x_L^*, x_R^*) is a PSE in the game $\mathcal{G}'$. Then, (16) and (17) hold. By way of contradiction, suppose that it is not a PSE in the game $\mathcal{G}$, and without loss of generality, assume that there is an $\tilde{x}_L \in X \setminus [\tau_R, \tau_L]$ which is a better response than x_L^* to x_R^*. By Lemma 4, because a possible best response is greater than x_R^*, we can assume that $\tilde{x}^L > \tau_L$ and it satisfies:

$$\mathrm{E}\Pi_L(\tilde{x}_L, x_R^*) > \mathrm{E}\Pi_L(x_L^*, x_R^*). \tag{18}$$

By Lemma 5, there is an $x_L \in [\tau_R, \tau_L]$ such that

$$\mathrm{E}\Pi_L(x_L, x_R^*) > \mathrm{E}\Pi_L(\tilde{x}_L, x_R^*). \tag{19}$$

By (18) and (19), we obtain

$$\mathrm{E}\Pi_L(x_L, x_R^*) > \mathrm{E}\Pi_L(x_L^*, x_R^*). \tag{20}$$

However, because $x_L \in [\tau_R, \tau_L]$, (20) is a contradiction with (16). □

3.2 Preliminary Analysis

We start with describing the expected payoffs by using two continuous functions. Note that a discontinuity of $\mathrm{E}\Pi_i(x_L, x_R)$ arises only at $x_L = x_R$, and $\mathrm{E}\Pi_i(x_L, x_R)$ is continuous everywhere else. Thus, when $x_L < x_R$, the expected payoff for each party is a continuous function and similarly, when $x_L > x_R$, the expected payoff for each party is another continuous function.

For each $i = L, R$ and $x_i, x_{-i} \in X$, define:

$$\begin{array}{rcl} U_0^i(x_i, x_{-i}) & = & (1 - G(\sigma(x_L, x_R)))(v(h_i, x_i) + k^i) + G(\sigma(x_L, x_R))v(h_i, x_{-i}) \\ U_1^i(x_i, x_{-i}) & = & G(\sigma(x_L, x_R))(v(h_i, x_i) + k^i) + (1 - G(\sigma(x_L, x_R)))v(h_i, x_{-i}) \end{array}.$$

Then, for each $i = L, R$ and $x_i, x_{-i} \in X$,

$$\mathrm{E}\Pi_i(x_i, x_{-i}) = \begin{cases} U_0^i(x_i, x_{-i}) & \text{if } x_i < x_{-i} \\ v(h_i, x_i) + pk^i & \text{if } x_i = x_{-i} \\ U_1^i(x_i, x_{-i}) & \text{if } x_i > x_{-i} \end{cases}. \tag{21}$$

As we stated, in Roemer (1997), p is assumed to be $\frac{1}{2}$. Under this assumption, a tie breaks at an equal probability for each party, which probably is one natural scenario. In the next lemma, we specify a condition under which a tie is *not* a PSE.

Lemma 7.

- (1) Let $p \neq \frac{1}{2}$. Then, $(x_L, x_R) = (\hat{\tau}, \hat{\tau})$ is not a PSE.
- (2) Let $\bar{x} \neq \hat{\tau}$. Then, $(x_L, x_R) = (\bar{x}, \bar{x})$ is not a PSE.

Proof. Part (1): By Lemma 2, $G\big(\sigma(\hat{\tau}, \hat{\tau})\big) = \frac{1}{2}$,

$$\lim_{x_L \to \hat{\tau}} \mathrm{E}\Pi_L(x_L, \hat{\tau}) = v(h_L, \hat{\tau}) + \frac{1}{2}k_L$$
$$\lim_{x_R \to \hat{\tau}} \mathrm{E}\Pi_R(\hat{\tau}, x_R) = v(h_R, \hat{\tau}) + \frac{1}{2}k_R.$$

If $p < \frac{1}{2}$, then $\mathrm{E}\Pi_L(\hat{\tau}, \hat{\tau}) = v(h_L, \hat{\tau}) + pk_L < \lim_{x_L \to \hat{\tau}} \mathrm{E}\Pi_L(x_L, \hat{\tau})$ and Party L has an incentive to unilaterally deviate from $\hat{\tau}$ to $\hat{\tau} \pm \varepsilon$, for $\varepsilon > 0$ sufficiently small. By analogous argument, if $p > \frac{1}{2}$, Party R has an incentive to deviate from $\hat{\tau}$. Hence, if $p \neq \frac{1}{2}$, the policy profile $(\hat{\tau}, \hat{\tau})$ cannot be a PNE.

Part (2): Let $\bar{x} \neq \hat{\tau}$ be the policy chosen by parties, such that $x_L = x_R = \bar{x}$, and $G\big(l(\bar{x})\big) \neq 1 - G\big(l(\bar{x})\big)$.

Party L has no incentive to deviate from $x_L = \bar{x}$ only if

$$p \geq \max\left\{G\big(l(\bar{x})\big), 1 - G\big(l(\bar{x})\big)\right\},$$

because

$$\mathrm{E}\Pi_L(\bar{x}, \bar{x}) \geq \max\left\{\lim_{\varepsilon \to 0} \mathrm{E}\Pi_L(\bar{x} - \varepsilon, \bar{x}), \lim_{\varepsilon \to 0} \mathrm{E}\Pi_L(\bar{x} + \varepsilon, \bar{x})\right\}.$$

But $p \geq \max\left\{G\big(l(\bar{x})\big), 1 - G\big(l(\bar{x})\big)\right\}$ immediately implies that

$$1 - p \leq \min\left\{G\big(l(\bar{x})\big), 1 - G\big(l(\bar{x})\big)\right\} < \max\left\{G\big(l(\bar{x})\big), 1 - G\big(l(\bar{x})\big)\right\},$$

in which case for Party R,

$$\mathrm{E}\Pi_R(\bar{x}, \bar{x}) < \max\left\{\lim_{\varepsilon \to 0} \mathrm{E}\Pi_R(\bar{x} - \varepsilon, \bar{x}), \lim_{\varepsilon \to 0} \mathrm{E}\Pi_R(\bar{x} + \varepsilon, \bar{x})\right\},$$

and Party R has an incentive to deviate from $x_R = \bar{x}$ whenever Party L has no incentive to deviate from $x_L = \bar{x}$. By symmetry, Party L has an incentive to deviate from the strategy profile $(\bar{x}, \bar{x})$ if Party R does not. Hence, $(\bar{x}, \bar{x})$ cannot be a PNE as at least one party can do better by deviating from $(\bar{x}, \bar{x})$. □

After Lemma 4, the relevant parts of the expected payoff are U_0^R and U_1^L.

Theorem 1 states that when conditions (NL) and (NR) hold, the equilibrium such that both parties choose $\hat{\tau}$ occurs. Because condition (NR) and (NL) respectively guarantees that $U_0^R(x_R, \hat{\tau})$ is increasing in x_R up to $\hat{\tau}$, and that $U_1^L(x_L, \hat{\tau})$ is decreasing in x_L up to $\hat{\tau}$, both parties choose $\hat{\tau}$ in equilibrium when $p = \frac{1}{2}$. The expected payoffs are continuous at $\hat{\tau}$, which maximizes the expected payoff for each party in response to the opponent choice of $\hat{\tau}$.

On the other hand, Theorem 2 and Theorem 3 state that when at $\hat{\tau}$, U_0^R is decreasing and U_1^L is increasing in response to the opponent's policy choice, then both U_0^R and U_1^L are bell-shaped and the maximizers constitute best responses. We denote a maximizer x of function $U_n^i(x, \bar{x})$ given $\bar{x}$ by $\mathbf{x}_n^i(\bar{x})$ for each $i = L, R$, and $n = 0, 1$. For every $i = L, R$ and $x_{-i} \in X$, let

$$\widehat{x}_i(x_{-i}) = \operatorname*{argmax}_{x \in X} \mathrm{E}\Pi_i(x, x_{-i}).$$

Then, $\widehat{x}_i$ is Party i's best response to Party $-i$'s strategy x_{-i}.

By Lemmas 4 and 7, we know that if equilibrium policies of two parties are not equal to $\hat{\tau}$, then we must have $x_R^* < x_L^*$. By (21), when $x_R^* < x_L^*$,

$$\mathrm{E}\Pi_L(x^*L, x_R^*) = U_1^L(x^*L, x_R^*) \text{ and } \mathrm{E}\Pi_R(x^*R, x_L^*) = U_0^R(x^*R, x_L^*).$$

Since $\mathbf{x}_0^R(x_L^*)$ and $\mathbf{x}_1^L(x_R^*)$ are respectively the maximizers of $U_0^R(x, x_L^*)$ and $U_1^L(x, x_R^*)$, when they are well-defined and $\mathbf{x}_0^R(x_L^*) < \mathbf{x}_1^L(x_R^*)$, $(\mathbf{x}_1^L(x_R^*), \mathbf{x}_0^R(x_L^*))$ is a PSE, which leads us to $\widehat{x}_L(x_R^*) = \mathbf{x}_1^L(x_R^*)$ and $\widehat{x}_R(x_L^*) = \mathbf{x}_0^R(x_L^*)$. Theorem 2 and Theorem 3 present the conditions under which $\mathbf{x}_0^R(x_L^*)$ and $\mathbf{x}_1^L(x_R^*)$ are well defined and $\mathbf{x}_0^R(x_L^*) < \mathbf{x}_1^L(x_R^*)$ holds.

