Equilibrium, Price Manipulation and Dynamic Informed Trading in Securities Markets*

Shino Takayama†

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Abstract

This paper studies a dynamic version of the model proposed in Glosten and Milgrom (1985) with a long-lived informed trader. When the same individual can buy, and then sell the same asset, the trader may profit from price manipulation. We first study a theoretical model. We provide a sufficient condition under which an equilibrium exists uniquely. We show that when manipulations arise, the value function is steep. Further, within the unique equilibrium, we show that bid and ask prices are monotonically increasing in the market maker's prior belief, and the bid-ask spread is the largest when the market maker is unsure about the state. Finally, we propose a computational method to solve for the equilibrium in a T-period model. Our simulation results confirm our theoretical findings.

Key Words: Market microstructure; Glosten–Milgrom; Dynamic trading; Price formation; Sequential trade; Asymmetric information; Bid–ask spreads.
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†Takayama; School of Economics, University of Queensland, Level 6 - Colin Clark Building (39), St Lucia, QLD 4072, Australia; e-mail: s.takayama@economics.uq.edu.au; tel: +61-7-3346-7379; fax: +61-7-3365-7299.

1 Introduction

This paper considers dynamic trading in a model proposed by Glosten and Milgrom (1985) with a long-lived informed trader. When the same individual can buy, and in the future sell, the same asset, the trader may profit from this round-trip trade, which we refer to as price manipulation. The history of market manipulation is long, perhaps as long as the one of organized exchanges. However, we do not know much about manipulation. Empirically it is difficult to obtain the data due to illegal nature of the practice, while theoretically it is difficult to analyze it because of intractability (see Putniņš, 2011). The main objective of this paper is to understand how and when the informed trader manipulates the market and how that affects price formation. To tackle at this problem, this paper studies a simplest possible case, provides conditions under which an equilibrium exists, and characterizes equilibria. Further, we develop a computer program to find equilibria.

The model in this paper works as follows. We adopt a sequential trade framework by Glosten and Milgrom (1985) with the trading of a risky asset over finitely many periods between the competitive market maker, the strategic informed trader, and liquidity traders. At the beginning of the game, nature chooses the liquidation value of a risky asset to be high or low and informs the informed trader, who then trades dynamically. In each period, there is a random determination of whether the informed trader or a liquidity trader trades.

The paper consists of two parts; theoretical and computational analyses. In the theoretical analyses, we provide a sufficient condition for the unique existence of equilibrium and show that manipulation arises only when the value functions are steep. Then we consider the two different settings. In the first setting, we split a unit time interval into subintervals, where the length of each subinterval is a function of the informed trading probability, and consider the situation where the probability of informed trading is sufficiently small and the number of trading rounds is sufficiently large. Our results show that when there are relatively few trading rounds, the equilibrium is unique, whereas when there are many trading rounds, there are multiple equilibria. The intuition is simple. As there are two types of informed trader, there are four possible regimes, depending on whether each type manipulates. By a single crossing property of the payoff difference between buy and sell orders, we can prove that there is one equilibrium strategy within each regime when only one type manipulates. However, when there are too many chances to re-trade, both types simultaneously manipulate and this gives more "freedom" for multiple regimes to coexist. This analysis explicitly derives bounds for the number of trading rounds for which these different situations arise.

The characterization of an equilibrium in this case is given by the following properties.

- The value functions are continuous, piecewise differentiable and monotone with respect to the market maker's belief.

- There are two regions of the market maker's belief, depending on whether the slopes of the value functions are steeper than one. We show that when the informed trader manipulates in equilibrium, the slope needs to be steeper than one, because today he loses one dollar believing that the benefit that he can obtain on the future payoff (namely, the slope of the value function) is more than one dollar. This property specifies a region where manipulation can possibly arise in equilibrium.

In the second setting, we study the 2-period model, which is the simplest setting of our framework. We show that the equilibrium exists uniquely, and that bid and ask prices are monotonically increasing. We further show that the changes in manipulation rates are bounded by the likelihood ratio in terms of market maker's prior belief. Then we characterize the equilibrium bid and ask prices. When manipulations do not arise, by a simple calculation, we can show that bid price is strictly convex and ask price is strictly concave. We also show that even when manipulations arise, bid and ask prices satisfy *arc-convexity* and *arc-concavity*, which require that the slope of each point with the edges of the interval $[0, 1]$ is increasing and decreasing. This indicates that the bid-ask spreads are the largest in the middle of the interval. This is intuitive. In the model, the market maker imposes bid-ask spreads to hedge a risk of trading with the informed trader. In other words, the bid-ask spreads exist due to the asymmetric information between the market maker and the informed trader. If the market maker is most uncertain about the state, spreads should be largest.