On the other hand, even if $\mathbf{x}_0^R(x_L^*) < \mathbf{x}_1^L(x_R^*)$ does not hold, there could still be a PSE. Theorem 1 provides the necessary and sufficient conditions under which at $\hat{\tau}$, U_0^R is increasing and U_1^L is decreasing in response to the opponent's strategy of $\hat{\tau}$. As we see in Lemma 9 and 10, Y_L and Y_R are the set of strategies $(x_L, x_R) = (x, x)$ at which $\mathrm{E}\Pi_L(x, x)$ and $\mathrm{E}\Pi_R(x, x)$ are increasing and decreasing, respectively. Thus, choosing the same policy does not constitute a PSE when a strategy x is in these sets. Note that when $\hat{\tau} \notin Y_L$ or Y_R, condition (NL) or (NR) holds.

Fix $\bar{x} \in X$ arbitrarily. We consider the relative values of the two continuous functions U_0^i and U_1^i for each $i = L, R$. Voter $\hat{h}$'s bliss point is a key factor in the positioning. For the purpose of proving the next lemma, by symmetry, we focus on Party L, and thus our interest here is U_0^L and U_1^L. Take $x, \bar{x} \in X$. By Lemma 4, we consider $x \geq \bar{x}$. Because the difference between $U_0^L(\mathcal{X}(\bar{x}), \bar{x})$ and $U_1^L(\mathcal{X}(\bar{x}), \bar{x})$ is

$$(1 - 2G(\sigma(\mathcal{X}(\bar{x}), \bar{x})))(v(h_L, x) - v(h_L, \bar{x}) + k^L) = 0. \tag{22}$$

The two functions, U_0^L and U_1^L, intersect at $\mathcal{X}(\bar{x})$. Further, because $\sigma(\mathcal{X}(\bar{x}), \bar{x}) < \sigma(x, \bar{x})$, if $x < \mathcal{X}(\bar{x})$, the fact that G is strictly increasing implies that:

$$\begin{cases} U_0^L(x, \bar{x}) < U_1^L(x, \bar{x}) & \text{if } x < \mathcal{X}(\bar{x}) \\ U_0^L(x, \bar{x}) = U_1^L(x, \bar{x}) & \text{if } x = \mathcal{X}(\bar{x}). \end{cases} \tag{23}$$

Now that we know the relative values of the two functions U_0^L and U_1^L, we study the shapes of these functions. By taking the first derivative of $U_0^L(x, \bar{x})$ with respect to $x \in X$, we obtain:

$$u_0^L(x, \bar{x}) \equiv -\sigma_1'(x, \bar{x})G'(\sigma(x, \bar{x}))(v(h_L, x) - v(h_L, \bar{x}) + k^i) + v'(h_L, x)(1 - G(\sigma(x, \bar{x})))$$

and similarly for $U_1^L(x, \bar{x})$ with respect to $x \in X$,

$$u_1^L(x, \bar{x}) \equiv \sigma_1'(x, \bar{x})G'(\sigma(x, \bar{x}))(v(h_L, x) - v(h_L, \bar{x}) + k^L) + v'(h_L, x)G(\sigma(x, \bar{x})).$$

Lemma 8. Let $\bar{x} \in [\tau_R, \tau_L]$ and take $\epsilon > 0$ sufficiently small.

- $U_0^L(x, \bar{x})$ is increasing in x when $x \in [\bar{x} - \epsilon, \bar{x}]$.
- $U_1^R(x, \bar{x})$ is decreasing in x when $x \in [\bar{x}, \bar{x} + \epsilon]$.

Proof. Finally, we prove $U_0^L(x, \bar{x})$ is increasing in x for $x \in [\bar{x} - \epsilon, \bar{x}]$. Consider u_1^L and $x \in [\bar{x} - \epsilon, \bar{x}]$. Note that $-\sigma_1'(x, \bar{x})$ and $G'(\sigma(x, \bar{x}))$ are positive, and $v(h_L, x) - v(h_L, \bar{x}) + k^L$ is positive for sufficiently small ϵ. Because $v'(h_L, x)$ is positive for $x < \tau_L$, we obtain $u_0^L(x, \bar{x}) > 0$ for $x \in [\bar{x} - \epsilon, \bar{x}]$. By symmetry, we can prove the result for $U_1^R(x, \bar{x})$. □

Next, we study the shapes of $U_1^L(x, \bar{x})$ and $U_0^R(x, \bar{x})$. Suppose that $Y_L \subset X$ is non-empty. Let $\bar{x} \in Y_L$. Because $v'(h_L, \tau_L) = 0$, the following holds:

$$\frac{\sigma_1'(\tau_L, \bar{x}) G'(\sigma(\tau_L, \bar{x}))}{G(\sigma(\tau_L, \bar{x}))} < \frac{-v'(h_L, \tau_L)}{v(h_L, \tau_L) - v(h_L, \bar{x}) + k^L}. \tag{24}$$

Because $\bar{x} \in Y_L$, we have

$$\frac{\sigma_1'(\bar{x}, \bar{x}) G'(\sigma(\bar{x}, \bar{x}))}{G(\sigma(\bar{x}, \bar{x}))} > \frac{-v'(h_L, \bar{x})}{k^L}. \tag{25}$$

Therefore,

$$\frac{\sigma_1'(\bar{x}, \bar{x}) G'(\sigma(\bar{x}, \bar{x}))}{G(\sigma(\bar{x}, \bar{x}))} > \frac{-v'(h_L, \bar{x})}{k^L} = \frac{-v'(h_L, \bar{x})}{v(h_L, \bar{x}) - v(h_L, \bar{x}) + k^L}. \tag{26}$$

For any $x > \bar{x}$, we have $v(h_L, x) - v(h_L, \bar{x}) > 0$ and $0 > -v'(h_L, x) > -v'(h_L, \bar{x})$ and hence,

$$\frac{-v'(h_L, x)}{v(h_L, x) - v(h_L, \bar{x}) + k^L} > \frac{-v'(h_L, \bar{x})}{v(h_L, \bar{x}) - v(h_L, \bar{x}) + k^L}. \tag{27}$$

Therefore, the RHS of (24) and (25), that is $\frac{-v'(h_L, x)}{v(h_L, x) - v(h_L, \bar{x}) + k^L}$, is an increasing function of $x \in [\hat{\tau}, \tau_L]$. By Assumption 1, the LHS of (24) and (25), that is $\frac{\sigma_1'(x, \bar{x}) G'(\sigma(x, \bar{x}))}{G(\sigma(x, \bar{x}))}$ is a decreasing function of $x \in [\hat{\tau}, \tau_L]$. Because both sides of (24) and (25) are continuous in $x \in X$, there is an $\mathbf{x}_1^L(\bar{x}) = x_L \in (\bar{x}, \tau_L)$ to satisfy

$$\frac{\sigma_1'(x_L, \bar{x}) G'(\sigma(x_L, \bar{x}))}{G(\sigma(x_L, \bar{x}))} = \frac{-v'(h_L, x_L)}{v(h_L, x_L) - v(h_L, \bar{x}) + k^L}, \tag{28}$$

Further, as stated above, because v is strictly concave, the RHS of (28) is increasing in x when $\bar{x} \leq x \leq \tau_L$, while Assumption 1 guarantees that the LHS of (28) is decreasing in x. Therefore, $\mathbf{x}_1^L(\bar{x})$ is unique. These observations are summarized in the following lemma.

Lemma 9. If Y_L is non-empty, then in response to $\bar{x} \in Y_L$, $\mathbf{x}_1^L(\bar{x})$ satisfying $\frac{\mathrm{d}U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x})}{\mathrm{d}x_L} = 0$ exists uniquely in the interval $(\bar{x}, \tau_L)$.

A similar argument works for Party R. When Y_R is non-empty, for $\bar{x} \in Y_R$, there is a unique $\mathbf{x}_0^R(\bar{x})$ such that:

$$\frac{\sigma_1'(\mathbf{x}_0^R(\bar{x}), \bar{x}) G'(\sigma(\mathbf{x}_0^R(\bar{x}), \bar{x}))}{1 - G(\sigma(\mathbf{x}_0^R(\bar{x}), \bar{x}))} = \frac{v'(h_R, \mathbf{x}_0^R(\bar{x}))}{v(h_R, \mathbf{x}_1^R(\bar{x})) - v(h_R, \bar{x}) + k^R}. \tag{29}$$

Lemma 10. If Y_R is non-empty, then in response to $\bar{x} \in Y_R$, $\mathbf{x}_0^R(\bar{x})$ satisfying $\frac{\mathrm{d}U_0^R(\mathbf{x}_0^R(\bar{x}), \bar{x})}{\mathrm{d}x_L} = 0$ exists uniquely in the interval $(\tau_R, \bar{x})$.

4 Proofs of the Main Theorems

4.1 Proof of Theorem 1

In this section, we study the condition under which both parties choose the same policy.

Lemma 11. For all $x_L, x_R \in (\tau_R, \tau_L)$, $u_1^L(x_L, x_R) < 0$ implies $u_1^L(x, x_R) < 0$ for all $x \in (x_L, \tau_L]$, and $u_0^L(x_L, x_R) > 0$ implies $u_1^L(x, x_R) > 0$ for all $x \in [\tau_R, x_L)$.