Our simulation results in the later part confirm the theoretical findings. We first develop a computational method using linear interpolation to find equilibria in a general T-period model. Our simulation numerically demonstrates the intuitions for the theoretical results. Our simulation demonstrates the characteristics of equilibrium bid and ask spreads and also shows that when the number of trading rounds is not so large, the equilibrium with manipulation is unique; however when the number of trading rounds is large, there are multiple equilibria where manipulation arises.

1.1 Related Literature

There is a vast empirical literature on market manipulation. For instance, Aggarwal and Wu (2006) suggest that stock market manipulation may have important impacts on market efficiency. According to their empirical findings, while manipulative activities appear to have declined in the main security exchanges, they remain a serious issue in both developed and emerging financial markets, especially in over-the-counter markets.[1]

[1] See Jordan and Jordan (1996) on the cornering of the Treasury note auction market by Solomon Brothers in May 1991, Felixson and Pelli (1999) on closing price manipulation in the Finnish stock market, Mahoney (1999) on stock price manipulation leading up to the US Securities Exchange Act of 1934, Vitale (2000) on manipulation in the foreign exchange market and Merrick *et al.* (2005) on manipulation involving a delivery squeeze on a London-traded bond futures contract. For a useful survey, see Putniņš (2011).

The theoretical literature begins with market manipulation by uninformed traders.[2] Allen and Gale (1992) provide a model of strategic trading in which some equilibria involve manipulation. Furthermore, Allen and Gorton (1992) consider a model of pure trade-based uninformed manipulation in which asymmetry in buys and sells by liquidity traders creates the possibility of manipulation. The first paper to consider manipulation by an informed agent within the discrete-time Glosten–Milgrom framework is Chakraborty and Yilmaz (2004). They show that when the market faces uncertainty about the existence of informed traders, and when there are many trading periods, long-lived informed traders will manipulate in every equilibrium. Takayama (2010) furthers this analysis by providing a lower bound for the number of trading periods necessary for the existence of manipulation in equilibrium and shows that if the number of trading periods exceeds this lower bound, every equilibrium involves manipulation. The literature has investigated conditions based on the relations between prices and trades that rule out manipulation (Jarrow, 1992; Huberman and Stanzl, 2004). This paper also relates to that issue: that is, our algorithm provides the method to study these relations. We then respond to the questions of when manipulation arises and when the equilibrium with manipulation is unique.[3]

In addition to the Glosten–Milgrom framework, another reference framework is proposed by Kyle (1985). Back (1992) extends the analysis in Kyle (1985) to a continuous-time version. While the uniqueness of the optimal informed trader's strategy either in the original Kyle model or Back (1992) remains unknown, McLennan *et al.* (2017), Boulatov and Taub (2013), and Boulatov *et al.* (2005) prove the uniqueness under some technical assumptions. Back and Baruch (2004) study the equivalence of the Glosten–Milgrom model and the Kyle model in a continuous-time setting, and show that the equilibrium in the Glosten–Milgrom model is approximately the same as that in the Kyle model when the trade size is small and uninformed trades occur frequently. Given these two closely related frameworks, our analysis, as a proxy for a continuous-time model, shows the possibility of multiple equilibria and provides insights concerning uniqueness within these frameworks. Finally, Back and Baruch (2004) conclude that the continuous-time Kyle model is more tractable than the Glosten–Milgrom model, although most markets follow a sequential trade model. The model in this paper is a discrete-time version of the Back and Baruch (2004)'s Glosten–Milgrom model. This paper is therefore useful for opening the black box lying between these two alternative frameworks, and especially in considering the uniqueness of the informed trader's dynamic strategy and price manipulation in the presence of bid–ask spreads.

The model in this paper is a theoretical depiction of quote-driven markets with competitive market-

[2] Although our main concern in this paper is the dynamic strategic informed trader, there are works that study the dynamic informed market maker. For example, see Calcagno and Lovo (2006).

[3] Our focus in this paper is trade-based manipulation and its effect on prices. Goldstein and Guembel (2008) study manipulation and and its effect on a firm's investment problem. Then, they show that the informed trader can profit from manipulation, and then identify a fundamental limitation inherent in the allocational role of stock prices.

making, as in Glosten and Milgrom (1985) and Ozsoylev and Takayama (2010). Quote-driven markets are common in over-the-counter (OTC) markets. The Nasdaq and London SEAQ are two examples of quote-driven markets with competitive market-making, although the Nasdaq has become more of an order-driven market like the NYSE. OTC markets have drawn renewed interests in the literature, particularly after the 2007-2008 subprime lending crisis as opaque financial markets (see Monnet and Quintin, 2017). Pagano and Volpin (2012) provides the model to study the impact of transparency on the market for structured debt producers with the primary and secondary markets. Bolton *et al.* (2016) describes that OTC markets emerge even in the presence of well-functioning centralized exchanges. This paper provides a richer framework for studying the intertemporal price manipulation within these markets.