Proof. Suppose $u_1^L(x_L, x_R) < 0$, then

$$\frac{\sigma_1'(x_L, x_R)G'(\sigma(x_L, x_R))}{G(\sigma(x_L, x_R))} < -\frac{v'(h_L, x_L)}{v(h_L, x_L) - v(h_L, x_R) + k^L}. \tag{30}$$

When Assumption 1 holds, the LHS of (30) is decreasing in x_L. Since $v''(h_L, x_L) < 0$ by the strict concavity of φ for all $x_L \in X$, and $v'(h_L, x_L) \leq 0$ for all $x_L \leq \tau_L$, the LHS of (30) is increasing in $x \in [x_L, \tau_L]$. Hence, for all $x \in (x_L, \tau_L]$, we have $u_1^L(x, x_R) < 0$. The proof for the second statement is similar. □

Proof of Theorem 1.
(If part) Suppose both (NL) and (NR) hold and $p = \frac{1}{2}$. Then, by (4) of Lemma 2, we have (NL') $\frac{\sigma_1'(\hat{\tau},\hat{\tau})G'(\sigma(\hat{\tau},\hat{\tau}))}{G(\sigma(\hat{\tau},\hat{\tau}))} \leq \frac{-v'(h_L,\hat{\tau})}{k^L}$ and (NR') $\frac{\sigma_2'(\hat{\tau},\hat{\tau})G'(\sigma(\hat{\tau},\hat{\tau}))}{1-G(\sigma(\hat{\tau},\hat{\tau}))} \leq \frac{v'(h_R,\hat{\tau})}{k^R}$. We will show that $(\hat{\tau}, \hat{\tau})$ is a PSE, or equivalently, that $\widehat{x}_L(\hat{\tau}) = \widehat{x}_R(\hat{\tau}) = \hat{\tau}$. By symmetry, we only prove $\widehat{x}_L(\hat{\tau}) = \hat{\tau}$.

The first part of Lemma 4 shows that for $x_L < \hat{\tau}$,

$$\mathrm{E}\Pi_L(x_L, \hat{\tau}) < \mathrm{E}\Pi_L(\hat{\tau}, \hat{\tau}). \tag{31}$$

Because $u_L^1(\hat{\tau}, \hat{\tau}) < 0$, (31) also holds for all $x_L \in (\hat{\tau}, \tau_L]$ by Lemma 11. Finally, since condition (NL') holds and $p = \frac{1}{2}$, (31) also holds for all $x_L > \tau_L$.

We have thus shown that $\hat{\tau}$ is maximizer of $\mathrm{E}\Pi_L(x_L, \hat{\tau})$ in X and thus, $\widehat{x}_L(\hat{\tau}) = \hat{\tau}$. The proof that $\widehat{x}_R(\hat{\tau}) = \hat{\tau}$ is identical.
(Only if part) Suppose $(\hat{\tau}, \hat{\tau})$ is a PSE. By Lemma 7, it must be the case that $p = \frac{1}{2}$. Suppose to the contrary that condition (NL') is false, and then $u_1^L(\hat{\tau}, \hat{\tau}) > 0$ holds. Then, there must exist some $\varepsilon > 0$ sufficiently small such that $\mathrm{E}\Pi_L(\hat{\tau} + \varepsilon, \hat{\tau}) > \mathrm{E}\Pi_L(\hat{\tau}, \hat{\tau})$, and $\widehat{x}_L(\hat{\tau}) \neq \hat{\tau}$. Hence $(\hat{\tau}, \hat{\tau})$ cannot be a PSE if condition (NL') does not hold. The proof for the necessity of condition (NR') for $(\hat{\tau}, \hat{\tau})$ to be a PSE is identical. □

4.2 Proofs of Theorem 2 and Theorem 3

4.2.1 Common Argument for the Two Theorems

To complete the proof of Theorem 2, we use a variant of *C-security* proposed in McLennan, Monteiro and Tourky (2011). For each $i = L, R$, define a function, $\underline{u}_i : X \to \mathbb{R}$ as follows:

$$\underline{u}_L(d_L, x_R) = \liminf_{\tilde{x}_R \to x_R} \mathrm{E}\Pi_L(d_L, \tilde{x}_R) \text{ and } \underline{u}_R(d_R, x_L) = \liminf_{\tilde{x}_L \to x_L} \mathrm{E}\Pi_R(d_R, \tilde{x}_L).$$

This is the payoff that strategy d_i can almost guarantee to player i if his opponents play any strategies close enough to x_{-i}. For $\alpha \in \mathbb{R}$, define:

$$\begin{aligned} B_i^\alpha(x_i, x_{-i}) &= \{y_i \in X_i : \underline{u}_i(y_i, x_{-i}) \geq \alpha\}; \text{ and} \\ C_i^\alpha(x_i, x_{-i}) &= \text{con } B_i^\alpha(x_i, x_{-i}), \end{aligned}$$

where $\text{con } Z$ is the convex hull of the set Z.

Definition (C-security)**.** The game is C-secure on $Z \subset X \times X$ if there is an $\alpha = (\alpha_L, \alpha_R) \in \mathbb{R}^2$ such that

(1) for each $i = L, R$ and any $z \in Z$, $B_i^{\alpha_i}(z)$ is nonempty; and

(2) for any $z \in Z$, there is some player $i \in \{L, R\}$ such that $z_i \notin C_i^{\alpha_i}(z)$.

The game $\mathcal{G}$ is *C-secure* at $(x_L, x_R) \in X^2$ if it is *C-secure* in some neighborhood of (x_L, x_R).

The following theorem restates the main theorem in McLennan, Monteiro and Tourky (2011).

Theorem (Theorem MMT)**.** If the game $\mathcal{G}$ is C-secure at each $(x_L, x_R) \in X^2$ that is not a Nash equilibrium, then $\mathcal{G}$ has a PSE.

In what follows, we show that when $[\tau_R, \hat{\tau}] \subseteq Y_L$ and $[\hat{\tau}, \tau_L] \subseteq Y_R$, the game is C-secure. This lemma is essential to apply Theorem MMT.

Lemma 12. Suppose $[\tau_R, \bar{y}] \subseteq Y_L$ and $[\bar{y}, \tau_L] \subseteq Y_R$ for some $\bar{y} \in [\tau_R, \tau_L]$, respectively. If $U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) > U_0^L(\bar{x}, \bar{x})$ for every $\bar{x} \in [\tau_R, \hat{\tau}]$ and $U_0^R(\mathbf{x}_0^R(\bar{x}), \bar{x}) > U_1^R(\bar{x}, \bar{x})$ for every $\bar{x} \in [\hat{\tau}, \tau_L]$, then the game $\mathcal{G}'$ is C-secure at each $(x_L, x_R) \in [\tau_R, \tau_L]^2$ that is not a Nash equilibrium.

Proof. Let $(\bar{x}_L, \bar{x}_R) \in [\tau_R, \tau_L]^2$ that is not PSE. We start by deriving $\underline{u}_L$ and $\underline{u}_R$. Since $\mathrm{E}\Pi_i(x_i, x_{-i})$ is continuous at all strategy profiles (x_L, x_R) for which $x_L \neq x_R$, and discontinuous at (x_L, x_R) where $x_L = x_R$ (with the exception of $x_L = x_R = \hat{\tau}$), it is easily shown that for every $i = L, R$ and $x_i, x_{-i} \in [\tau_R, \tau_L]$,

$$\underline{u}_i(x_i, x_{-i}) = \begin{cases} \mathrm{E}\Pi_i(x_i, x_{-i}) & \text{if } x_i \neq x_{-i} \\ \min\{U_0^i(x_{-i}, x_{-i}), U_1^i(x_{-i}, x_{-i})\} & \text{otherwise .} \end{cases}$$

Suppose $\bar{x}_L \neq \bar{x}_R$. By assumption, the maximizer $\mathbf{x}_1^L(\bar{x}_R)$ of $U_1^L(x, \bar{x}_R)$ and the maximizer $\mathbf{x}_0^R(\bar{x}_L)$ of $U_0^R(x, \bar{x}_L)$ are well-defined. Because $(\bar{x}_L, \bar{x}_R)$ is not a PSE, $\bar{x}_L = \widehat{x}_L(\bar{x}_R)$ and $\bar{x}_R = \widehat{x}_R(\bar{x}_L)$ do not simultaneously hold. Then, suppose $\bar{x}_L \neq \widehat{x}_L(\bar{x}_R)$ and let $\alpha^L = U_1^L(\mathbf{x}_1^L(\bar{x}_R), \bar{x}_R)$. Then, for every $x \in [\tau_R, \tau_L]$,

$$U_1^L(x, \bar{x}_R) < U_1^L(\mathbf{x}_1^L(\bar{x}_R), \bar{x}_R) = \mathrm{E}\Pi_L(\mathbf{x}_1^L(\bar{x}_R), \bar{x}_R). \tag{32}$$

By Lemma 4, for every $x < \bar{x}_R$, there is an $x_L \geq \bar{x}_R$ such that

$$U_1^L(x_L, \bar{x}_R) = \mathrm{E}\Pi_L(x_L, \bar{x}_R) > U_0^L(x, \bar{x}_R) = \mathrm{E}\Pi_L(x, \bar{x}_R)$$

and because $\mathbf{x}_1^L(\bar{x}_R)$ is the maximizer of $U_1^L(x, \bar{x}_R)$, for every $x < \bar{x}_R$,

$$\begin{aligned} U_0^L(x, \bar{x}_R) &= \mathrm{E}\Pi_L(x, \bar{x}_R) \\ &< U_1^L(x_L, \bar{x}_R) = \mathrm{E}\Pi_L(x_L, \bar{x}_R) \\ &\leq U_1^L(\mathbf{x}_1^L(\bar{x}_R), \bar{x}_R) = \mathrm{E}\Pi_L(\mathbf{x}_1^L(\bar{x}_R), \bar{x}_R). \end{aligned} \tag{33}$$

Because $\mathrm{E}\Pi_L(x, \bar{x}_R)$ is continuous when $x \neq \bar{x}_R$, there is some y in the ϵ-neighborhood of $\bar{x}_R$ such that for every $x < \bar{x}_R$,

$$\mathrm{E}\Pi_L(x, y) < \mathrm{E}\Pi_L(\mathbf{x}_1^L(\bar{x}), \bar{x}). \tag{34}$$

Therefore, $[\tau_R, y] \cap B_L^{\alpha_L}(x_L) = \varnothing$. As a result, $[\tau_R, y] \cap C_L^{\alpha_L}(x_L) = \varnothing$. Thus, we have shown that the game $\mathcal{G}'$ is C-secure at each non-PSE (x_R, x_L) when $x_R \neq x_L$.