The remainder of the paper is organized as follows. Section 2 details the model. Section 3 provides some theoretical results. Section 4 describes the algorithm and illustrates the results from the numerical simulations.

2 The Model

In this section, we present our model, which is a discrete-time version of the Back and Baruch (2004)'s Glosten–Milgrom model.[4] There is a single risky asset and a numeraire. The terminal value of the risky asset, denoted $\tilde{\theta}$, is a random variable that can take a low or high value, i.e., L or H, where $L = 0$ and $H = 1$. We assume that $\Pr(\tilde{\theta} = H) = \delta_0$ for some $\delta_0 \in (0, 1)$. There is a single long-lived informed trader who learns $\tilde{\theta}$ prior to the beginning of trading.

Trade occurs in finitely many periods $t = 1, 2, \ldots, T$. In each period, a single trader comes to the market, the market maker quotes bid and ask prices for the risky asset, and the trader either buys one unit or sells one unit. The agent who goes to the market in period t is a random variable unobserved by the market maker, such that with probability μ the informed trader is selected. If the informed trader is not selected, the agent selected is a "noise" or "liquidity" trader who (regardless of the quoted prices) buys with probability γ and sells with probability $1 - \gamma$. The identities of the selected traders, and the values of the liquidity trader's trades, in the various periods are independent random variables. Prior to period t, there is no disclosure of information concerning the values of the random variables in that period.

We focus on equilibria where the market maker's belief and the number of remaining time periods

[4]The model in Chakraborty and Yilmaz (2004) is different from our model. In their model, in the beginning of the entire game, the informed trader or an uninformed trader is chosen. Once the informed trader is chosen, he trades in every period. In our model, in every period, there is a random determination on the trader's type. We use this setting, to compare the characteristics of prices in our analyses with the ones in Back and Baruch (2004). Further, our setting allows us to consider the first case in this paper, which we split a unit interval to subintervals, whose limit could approximate the continuous-time setting.

determine an equilibrium strategy. The set of possible actions for the informed trader is denoted by $\{B, S\}$, in which B is a buy order and S is a sell order. Here, the market maker's belief b is the probability, from the market maker's point of view, that the state is high, H, going into period t. The market maker's ask and bid prices are functions of the market maker's belief and given by the functions $\alpha_t : [0,1] \to [0,1]$ and $\beta_t : [0,1] \to [0,1]$, respectively. For each type θ of informed trader, a trading strategy $\sigma_{\theta t} : [0,1] \to \Delta(\{B,S\})$ for $\theta \in \{H, L\}$ specifies a probability distribution over trades in period t with respect to the bid and ask prices posted in period t. In period t, the type-H informed trader buys the security with probability $\sigma_{Ht}(b)$ and sells with probability $1 - \sigma_{Ht}(b)$, and the type-L trader buys and sells with probabilities $1 - \sigma_{Lt}(b)$ and $\sigma_{Lt}(b)$, respectively.

The market maker's posterior belief after observing an order is updated using Bayes' rule on the posterior probability that $\tilde{\theta} = H$. Since the value of the asset is either 0 or 1, the market maker's prior belief b is equal to the expected value of the risky asset conditional on his information going into period t. Define the bid and ask functions $A, B : [0,1]^3 \to [0,1]$ by the formulas:

$$A(b,x,y) = \frac{[(1-\mu)\gamma + \mu x]b}{(1-\mu)\gamma + \mu b x + \mu(1-b)(1-y)} \quad \text{and} \quad B(b,x,y) = \frac{[(1-\mu)(1-\gamma) + \mu(1-x)]b}{(1-\mu)(1-\gamma) + \mu b(1-x) + \mu(1-b)y}.$$

Now, the market maker is really to be thought of as a competitive market of risk-neutral market makers, for instance, a pair of market makers in Bertrand competition or a continuum of identical market makers. The equilibrium condition for the market maker is zero expected profits, which amounts to setting ask and bid prices equal to the posterior expected values of the asset.[5]

Now, we define the Markov equilibrium as follows.[6]

Definition 1. *A* Markov equilibrium *is a collection of functions* $\{\alpha_t, \beta_t, \sigma_{Ht}, \sigma_{Lt}\}_{t=1,\cdots,T}$ *with* $\alpha_t, \beta_t : [0,1] \to [0,1]$*,* $\sigma_{Ht}, \sigma_{Lt} : [0,1] \to \Delta(\{B,S\})$ *and* $J_t, V_t : [0,1] \to \mathbb{R}$ *such that for each* $t = 1, \ldots, T$ *and* $b \in [0,1]$*,*

(E1) $\alpha_t(b) = A(b, \sigma_{Ht}(b), \sigma_{Lt}(b))$ *and* $\beta_t(b) = B(b, \sigma_{Ht}(b), \sigma_{Lt}(b))$.