Now, suppose that $\bar{x}_L = \bar{x}_R = \bar{x}$. Let $\alpha^L = U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x})$. By Lemma 9, $\mathbf{x}_1^L(\bar{x})$ exists uniquely and $\mathbf{x}_1^L(\bar{x}) > \bar{x}$. Then, $\widehat{x}_L(\bar{x}) \in B_L^{\alpha_L}(\bar{x}_L, \bar{x}_R)$. Further, by Lemma 4, there is $b(\bar{x}) > \bar{x}$ such that for every $x < \bar{x}$,

$$\mathrm{E}\Pi_L(x, \bar{x}) < \mathrm{E}\Pi_L(b(\bar{x}), \bar{x}). \tag{35}$$

Because $b(\bar{x}) > \bar{x}$, we have

$$\mathrm{E}\Pi_L(b(\bar{x}), \bar{x}) = U_1^L(b(\bar{x}), \bar{x}) \leq U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) = \mathrm{E}\Pi_L(\mathbf{x}_1^L(\bar{x}), \bar{x}). \tag{36}$$

Because $\mathbf{x}_1^L(\bar{x}) > \bar{x}$ and $\mathrm{E}\Pi_L(x, \bar{x})$ is continuous when $x < \bar{x}$, there is some y in the ϵ-neighborhood of $\bar{x}$ such that for every $x < \bar{x}$,

$$\mathrm{E}\Pi_L(x, y) < \mathrm{E}\Pi_L(\mathbf{x}_1^L(\bar{x}), \bar{x}). \tag{37}$$

Also, by assumption and Lemma 9, we have

$$U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) > \max\{U_0^L(\bar{x}, \bar{x}), U_1^L(\bar{x}, \bar{x})\}. \tag{38}$$

Because U_L^0 and U_1^L are continuous, for some y, we have

$$U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) > \max\{U_0^L(\bar{x}, y), U_1^L(\bar{x}, y)\}. \tag{39}$$

Thus, for all $x \leq \bar{x}$, we have

$$\mathrm{E}\Pi_L(x, y) < \mathrm{E}\Pi_L(\mathbf{x}_1^L(\bar{x}), \bar{x}) = \alpha^L. \tag{40}$$

Therefore, $[\tau_R, y] \cap B_L^{\alpha_L}(x_L) = \varnothing$. As a result, $[\tau_R, y] \cap C_L^{\alpha_L}(x_L) = \varnothing$. Thus, we have shown that the restricted game $\mathcal{G}'$ is C-secure at each non-PSE (x_R, x_L) when $x_R = x_L$. This completes the proof. □

4.2.2 Proof of Theorem 2

Lemma 13.

- If $[\tau_R, \hat{\tau}] \subseteq Y_L$, then in response to $\bar{x} \in [\tau_R, \hat{\tau}]$, $\widehat{x}_L(\bar{x})$ exists in $(\bar{x}, \tau_L)$ and

$$\mathrm{E}\Pi_L(\widehat{x}_L(\bar{x}), \bar{x}) = U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) > U_0^L(\bar{x}, \bar{x}).$$

- If $[\hat{\tau}, \tau_L] \subseteq Y_R$, then in response to $\bar{x} \in [\hat{\tau}, \tau_L]$, $\widehat{x}_R(\bar{x})$ exists in $(\tau_R, \bar{x})$ and

$$\mathrm{E}\Pi_R(\widehat{x}_R(\bar{x}), \bar{x}) = U_0^R(\mathbf{x}_0^R(\bar{x}), \bar{x}) > U_1^R(\bar{x}, \bar{x}).$$

Proof. First, suppose $\bar{x} \leq \hat{\tau}$. By Lemma 9, $\mathbf{x}_1^L(\bar{x})$ uniquely exists at $\mathbf{x}_1^L(\bar{x}) > \bar{x}$, implying $U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) > U_1^L(\bar{x}, \bar{x})$. By Lemma 3, since $\bar{x} \leq \hat{\tau}$, $\mathcal{X}(\bar{x}) \geq \bar{x}$. By Lemma 9, $U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) > U_1^L(\bar{x}, \bar{x}) \geq U_0^L(\bar{x}, \bar{x})$ by (23). Thus,

$$\mathrm{E}\Pi_L(\widehat{x}_L(\bar{x}), \bar{x}) = U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) > U_0^L(\bar{x}, \bar{x}).$$

By symmetry, we can prove the second statement. □

Proof of Theorem 2. By Lemma 13, Lemma 12 is applicable. Thus, by Lemma 12, the game $\mathcal{G}'$ is C-secure. Thus, by Theorem MMT, there exists a PSE in the restricted game $\mathcal{G}'$. By Proposition 1, this PSE is also a PSE in the game $\mathcal{G}$. □

4.2.3 Proof of Theorem 3

Lemma 14. Suppose that $k^L < \frac{G(\sigma(\tau_L,\hat{\tau}))(v(h_L,\tau_L)-v(h_L,\hat{\tau}))}{\frac{1}{2}-G(\sigma(\tau_L,\hat{\tau}))}$ so that there is a $\bar{y} > \hat{\tau}$ that solves

$$k^L = \frac{G(\sigma(\tau_L, \bar{y}))(v(h_L, \tau_L) - v(h_L, \bar{y}))}{1 - G(\sigma(\bar{y}, \bar{y})) - G(\sigma(\tau_L, \bar{y}))}.$$

Then, $U_0^L(\bar{y}, \bar{y}) < U_1^L(\tau_L, \bar{y})$, and for every $\bar{x} < \bar{y}$, there is an x satisfying $U_0^L(\bar{x}, \bar{x}) = U_1^L(x, \bar{x})$ or $U_0^L(\bar{x}, \bar{x}) < U_1^L(x, \bar{x})$ for every $x > \bar{x}$.

Proof. As described immediately after Theorem 3, when $\bar{y} = \tau_L$, the LHS is zero, and thus the RHS, k^L, is greater than the LHS. When $\bar{y}$ decreases from τ_L to $\hat{\tau}$, the LHS increases. Thus, by the intermediate value theorem, we can guarantee such a $\bar{y}$ to solve the equation. Note:

$$\begin{aligned} U_0^L(\bar{y}, \bar{y}) &= v(h_L, \bar{y}) + (1 - G(\sigma(\bar{y}, \bar{y})))k^L \\ U_1^L(\hat{x}, \bar{y}) &= G(\sigma(\hat{x}, \bar{y}))(v(h_L, \hat{x}) + k^L) + (1 - G(\sigma(\hat{x}, \bar{y})))v(h_L, \bar{y}). \end{aligned}$$

The first statement can be obtained by comparing the two in (4.2.3). Because $\bar{y} > \hat{\tau}$, $U_0^L(\bar{y}, \bar{y}) > U_1^L(\bar{y}, \bar{y})$. When $U_0^L(\bar{y}, \bar{y}) < U_1^L(\tau_L, \bar{y})$, $U_0^L(\bar{y}, \bar{y}) = U_1^L(\hat{x}, \bar{y})$ holds for some $\hat{x}$ by the continuity of $U_1^L(x, \bar{y})$ with respect to x. Then,

$$G(\sigma(\hat{x}, \bar{y}))(v(h_L, \hat{x}) - v(h_L, \bar{y}) + k^L) = (1 - G(\sigma(\bar{y}, \bar{y})))k^L. \tag{41}$$

As $\bar{x} < \bar{y}$, $\sigma(\bar{y}, \bar{y}) < \sigma(\bar{x}, \bar{x})$. Applying this to the RHS of (41),

$$G(\sigma(\hat{x}, \bar{y}))(v(h_L, \hat{x}) - v(h_L, \bar{y}) + k^L) > (1 - G(\sigma(\bar{x}, \bar{x})))k^L. \tag{42}$$

Because $v(h_L, \bar{y}) < v(h_L, \bar{x})$, substituting this in the LHS of (42) yields

$$G(\sigma(\hat{x}, \bar{y}))(v(h_L, \hat{x}) - v(h_L, \bar{x}) + k^L) > (1 - G(\sigma(\bar{x}, \bar{x})))k^L. \tag{43}$$

Suppose $U_0^L(\bar{x}, \bar{x}) \geq U_1^L(x, \bar{x})$ for some $x > \bar{x}$. If the equality holds, we are done. So, suppose that $U_0^L(\bar{x}, \bar{x}) > U_1^L(x, \bar{x})$, which implies

$$G(\sigma(x, \bar{x}))(v(h_L, x) - v(h_L, \bar{x}) + k^L) < (1 - G(\sigma(\bar{x}, \bar{x})))k^L. \tag{44}$$

Because $\bar{x} < \bar{y}$ and $\sigma(x, \bar{x}) > \sigma(x, \bar{y})$, substituting this in the LHS of (44) yields

$$G(\sigma(x, \bar{y}))(v(h_L, x) - v(h_L, \bar{x}) + k^L) < (1 - G(\sigma(\bar{x}, \bar{x})))k^L. \tag{45}$$

The LHS of (45) is continuous in x. Applying the intermediate value theorem to (43) and (45), there is some $z \in (x, \hat{x})$ satisfying

$$G(\sigma(z, \bar{y}))(v(h_L, z) - v(h_L, \bar{x}) + k^L) = (1 - G(\sigma(\bar{x}, \bar{x})))k^L. \tag{46}$$

Noting that $x > \bar{x}$, this completes the proof. $\square$

In an exactly symmetrical way, we can obtain the following result.

Lemma 15. Suppose that $k^R < \frac{(1-G(\sigma(\tau_R,\hat{\tau})))(v(h_R,\tau_R)-v(h_R,\hat{\tau}))}{G(\sigma(\tau_R,\hat{\tau}))-\frac{1}{2}}$ so that there is a $\bar{y} \in (\tau_R, \hat{\tau})$ that solves

$$k^R = \frac{(1 - G(\sigma(\tau_R, \bar{y})))(v(h_R, \tau_R) - v(h_R, \bar{y}))}{G(\sigma(\bar{y}, \bar{y})) + G(\sigma(\tau_R, \bar{y})) - 1}.$$

Then, $U_0^R(\tau_R, \bar{y}) > U_1^R(\bar{y}, \bar{y})$, and for every $\bar{x} > \bar{y}$, there is an x satisfying $U_0^R(x, \bar{x}) = U_1^R(\bar{x}, \bar{x})$ or $U_0^R(\bar{x}, \bar{x}) > U_1^R(x, \bar{x})$ for every $x < \bar{x}$.

Lemma 16.

- Let $\bar{y}$ be as defined in Lemma 14. If $U_0^L(\bar{y}, \bar{y}) < U_1^L(\tau_L, \bar{y})$, then $\widehat{x}_L(\bar{x})$ exists in response to $\bar{x} \in [\tau_R, \bar{y}]$ and $U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) \geq U_0^L(\bar{x}, \bar{x})$.
- Let $\bar{y}$ be as defined in Lemma 15. If $U_0^R(\tau_R, \bar{y}) > U_1^R(\bar{y}, \bar{y})$, then $\widehat{x}_R(\bar{x})$ exists in response to $\bar{x} \in [\bar{x}, \tau_L]$ and $U_0^R(\mathbf{x}_0^R(\bar{x}), \bar{x}) \geq U_1^R(\bar{x}, \bar{x})$.

Proof. First, suppose $\bar{y} > \hat{\tau}$. Let $\bar{x} \in [\tau_R, \bar{y}]$. By Lemma 9, $\mathbf{x}_1^L(\bar{x})$ uniquely exists and $\mathbf{x}_1^L(\bar{x}) > \bar{x}$. By Lemma 14, there is some x satisfying $U_1^L(x, \bar{x}) = U_0^L(\bar{x}, \bar{x})$ or $U_1^L(x, \bar{x}) = U_0^L(\bar{x}, \bar{x})$ holds for every $x > \bar{x}$. Hence, $U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) \geq U_1^L(x, \bar{x}) \geq U_0^L(\bar{x}, \bar{x})$. Thus,

$$\mathrm{E}\Pi_L(\widehat{x}_L(\bar{x}), \bar{x}) = U_1^L(\mathbf{x}_1^L(\bar{x}), \bar{x}) \geq U_0^L(\bar{x}, \bar{x}).$$

By symmetry, we can prove the second statement. $\square$

Proof of Theorem 3. Because the proof is symmetrical, we only prove the result when $\bar{x} > \hat{\tau}$. By Lemma 10, $\widehat{x}_R(\bar{x})$ exists in response to $\bar{x} \in [\hat{\tau}, \tau_L]$. By Lemma 16, $\mathrm{E}\Pi_L(\widehat{x}_L(\bar{x}), \bar{x}) \geq U_0^L(\bar{x}, \bar{x})$ holds in response to $\bar{x}$. Finally, by Lemma 12 and Theorem MMT, there exists a PSE. □

5 The Existence of Mixed Strategy Equilibrium

We have given conditions under which a PSE exists. The next proposition is the existence result including an MSE. We borrow the argument from Simon and Zame (1990). Simon and Zame (1990) introduce an endogenous sharing rule which guarantees the existence of an MSE. Our strategy then is to show that an MSE in the game with an endogenous sharing rule is an MSE in the original game and a tie happens with a probability zero, so that a sharing rule indeed does not matter in the original game.

Proposition 2. An equilibrium (a PSE or/and an MSE) exists unless conditions (NL) and (NR) hold and $p \neq \frac{1}{2}$.

For every $(x_L, x_R) \in X \times X$, define a *payoff correspondence* $Q : X \times X \to \mathbb{R}^2$ to be

$$Q(x_L, x_R) = (\mathrm{E}\Pi_L(x_L, x_R), \mathrm{E}\Pi_L(x_L, x_R)).$$

A *sharing rule* is a Borel measurable function $q : X \times X \to \mathbb{R}^2$ such that $q(x_L, x_R) \in Q(x_L, x_R)$ for every $(x_L, x_R) \in X \times X$. Notice that the payoff correspondence Q is discontinuous when $x_L = x_R$.

A mixed strategy for party $i = L, R$ is a probability measure on X and a mixed strategy profile is a pair (α_L, α_R) of mixed strategies. A *solution* for the game $\mathcal{G}$ is a sharing rule q and a mixed strategy profile (α_L, α_R) such that q is a Borel measurable selection from the payoff correspondence Q, and (α_L, α_R) is a profile of mixed strategies such that for each $i = L, R$ and each probability measure β_i on X,

$$\int q_i(x_L, x_R) \mathrm{d}(\alpha_i \times \alpha_{-i}) \geq \int q_i(x_L, x_R) \mathrm{d}(\beta_i \times \alpha_{-i}). \tag{47}$$

Each sharing rule q defines a game $\mathcal{G}_q$, while a solution for $\mathcal{G}$ is a sharing rule q and a profile of mixed strategies (α_L, α_R) which constitutes a Nash equilibrium in the game $\mathcal{G}_q$. By the main theorem of Simon and Zame (1990), every game with an endogenous sharing rule has a solution.

Proof of Proposition 2. When both conditions (NL) and (NR) hold and $p = \frac{1}{2}$, by Theorem 1, a PSE exists. Fan-Glicksberg's fixed point theorem[3] is an extension of Kakutani's fixed

[3]For details, see page 108 of McLennan (2014).

point theorem to correspondences with infinite dimensional domains, stating that if V is a locally convex topological vector space, $X \times X \subset V$ is nonempty, convex, and compact, and $F : X \times X \to X \times X$ is an upper semicontinuous convex valued correspondence, then F has a fixed point.

Let X^* be a dense subset of $X \times X$ and a bounded continuous function $\psi : X^* \to \mathbb{R}^2$. Let $C_\psi : X \times X \to \mathbb{R}^2$ be the correspondence whose graph is the closure of the graph of ψ and define $Q_\psi(x_L, x_R)$ to be the convex hull of $C_\psi(x_L, x_R)$ for each $(x_L, x_R) \in X \times X$. We claim that Q_ψ defined above is bounded, upper semi-continuous, has nonempty convex, compact values and $Q_\psi(x_L, x_R) = \psi(x_L, x_R)$ for each $(x_L, x_R) \in X^*$. Any selection q from the correspondence Q_ψ agrees with ψ on X^*, and thus, every sharing rule is an extension of the given payoff function ψ on X^* to the entire space $X \times X$. Therefore, for each $i = L, R$, and α_{-i}, a best response to α_{-i} satisfying (47) exists and by Fan-Glicksberg's fixed point theorem, a fixed point exists. Then, its fixed point is a Nash equilibrium in the game $\mathcal{G}$ with a sharing rule q.

Now, we show that a tie happens in equilibrium with probability zero. On the contrary, suppose that for some $\bar{x} \in X$, a probability measure β_i on X satisfies $\beta_i(\bar{x}) > 0$ for each $i = L, R$ and $\beta = (\beta_L, \beta_R)$ constitutes an MSE. Then, construct α_L such that $\alpha_L(\bar{x} - \epsilon) = \beta_L(\bar{x})$ and $\alpha_L(\bar{x}) = 0$. Then,

$$\int q_L(x_L, x_R)\mathrm{d}(\alpha_L \times \beta_R) > \int q_L(x_L, x_R)\mathrm{d}(\beta_L \times \beta_R). \tag{48}$$

This is a contradiction with the assumption that $\beta = (\beta_L, \beta_R)$ constitutes an MSE.