(E2)

$$\sigma_{Ht}(b) = \begin{cases} 0, & 1 - \alpha_t(b) + J_{t+1}(\alpha_t(b)) < \beta_t(b) - 1 + J_{t+1}(\beta_t(b)), \\ 1, & 1 - \alpha_t(b) + J_{t+1}(\alpha_t(b)) > \beta_t(b) - 1 + J_{t+1}(\beta_t(b)), \end{cases}$$

and

$$\sigma_{Lt}(b) = \begin{cases} 0, & -\alpha_t(b) + V_{t+1}(\alpha_t(b)) > \beta_t(b) + V_{t+1}(\beta_t(b)), \\ 1, & -\alpha_t(b) + V_{t+1}(\alpha_t(b)) < \beta_t(b) + V_{t+1}(\beta_t(b)). \end{cases}$$

[5] Biais *et al.* (2000) justify this assumption by showing that when there are infinitely many market makers, their expected profit converges to zero. More recently, Calcagno and Lovo (2006) show that a market maker's equilibrium expected payoff is zero if he is uninformed.

[6] Imagine that the market maker observes the sequence of realized trades for up until period t and updates his belief. Observing the history of realized trades, the informed trader can also find out what the market maker's belief b is. In this sense, this is a reduced form of equilibrium.

(E3)

$$J_t(b) = \mu\left[\sigma_{Ht}(b)(1-\alpha_t(b) + J_{t+1}(\alpha_t(b))) + (1-\sigma_{Ht}(b))(\beta_t(b) - 1 + J_{t+1}(\beta_t(b)))\right]$$

$$+(1-\mu)\left[\gamma J_{t+1}(\alpha_t(b)) + (1-\gamma) J_{t+1}(\beta_t(b))\right],$$

and

$$V_t(b) = \mu\left[(1-\sigma_{Lt}(b))(-\alpha_t(b) + V_{t+1}(\alpha_t(b))) + \sigma_{Lt}(b)(\beta_t(b) + V_{t+1}(\beta_t(b)))\right]$$

$$+(1-\mu)\left[\gamma V(\alpha_t(b)) + (1-\gamma) V(\beta_t(b))\right].$$

We see that (E1) states that the ask and bid prices are Bayesian updatings of b conditional on the type of order received; (E2) states that both types of informed seller optimize their order, taking into account the effect on the expected profits from future trades; and (E3) specifies the recursive computation of value functions. Implicitly we are assuming that the functions J_{T+1} and V_{T+1} are identically zero.

Lemma 1. *Suppose that* $b \in (0,1)$. *Then* $A(b,x,y)$ *is a strictly increasing function of* x *and* y, $B(b,x,y)$ *is a strictly decreasing function of* x *and* y, *and* $A(b,x,y) > (=, <)\ B(x,y)$ *if and only if* $x + y > (=, <)\ 1$. *If* $x + y > 1$, *then*

$$0 < \frac{\partial A}{\partial x}(b,x,y) < \frac{\partial A}{\partial y}(b,x,y) \quad \textit{and} \quad 0 > \frac{\partial B}{\partial y}(b,x,y) > \frac{\partial B}{\partial x}(b,x,y). \tag{1}$$

Proof. The monotonicity claims are obvious, and it is easy to see that $A(b,z,1-z) = b = B(b,z,1-z)$. If $x + y > 1$, then

$$A(b,x,y) > A(b,x,1-x) = b = B(b,x,1-x) > B(b,x,y)$$

by monotonicity, and similarly if $x + y < 1$. To verify the claims concerning the partial derivatives we compute that:

$$\frac{\partial A}{\partial x}(b,x,y) = \frac{[(1-\mu)\gamma + \mu(1-y)]b(1-b)}{[(1-\mu)\gamma + \mu bx + \mu(1-b)(1-y)]^2}, \qquad \frac{\partial B}{\partial x}(b,x,y) = -\frac{[(1-\mu)(1-\gamma) + \mu y]b(1-b)}{[(1-\mu)(1-\gamma) + \mu b(1-x) + \mu(1-b)y]^2},$$

$$\frac{\partial A}{\partial y}(b,x,y) = \frac{[(1-\mu)\gamma + \mu x]b(1-b)}{[(1-\mu)\gamma + \mu bx + \mu(1-b)(1-y)]^2}, \qquad \frac{\partial B}{\partial y}(b,x,y) = -\frac{[(1-\mu)(1-\gamma) + \mu(1-x)]b(1-b)}{[(1-\mu)(1-\gamma) + \mu b(1-x) + \mu(1-b)y]^2}.$$

□

3 Theoretical Results

Our idea of finding equilibria in this model is backward. In the terminal period, the informed trader does not have a chance to re-trade and thus do not manipulate. Thus, we know what the period-T

value functions of prior beliefs are. As we explain further in Section 4, given the next period value functions, we solve for the current period value functions. In this way, we focus on the decision making problem of the informed trader in period t, given the period-$(t+1)$ value functions.