To complete the proof of the first statement, it remains to show the non-existence of equilibrium when $p \neq \frac{1}{2}$ and conditions (NL) and (NR) hold. By Lemma 7, there is no PSE. We show that there is no MSE. Suppose that there is one, which we denote by (α_L, α_R) and there is no $x \in X$ such that (α_L, α_R) is totally mixed and so $\alpha_i(x) = 1$ for each $i = L, R$. Then, for each $i = L, R$ and each probability measure β_i on X,

$$\int q_i(x_L, x_R)\mathrm{d}(\alpha_i \times \alpha_{-i}) \geq \int q_i(x_L, x_R)\mathrm{d}(\beta_i \times \alpha_{-i}). \tag{49}$$

Take some $\bar{x} \in X$ and let a probability measure β_i on X be such that for each $i = L, R$, $\beta_i(\bar{x}) = 1$ and $\beta_i(x) = 0$ for every $x \neq \bar{x}$. Then, for each $i = L, R$,

$$\int q_i(\bar{x}, \bar{x})\mathrm{d}(\beta_i \times \alpha_{-i}) > \int q_i(\bar{x}, \bar{x})\mathrm{d}(\alpha_i \times \alpha_{-i}). \tag{50}$$

This is a contradiction with Lemma 7. □

6 The Results of Party Polarization

6.1 The Four Propositions

In this section, we present four propositions about equilibrium policy choices. The first three propositions demonstrate the conditions about office rent under which polarization, right-sided differentiation, and left-sided differentiation arise in equilibrium, where right-sided differentiation is a situation whereby both parties choose policies greater than the median of the distribution of the median voter's bliss point. The last proposition shows that the further away the party's bliss point is from the center, then the further away the party's optimal policy is from the center.

The intuition of the first three propositions can be summarized as follows. If the degree of parties' office motives is sufficiently high, there exists a PSE, and each party announces a policy located on the center. However, as the degree of the office rent decreases, an equilibrium in pure strategies may fail to exist. When one party's office rent is higher than the other, there is a situation such that equilibrium policies are both biased toward the bliss point of one party whose office rent is relatively lower. Then, the other party chooses a policy between the opponent's policy and the center.

For each $\bar{y} \in X$, define

$$Z_L(\bar{y}) = \{x \in X : u_1^L(\bar{y}, x) > 0 \text{ and } x \leq \bar{y}\},$$

and

$$Z_R(\bar{y}) = \{x \in X : u_0^R(\bar{y}, x) < 0 \text{ and } x \geq \bar{y}\}.$$

Then, $Z_L(\bar{y})$ is the set of Party R's strategies x_R for which $\mathrm{E}\Pi_L(x, x_R)$ is increasing at $x = \bar{y}$, while $Z_R(\bar{y})$ is the set of Party L's strategies x_L for which $\mathrm{E}\Pi_R(x, x_L)$ is decreasing at $x = \bar{y}$. This is a generalization of Y_L and Y_R in the sense that $\hat{\tau} \in Y_L$ if and only if $\hat{\tau} \in Z_L(\hat{\tau})$ and $\hat{\tau} \in Y_R$ if and only if $\hat{\tau} \in Z_R(\hat{\tau})$.

Proposition 3.

- If a PSE (x_L^*, x_R^*) satisfies $x_R^* < \hat{\tau} < x_L^*$, then $x_R^* \in Z_L(\hat{\tau})$ and $x_L^* \in Z_R(\hat{\tau})$.
- Conversely, suppose $[\tau_R, \hat{\tau}] \subseteq Y_L \cap Z_L(\hat{\tau})$ and $[\hat{\tau}, \tau_L] \subseteq Y_R \cap Z_R(\hat{\tau})$. Then, a PSE (x_L^*, x_R^*) satisfies $x_R^* < \hat{\tau} < x_L^*$.

Proof. Suppose that the parties choose policies on different sides of the center such that $x_R^* < \hat{\tau} < x_L^*$. Then, it must be the case that $U_1^L(x, x_R^*)$ is increasing at $x = \hat{\tau}$ and $U_0^R(x, x_L^*)$ is decreasing at $x = \hat{\tau}$. Thus,

$$\frac{\sigma_1'(\hat{\tau}, x_R^*) G'(\sigma(\hat{\tau}, x_R^*))}{G(\sigma(\hat{\tau}, x_R^*))} > \frac{-v'(h_L, \hat{\tau})}{v(h_L, \hat{\tau}) - v(h_L, x_R^*) + k^L},$$

and

$$\frac{\sigma_1'(\hat{\tau}, x_L^*)G'(\sigma(\hat{\tau}, x_L^*))}{1 - G(\sigma(\hat{\tau}, x_L^*))} > \frac{v'(h_R, \hat{\tau})}{v(h_R, \hat{\tau}) - v(h_R, x_L^*) + k^R}.$$

Thus, we obtain the first statement.

On the other hand, suppose $[\tau_R, \hat{\tau}] \subseteq Y_L$ and $[\hat{\tau}, \tau_L] \subseteq Y_R$. By Theorem 2, there exists a PSE (x_L^*, x_R^*). On the contrary, suppose that $x_L^* < \hat{\tau}$. By Lemmas 4 and 7, $x_R^* < x_L^* < \hat{\tau}$. However, because $x_R^* \in Z_L(\hat{\tau})$, $\mathrm{E}\Pi_L(x, x_R^*)$ is increasing at $x = \hat{\tau}$ and $\mathbf{x}_1^L(x_R^*) > \hat{\tau}$, which contradicts $x_L^* < \hat{\tau}$. □

Even when equilibrium policies of two parties are different at $x_R^* = \mathbf{x}_0^R(x_L^*)$ and $x_L^* = \mathbf{x}_1^L(x_R^*)$, there are two possible situations in the sense that equilibrium policies are located on different sides or on the same side of the center. In the first proposition, we have shown the condition under which both parties choose policies on the different sides of $\hat{\tau}$. This is polarization. In the next two propositions, we provide the conditions under which one-sided differentiation arises, in which both parties choose policies on the same side with respect to $\hat{\tau}$.

Proposition 4.

- If a PSE (x_L^*, x_R^*) satisfies $\hat{\tau} < x_R^* < x_L^*$, then $x_L^* \notin Z_R(\hat{\tau})$.
- Conversely, suppose that the conditions of Theorem 3-1 hold. When $[\tau_R, \hat{\tau}] \subseteq Z_L(\hat{\tau})$ and $(\hat{\tau}, \tau_L) \cap Z_R(\hat{\tau}) = \varnothing$, a PSE (x_L^*, x_R^*) satisfies $\hat{\tau} < x_R^* < x_L^*$.

Proof. Suppose that a PSE (x_L^*, x_R^*) satisfies $\hat{\tau} < x_R^* < x_L^*$. Then, it must be the case that $U_0^R(x, x_L^*)$ is increasing at $x = \hat{\tau}$. Thus,

$$\frac{\sigma_1'(\hat{\tau}, x_L^*)G'(\sigma(\hat{\tau}, x_L^*))}{1 - G(\sigma(\hat{\tau}, x_L^*))} < \frac{v'(h_R, \hat{\tau})}{v(h_R, \hat{\tau}) - v(h_R, x_L^*) + k^R}.$$

Thus, we obtain the first statement.

To show the second statement, note that by Theorem 3, there is a PSE (x_L^*, x_R^*). On the contrary, suppose $\hat{\tau} \geq x_R^*$. Because $[\tau_R, \hat{\tau}] \subseteq Z_L(\hat{\tau})$, $x_R^* \in Z_L(\hat{\tau})$ and then $U_1^L(x, x_R^*)$ is increasing at $x = \hat{\tau}$. Thus, $x_L^* > \hat{\tau}$ as it maximizes $U_1^L(x, x_R^*)$. However, as $x_L^* \notin Z_R(\hat{\tau})$, $\mathrm{E}\Pi_R(x, x_L^*)$ is increasing at $x = \hat{\tau}$ and this contradicts the assumption that x_R^* constitutes a PSE. □

Symmetrically with Proposition 4, we can obtain the following proposition such that both parties choose policies on the left-hand side of $\hat{\tau}$.

Proposition 5.

- If a PSE (x_L^*, x_R^*) satisfies $x_R^* < x_L^* < \hat{\tau}$, then $x_R^* \notin Z_L(\hat{\tau})$.

- Conversely, suppose that the conditions of Theorem 3-2 hold. When $[\hat{\tau}, \tau_L] \subseteq Z_R(\hat{\tau})$ and $(\tau_R, \bar{y}) \cap Z_L(\hat{\tau}) = \varnothing$, a PSE (x_L^*, x_R^*) satisfies $x_R^* < x_L^* < \hat{\tau}$.

We have provided conditions under which polarization, left-sided, and right-sided differentiation arise in equilibrium. In the next proposition, we conduct comparative static analysis on the difference of incomes and policy difference. When two parties choose different policies, we show that as the income difference of the two parties' supporters increases, the distance between the parties' policy choices increases as well. This result is a straightforward outcome of the strict concavity of v and Assumption 1.

Proposition 6. Suppose that (x_L^*, x_R^*) is a PSE and $x_L^* \neq x_R^*$. Holding all others constant, as h_L decreases, x_L^* increases. Similarly, holding all others constant, as h_R increases, x_R^* decreases.

Proof. For every h and $x \in X$, we have

$$-v'(h,x) = xh - \bar{h}\varphi'(x\bar{h}) \text{ and} \tag{51}$$

$$v(h,x) - v(h,\bar{x}) = (\bar{x} - x)h + \varphi(x\bar{h}) - \varphi(\bar{x}\bar{h}). \tag{52}$$

When h decreases, $-v'(h,x)$ decreases by (51) and $v(h,x) - v(h,\bar{x})$ increases by (52) if $\bar{x} - x$ is negative. As a result, the RHS of (28) decreases when $\bar{x} - x$ is negative. Thus, the LHS becomes larger than the RHS. When x decreases, then the LHS increases by Assumption 1, while the RHS decreases. Thus, the difference becomes even larger. To equalize both sides, x must increase. □

6.2 Numerical Illustration

To elaborate on C-security and the three propositions in the previous section, we present some numerical examples. Here, we use a uniform distribution for G and it is assumed that $h_L = 0.2$, $h_R = 0.8$, $\bar{h} = 0.5$ and $\varphi(x\bar{h}) = 0.5\sqrt{x\bar{h}}$. Then, $\tau_R = 0.049$, $\hat{\tau} = 0.125$, and $\tau_L = 0.781$.