Define the difference in payoffs between trading for and against the information is monotone across the relevant region of prior beliefs as follows:

$$\begin{aligned} D_H(b,x,y) &= -A(b,x,y) + J_{t+1}(A(b,x,y)) - B(b,x,y) - J_{t+1}(B(b,x,y)) + 2; \\ D_L(b,x,y) &= B(b,x,y) + V_{t+1}(B(b,x,y)) + A(b,x,y) - V_{t+1}(A(b,x,y)). \end{aligned} \tag{2}$$

Then $D_\theta(b,x,y)$ is the difference in payoffs between trading for and against the information given prices $A(b,x,y)$ and $B(b,x,y)$ for each type $\theta \in \{H, L\}$. Notice that for any $\sigma \in [0,1]$ and prior $b \in [0,1]$, by Bayes' rule, both the bid and ask prices are equal to b. Thus, we have

$$D_L(b,\sigma,1-\sigma) > 0 \quad \text{and} \quad D_H(b,\sigma,1-\sigma) > 0. \tag{3}$$

Theorem 1. *Suppose that the value functions in every period s satisfy the following conditions:*

(C) *J_s and V_s are continuous and piecewise differentiable;*

(M) *J_s is strictly decreasing and V_s is strictly increasing;*

(SH) *there is a b_H such that for any $b_0, b_1 < b_H$, $\frac{J_s(b_1)-J_s(b_0)}{b_1-b_0} < -1$, and for any $b_0, b_1 > b_H$, $\frac{J_s(b_1)-J_s(b_0)}{b_1-b_0} > -1$;*

(SL) *there is a b_L such that for any $b_0, b_1 < b_L$, $\frac{V_s(b_1)-V_s(b_0)}{b_1-b_0} < 1$, and for any $b_0, b_1 > b_L$, $\frac{V_s(b_1)-V_s(b_0)}{b_1-b_0} > 1$.*

When at least one of $D_L(b,1,1)$ and $D_H(b,1,1)$ is positive, there exists a unique equilibrium, and bid and ask prices are monotonically increasing in b.

Theorem 1 provides a set of sufficiently conditions for the unique existence of equilibrium. Further this proposition states that in equilibrium, bid and ask prices are monotonically increasing with respect to prior beliefs, even in the presence of manipulation. This indicates that even if the informed trader manipulates in equilibrium, the effects of manipulation are not large enough to change monotonicity of bid and ask prices.

3.1 Results in the T-period model

Our aim in this section is to prove Theorem 1. We start with defining a manipulative strategy. We say that a strategy is manipulative if it involves the informed trader undertaking a trade in any period that yields a strictly negative short-term profit.

Definition 2. *For $\theta \in \{H, L\}$ we say that* the type-θ trader manipulates at b in period t *if* $\sigma_{\theta t}(b) < 1$.[7]

Our mode of analysis is backwards induction: we assume certain properties of the value functions J_{t+1} and V_{t+1}, and from this assumption, derive various properties of the functions α_t, β_t, σ_{Ht}, σ_{Lt}, J_t and V_t. For $t = T$ the equilibrium conditions have a unique closed-form solution, because J_{T+1} and V_{T+1} are identically zero. The next result is necessary to begin the process of backwards induction. As a direct consequence of optimization, we can prove that there is no manipulation in the last period. By using this, we obtain the following theorem.

Proposition 1. *The last-period value functions J_T and V_T satisfy* $(\mathbf{C})$, $(\mathbf{M})$, $(\mathbf{SH})$, *and* $(\mathbf{SL})$.

Fix $t < T$ and suppose that a family of continuous next-period value functions is given where each J_{t+1} and V_{t+1} satisfies the four properties. For $b \in [0, 1]$ let

$$\mathcal{E}(b) = \{(\sigma_{Ht}(b), \sigma_{Lt}(b)) : \text{for the } \alpha_t(b) \text{ and } \beta_t(b) \text{ given by (M1), (M2) holds}\}.$$

Lemma 2. *The set-valued mapping $\mathcal{E}$ has a closed graph.*

Proof of Proposition 2. The result follows by the continuity of the next-period value functions and Bayes' rule. □

Proposition 2. *If J_{t+1} and V_{t+1} are continuous, then $\mathcal{E}(b)$ is nonempty for each $b \in [0, 1]$.*

Proof of Proposition 2. For $(\sigma_H, \sigma_L) \in [0, 1]^2$ let $B(\sigma_H, \sigma_L)$ be the pair of posterior beliefs given by (M1). Evidently $B : [0, 1]^2 \to [0, 1]^2$ is a continuous function. For $(\alpha, \beta) \in [0, 1]^2$ let $BR^b(\alpha, \beta)$ be the set of pairs (σ_H, σ_L) satisfying (M2). Given that J_{t+1} and V_{t+1} are continuous, BR^b is an upper semicontinuous correspondence. Its value is always a Cartesian product of two elements of the set $\{\{0\}, [0, 1], \{1\}\}$, so it is convex valued. The composition $BR^b \circ B$ is thus an upper semicontinuous convex-valued correspondence, so Kakutani's fixed point theorem implies that it has a fixed point. □

Fix $b \in (0, 1)$ and $\sigma = (\sigma_H, \sigma_L) \in \mathcal{E}(b)$, and let $\alpha = A(b, \sigma_H, \sigma_L)$ and $\beta = B(b, \sigma_H, \sigma_L)$ be the pair of equilibrium ask and bid prices associated with b and σ in period t.

Lemma 3. $\alpha > b > \beta$ *and* $\sigma_H + \sigma_L > 1$. *In particular,* $\sigma_H, \sigma_L > 0$.

Proof. Suppose that $\alpha \leq \beta$. Bayes' rule implies that $0 < \alpha, \beta < 1$, so $1 - \alpha > \beta - 1$ and $-\alpha < \beta$, and the monotonicity condition $(\mathbf{M})$ gives

$$\begin{aligned} 1 - \alpha + J_{t+1}(\alpha) &> \beta - 1 + J_{t+1}(\beta); \\ -\alpha + V_{t+1}(\alpha) &< \beta + V_{t+1}(\beta). \end{aligned}$$

[7]This is the same definition used by Chakraborty and Yilmaz (2004). Back and Baruch (2004) use the term "bluffing" instead, while Huberman and Stanzl (2004) define price manipulation as a round-trip trade. For additional discussion on how to define price manipulation, see Kyle and Viswanathan (2008).

Now optimisation implies that $\sigma_H = 1$ and $\sigma_L = 1$, and Bayes' rule gives $\alpha > b > \beta$, a contradiction. In turn, by Bayes' rule, $\alpha > b > \beta$ implies that $\sigma_H + \sigma_L > 1$. □

In equilibrium, the type-H trader does not sell with probability one and the type-L trader does not buy with probability one. This means that the informed trader either trades on his information or assigns a positive probability to both buy and sell orders. In the latter case, the informed trader is indifferent between buy and sell orders.

Interestingly, Lemma 3 indicates that even when the informed trader manipulates, a strictly positive amount of the true information comes out to the market in each period. This leads to a strictly positive bid and ask spread. This motivates consideration of the slopes of the value functions in the two points of bid and ask prices, because in our setting, bid and ask prices are the priors of market maker 's belief in the next period. By Lemma 3 the bid–ask spread $\alpha - \beta$ is strictly positive. If the type-H trader manipulates, it must be the case that $1 - \alpha + J_{t+1}(\alpha) = \beta - 1 + J_{t+1}(\beta)$, so we have:

$$\frac{J_{t+1}(\alpha) - J_{t+1}(\beta)}{\alpha - \beta} = \frac{\alpha + \beta - 2}{\alpha - \beta} = -1 - \frac{2 - 2\alpha}{\alpha - \beta} < -1. \tag{4}$$

Similarly, if the type-L trader manipulates, we have:

$$\frac{V_{t+1}(\alpha) - V_{t+1}(\beta)}{\alpha - \beta} = \frac{\alpha + \beta}{\alpha - \beta} = 1 + \frac{2\beta}{\alpha - \beta} > 1. \tag{5}$$

Given $\beta < b < \alpha$, and slopes of the value functions between the bid and ask prices larger than one, **(SH)** and **(SL)** imply the following result.

Proposition 3. *In equilibrium, the following hold.*

H. *If the type-H trader manipulates at b, then* $\frac{J_{t+1}(\beta_0) - J_{t+1}(\beta_1)}{\beta_0 - \beta_1} < -1$ *for any* $\beta_0, \beta_1 \leq \beta$.

L. *If the type-L trader manipulates at b, then* $\frac{V_{t+1}(\alpha_0) - V_{t+1}(\alpha_1)}{\alpha_0 - \alpha_1} > 1$ *for any* $\alpha_0, \alpha_1 \geq \alpha$.

Proposition 3 indicates that manipulation could arise only in a region where the value function is steep. This is intuitive: the informed trader manipulates when the change in the future payoff from manipulating is large. In other words, the informed trader knowingly makes loss when the value function is steep, because the manipulation makes a difference for the future payoff, which is represented by the slope of the value function.

We now classify equilibria according to the types of trader that sometimes trade against their information. An equilibrium σ is in *Regime* $\emptyset$ if $\sigma = (1, 1)$. It is in *Regime* L if $\sigma_L < 1$ and $\sigma_H = 1$; it is in *Regime* H if $\sigma_L = 1$ and $\sigma_H < 1$; and it is in *Regime* HL if $\sigma_L < 1$ and $\sigma_H < 1$. We say that a regime *arises* at a belief b if $\mathcal{E}(b)$ contains an equilibrium in that regime.