The first two panels of the first Figure (1a) and (1b) show the payoffs of Party L and Party R, when the opponent's strategy equals 0.1. In this example, C-security holds at $\bar{x}$, as clearly the maximizers $\widehat{x}_L(\bar{x})$ and $\widehat{x}_R(\bar{x})$ yield higher payoffs. However, when k^R increases to 2.00, as we see in the remaining two panels of the Figure (1c) and (1d), Party R's best response is not well defined in the sense that Party R would be better off by choosing as close as possible to $\bar{x}$, but once $\bar{x}$ is chosen, the payoff drops.

Next, Figure 2 provides the illustrative examples for party polarization. The first two panels, (2a) and (2b) present the payoffs in PSE when $(k_L, k_R) = (0.08, 0.0005)$. In the mathematical calculation, we have obtained $\bar{y} = 0.074$, and left-sided PSE where $x_L^* = 0.076$ and $x_R^* = 0.058$. In this example, Party L's best response x_L^* yields higher payoffs than $\hat{\tau}$ or x_R^* such that

$$\text{E}\Pi_L(x_L^*, x_R^*) = 0.3453 > \text{E}\Pi_L(\hat{\tau}, x_R^*) = 0.3449 > \text{E}\Pi_L(\hat{\tau}, x_R^*) = 0.3437.$$

On the other hand, in the second two panels, (2c) and (2d), we set $(k_L, k_R) = (0.01, 0.02)$, where we obtain polarized PSE. In this example, we have obtained $x_L^* = 0.316$ and $x_R^* = 0.091$. In this situation, both parties choose different policies on the different sides with respect to $\hat{\tau}$.

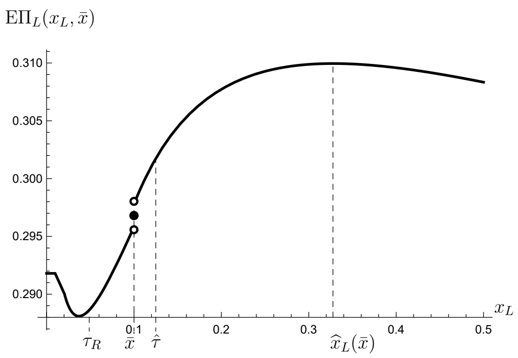
Finally, we will demonstrate how the required conditions in the propositions are met in the above two examples of $(k_L, k_R) = (0.08, 0.0005)$ and $(k_L, k_R) = (0.01, 0.02)$. Recall that when $(k_L, k_R) = (0.08, 0.0005)$, we have $(x_L^*, x_R^*) = (0.076, 0.058)$ (left-sided PSE) and when $(k_L, k_R) = (0.01, 0.02)$, we have $(x_L^*, x_R^*) = (0.316, 0.091)$ (polarized PSE). Figure 3 illustrates the conditions in Proposition 5.

The first two panels, Figure (3a) and (3b) present the first derivatives, $u_1^L(x_L, x_L)$ and $u_0^R(x_R, x_R)$ on $[\tau_R, \tau_L]$ when $(k_L, k_R) = (0.08, 0.0005)$. As Theorem 3 requires, we can see that that $u_1^L(x_L, x_L)$ is strictly positive on $[\tau_R, \bar{y}]$, indicating that $[\tau_R, \bar{y}] \subset Y_L$, while $u_0^R(x_R, x_R)$ is strictly negative on $[\bar{y}, \tau_L]$, indicating that $[\bar{y}, \tau_L] \subset Y_R$. The second two panels (3c) and (3d) show $u_1^L(\hat{\tau}, x_R)$ on $[\theta_R, \hat{\theta}]$ and $u_0^R(\hat{\tau}, x_L)$ on $[\hat{\tau}, \tau_L]$. In this situation, both parties would not choose the same policy. Party L chooses a policy greater than the one that Party R chooses. Figure (3c) shows that $u_1^L(\hat{\tau}, x_L)$ is strictly negative on $[\tau_R, \bar{y}]$, which indicates that $[\tau_R, \bar{y}] \cap Z_L(\hat{\tau}) = \varnothing$, as Proposition 5 requires. Further, Figure (3d) shows that $u_0^R(\hat{\tau}, x_R)$ is strictly negative on $[\hat{\tau}, \tau_L]$, which indicates that $[\hat{\tau}, \tau_L] \subset Z_R(\hat{\tau})$, as Proposition 5 requires. Under this circumstance, because both parties' payoffs are decreasing at $\hat{\tau}$ in response to the opponent's strategy so that both choose policies smaller than $\hat{\tau}$, both $u_1^L(\hat{\tau}, x_R)$ and $u_0^R(\hat{\tau}, x_L)$ are decreasing on $[\tau_R, \hat{\tau}]$ and $[\hat{\tau}, \tau_L]$. Thus, in this situation, as we see from the first two panels of Figure 2 (2a) and (2b), we obtain $x_L^* = 0.076$ and $x_R^* = 0.058$ as a PSE. As shown in Proposition 5, in this situation, both parties choose policies on the left side of $\hat{\tau}$.

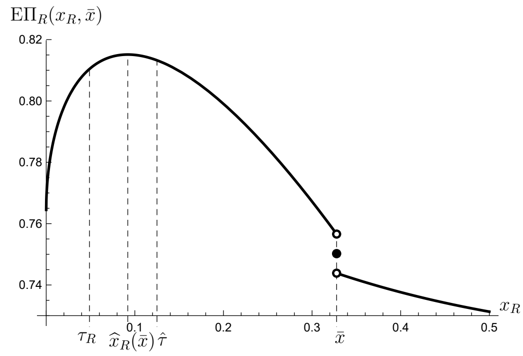
Next, Figure 4 illustrates the conditions in Proposition 3. In this case, we set $(k_L, k_R) = (0.01, 0.02)$. The first two panels, Figure (4a) and (4b) present $u_1^L(x_L, x_L)$ and $u_0^R(x_R, x_R)$ on $[\tau_R, \tau_L]$, in which we can see that $[\tau_R, \hat{\tau}] \subset Y_L$ and $[\hat{\tau}, \tau_L] \subset Y_R$. By Lemmas 9 and 10, we can guarantee a best response to each strategy in Y_L or Y_R. The second two panels, (4c) and (4d) show that $u_1^L(\hat{\tau}, x_R)$ is strictly positive on $[\tau_R, \hat{\tau}]$ and $u_0^R(\hat{\tau}, x_L)$ is strictly negative on $[\hat{\tau}, \tau_L]$. This indicates that $[\tau_R, \hat{\tau}] \subset Y_L \cap Z_L(\hat{\tau})$ and $[\hat{\tau}, \tau_L] \subset Y_R \cap Z_R(\hat{\tau})$. Then, Party R's payoff is decreasing at $\hat{\tau}$, while Party L's payoff is increasing at $\hat{\tau}$, in response to the opponent's strategy in Y_L or Y_R. In this way, both parties choose policies on the different sides of $\hat{\tau}$, as proved in Proposition 3. More specifically, in this example, we obtain $x_L^* = 0.316$ and $x_R^* = 0.091$, as we have seen in the first two panels of Figure 2, (2a) and (2b).

Figure 1: Payoffs ($\bar{x} = 0.1$)

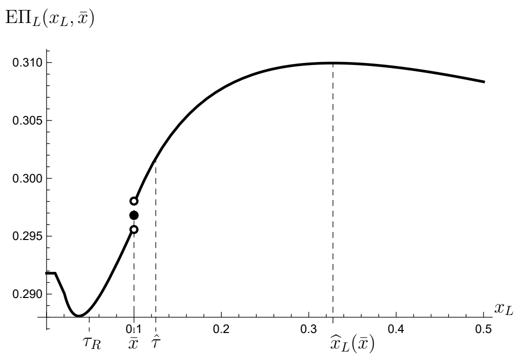
(a) L for $(k_L, k_R) = (0.01, 0.02)$



(b) R for $(k_L, k_R) = (0.01, 0.02)$



(c) L for $(k_L, k_R) = (0.01, 2.00)$



(d) R for $(k_L, k_R) = (0.01, 2.00)$

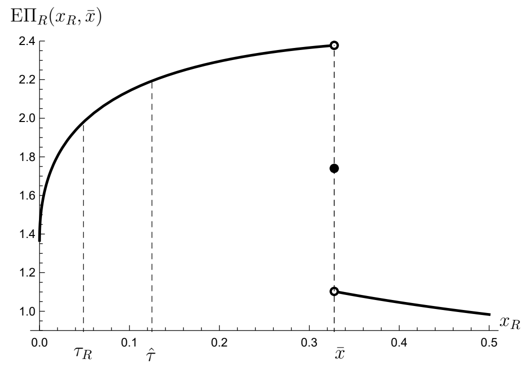
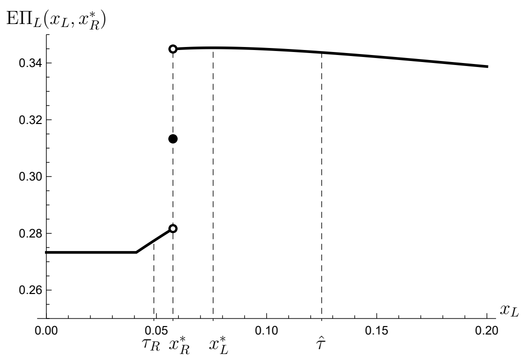
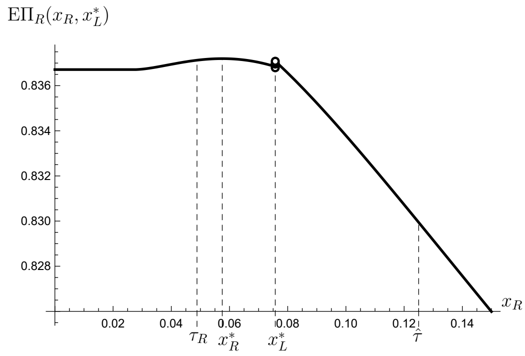