For each $\theta \in \{H, L\}$, define x^θ and y^θ by:

$$x^\theta := \begin{cases} \min\{x : D_\theta(b, x, 1) = 0\} & \text{if} \quad \{x : D_\theta(b, x, 1) = 0\} \neq \emptyset; \\ 1 & \text{otherwise,} \end{cases}$$

and

$$y^\theta := \begin{cases} \min\{y : D_\theta(b, 1, y) = 0\} & \text{if} \quad \{y : D_\theta(b, 1, y) = 0\} \neq \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Finally, in Proposition 4, we will show that within each regime of Regime H, Regime L and Regime $\emptyset$, an equilibrium is unique. Theorem 1 is a direct consequence of this proposition. To prove this proposition, in the following lemmas, we show hat when only one type manipulates, the payoff difference is monotonically decreasing with respect to each type's strategy in $[x^\theta, 1]$ or $[y^\theta, 1]$ for each $\theta \in \{H, L\}$ and further that y^θ also satisfies monotonicity.

The next lemma shows that given the other type's strategy, when only one type manipulates, the payoff difference is monotonically decreasing with respect to each type's strategy in $[x^\theta, 1]$ or $[y^\theta, 1]$ for each $\theta \in \{H, L\}$.

Lemma 4. *If $0 < \bar{x}, \bar{y} \leq 1$, $\bar{x} + \bar{y} > 1$, $\theta \in \{H, L\}$ and $D_\theta(b, \bar{x}, \bar{y}) = 0$, then*

- *the payoff difference $D_\theta(b, x, \bar{y})$ is strictly decreasing as x increases for all $x \geq \bar{x}$;*
- *the payoff difference $D_\theta(b, \bar{x}, y)$ is strictly decreasing as y increases for all $y \geq \bar{y}$.*

Proof of Lemma 4. By symmetry it suffices to prove that $D_L(b, x, \bar{y})$ and $D_H(b, x, \bar{y})$ are decreasing in x. First, consider $D_L(b, x, \bar{y})$. Now,

$$V_{t+1}(A(b, \bar{x}, \bar{y})) - V_{t+1}(B(b, \bar{x}, \bar{y})) = A(b, \bar{x}, \bar{y}) + B(b, \bar{x}, \bar{y}). \tag{6}$$

By Bayes' rule, $A(b, \bar{x}, \bar{y}) > B(b, \bar{x}, \bar{y})$. Dividing (6) by $A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})$ gives

$$\frac{V_{t+1}(A(b, \bar{x}, \bar{y})) - V_{t+1}(B(b, \bar{x}, \bar{y}))}{A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})} = \frac{A(b, \bar{x}, \bar{y}) + B(b, \bar{x}, \bar{y})}{A(b, \bar{x}, \bar{y}) - B(b, \bar{x}, \bar{y})} > 1.$$

This shows that $A(b, \bar{x}, \bar{y}) > b_L$. By Bayes' rule, $A(b, x, \bar{y})$ is monotonically increasing in x, so for any $x \geq \bar{x}$ and $\Delta > 0$ we have $A(b, x, \bar{y}), A(b, x + \Delta, \bar{y}) > b_L$ and consequently

$$\begin{aligned} &\big(V_{t+1}(A(b, x + \Delta, \bar{y})) - A(b, x + \Delta, \bar{y})\big) - \big(V_{t+1}(A(b, x, \bar{y})) - A(b, x, \bar{y})\big) \\ &\quad = (A(b, x + \Delta, \bar{y}) - A(b, x, \bar{y})) \left(\frac{V_{t+1}(A(b, x + \Delta, \bar{y}) - V_{t+1}(A(b, x, \bar{y}))}{A(b, x + \Delta, \bar{y})) - A(b, x, \bar{y})} - 1 \right) > 0. \end{aligned}$$

That is, $V_{t+1}(A(b, x, \bar{y})) - A(b, x, \bar{y})$ is an increasing function of x. On the other hand, Bayes' rule and the monotonicity condition ($\mathbf{M}$) imply that $B(b, x, \bar{y}) + V_{t+1}(B(b, x, \bar{y}))$ is a decreasing function of x. As

$$D_L(b, x, \bar{y}) = B(b, x, \bar{y}) + V_{t+1}(B(b, x, \bar{y})) + A(b, x, \bar{y}) - V_{t+1}(A(b, x, \bar{y})),$$

the result follows. Second, consider $D_H(b, x, \bar{y})$. Suppose that we have:

$$J_{t+1}(A(b,\bar{x},\bar{y})) - J_{t+1}(B(b,\bar{x},\bar{y})) = A(b,\bar{x},\bar{y}) + B(b,\bar{x},\bar{y}) - 2. \tag{7}$$