Figure 2: PSE

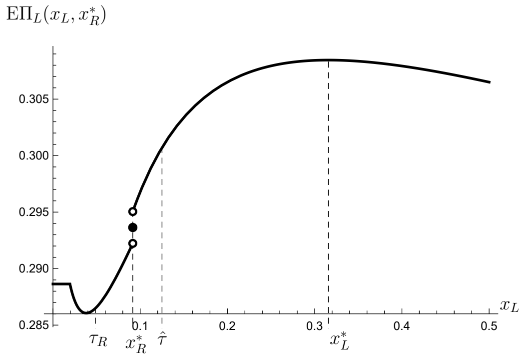
(a) L for $(k_L, k_R) = (0.08, 0.0005)$



(b) R for $(k_L, k_R) = (0.08, 0.0005)$



(c) R for $(k_L, k_R) = (0.01, 0.02)$



(d) R for $(k_L, k_R) = (0.01, 0.02)$

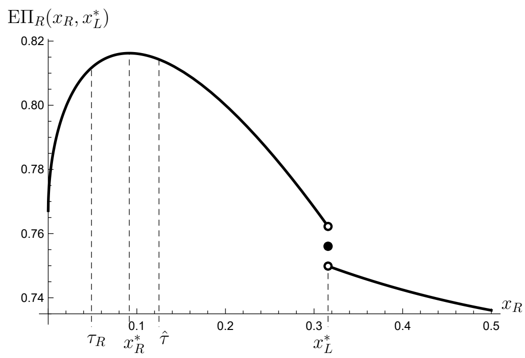
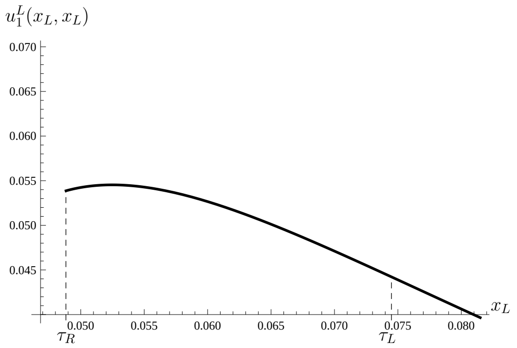
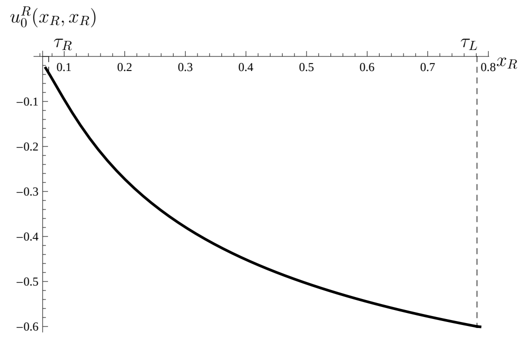


Figure 3: Payoffs in Left-Sided PSE

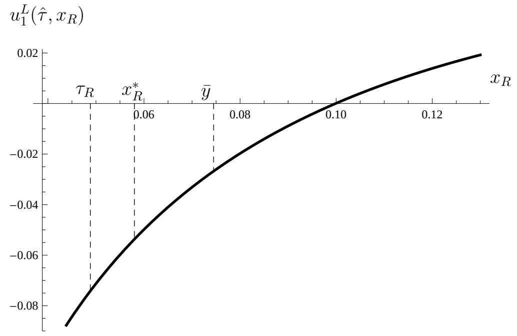
(a) $u_1^L(x_L, x_L)$ for Y_L



(b) $u_0^R(x_R, x_R)$ for Y_R



(c) $u_1^L(\hat{\tau}, x_R)$ for $Z_L(\hat{\tau})$



(d) $u_0^R(\hat{\tau}, x_L)$ for $Z_R(\hat{\tau})$

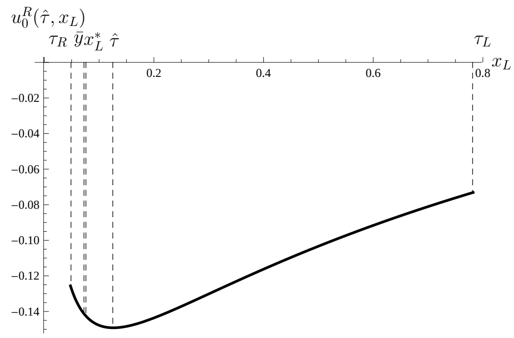
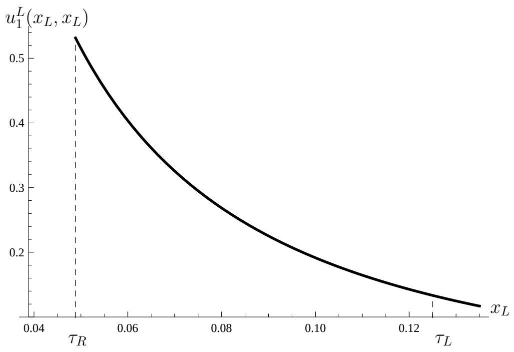
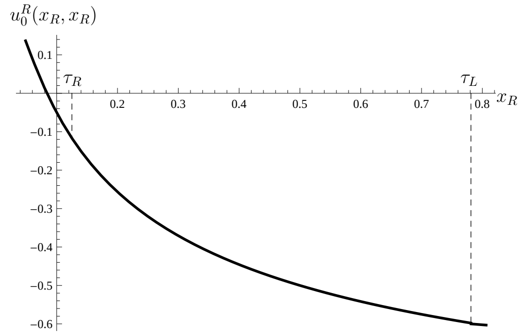


Figure 4: Payoffs in Poralized PSE

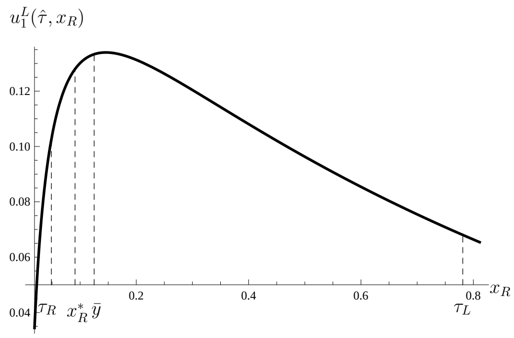
(a) $u_1^L(x_L, x_L)$ for Y_L



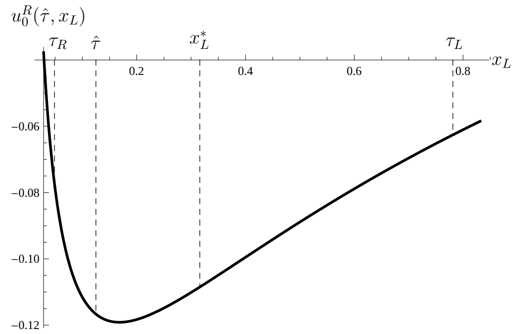
(b) $u_0^R(x_R, x_R)$ for Y_R



(c) $u_1^L(\hat{\tau}, x_R)$ for $Z_L(\hat{\tau})$



(d) $u_0^R(\hat{\tau}, x_L)$ for $Z_R(\hat{\tau})$



7 Discussion

After showing the conditions under which each type of equilibrium exists, a natural question that could arise is: *Under what conditions is a PSE unique?* The answer exists in the relationship between the hazard rate and the utility function. This relationship is summarized in the first order conditions (28) and (29). Let (x_L^*, x_R^*) be a PSE. Then, this satisfies (28) and (29). Consider $x_R' = x_R^* + \epsilon$ for $\epsilon > 0$. Then, both the LHS and the RHS of (28) increase for x_R'. Whether or not there is another x_L' to satisfy (28) for x_R' depends on the changes in both sides. In other words, it depends on the magnitude of the change in hazard rate and the change from $v(h_L, x_L^*)$ to $v(h_L, x_L')$.

Our results relating to party polarizations generalize the preceding works. We have demonstrated the conditions about office rent under which each type of polarization arises in equilibrium. Further, we have showed that when the income of the voter that each party tries to maximizes goes further away from the median, then the party's choice also goes further away from the middle point as a best response to the opponent's party choice. This result is in line with the empirical observation in Smidt (2015). Finally, by showing that polarized differentiation arises if and only if the game is C-secure, our analysis bridges the literature of political competition and the advances in equilibrium analysis of discontinuous games. Our analysis shows how the concept of C-security can be used in a political competition model and indeed provides a useful insight for party polarizations.

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