Similarly, dividing (7) by $A(b,\bar{x},\bar{y}) - B(b,\bar{x},\bar{y})$ gives

$$\frac{J_{t+1}(A(b,\bar{x},\bar{y})) - J_{t+1}(B(b,\bar{x},\bar{y}))}{A(b,\bar{x},\bar{y}) - B(b,\bar{x},\bar{y})} = \frac{A(b,\bar{x},\bar{y}) + B(b,\bar{x},\bar{y}) - 2}{A(b,\bar{x},\bar{y}) - B(b,\bar{x},\bar{y})} < -1.$$

This shows that $B(b,\bar{x},\bar{y}) < b_H$. Therefore, $(\mathbf{SH})$ implies that for any $\Delta > 0$ and $x \leq \bar{x}$,

$$\begin{aligned} &J_{t+1}(B(b,x+\Delta,\bar{y})) - J_{t+1}(B(b,x,\bar{y})) + B(b,x+\Delta,\bar{y}) - B(b,x,\bar{y}) \\ &= (B(b,x+\Delta,\bar{y}) - B(b,x,\bar{y}))\left(\frac{J_{t+1}(B(b,x+\Delta,\bar{y})) - J_{t+1}(B(b,x,\bar{y}))}{B(b,x+\Delta,\bar{y}) - B(b,x,\bar{y})} + 1\right) < 0, \end{aligned}$$

as by Bayes' rule, $B(b,x+\Delta,\bar{y}) < B(b,x,\bar{y}) < b_H$. That is, $J_{t+1}(B(b,x,\bar{y})) - B(b,x,\bar{y})$ is a decreasing function of x. On the other hand, Bayes' rule and $(\mathbf{M})$ imply that $J_{t+1}(A(b,x,\bar{y})) - A(b,x,\bar{y})$ is an increasing function of x. As

$$D_H(b,x,\bar{y}) = \big(1 - A(b,x,\bar{y}) + J_{t+1}(A(b,x,\bar{y}))\big) - \big(B(b,x,\bar{y}) - 1 + J_{t+1}(B(b,x,\bar{y}))\big),$$

the result follows. □

Let $\bar{x} = 1$ and $\bar{y} = 1$. Because Lemma 4 shows that given the other type's strategy, D_θ is monotone, this also holds when the other type trades honestly and thus the next lemma directly follows from Lemma 4.

Lemma 5. *For each $\theta \in \{H, L\}$,*

- *the payoff difference $D_\theta(b,x,1)$ is monotonically decreasing as x increases for all $x \geq x^\theta$;*
- *the payoff difference $D_\theta(b,1,y)$ is monotonically decreasing as y increases for all $y \geq y^\theta$.*

Further, for each $\theta \in \{H, L\}$, define $\tilde{y}_\theta : [0,1] \to [0,1]$ by $D_\theta(b,x,\tilde{y}_\theta(x)) = 0$ if $D_\theta(b,1,1) \leq 0$. Then, we obtain the following result.

Lemma 6. *For each $\theta \in \{H, L\}$, $\tilde{y}_\theta$ is continuous and is strictly decreasing in x.*

Proof of Lemma 6. Suppose that $\tilde{y}_\theta$ is well defined. Continuity of D_θ indicates that $\tilde{y}_\theta$ is also continuous for each $\theta \in \{H, L\}$. Suppose that $x_1 > x_2$ and $\tilde{y}_L(x_1) \geq \tilde{y}_L(x_2)$. By Lemma 4,

$$0 = D_L(b,x_1,\tilde{y}_L(x_1)) < D_L(b,x_2,\tilde{y}_L(x_1)) \leq D_L(b,x_2,\tilde{y}_L(x_2)),$$

which is a contradiction to $0 = D_L(b,x_2,\tilde{y}_L(x_2))$. By symmetry, the same holds for $\tilde{y}_H$. □

Thus, by the monotonicity of D_θ's proved in Lemma 5 and Lemma 6, we obtain the following result.

Proposition 4. *The following holds:*

(a) if Regime HL arises at b, then $D_L(b,1,1) < 0$ and $D_H(b,1,1) < 0$;

(b) if $D_L(b,1,1) \geq 0$ and $D_H(b,1,1) \geq 0$, then $\mathcal{E}(b) = \{(1,1)\}$, so that only regime $\emptyset$ arises at b;

(c) if $D_L(b,1,1) < 0$ and $D_H(b,1,1) \geq 0$, then $\mathcal{E}(b)$ is a singleton whose unique element is in Regime L;

(d) if $D_H(b,1,1) < 0$ and $D_L(b,1,1) \geq 0$, then $\mathcal{E}(b)$ is a singleton whose unique element is in Regime H;

(e) if $D_H(b,1,1) < 0$ and $D_L(b,1,1) < 0$, then at most one element within Regime H is in $\mathcal{E}(b)$ and at most one element within Regime L is in $\mathcal{E}(b)$.

Proof of Proposition 4. We give a proof of each statement below.
Proof of (a). Suppose that Regime HL arises at b. Then, there must exist $(\bar{x},\bar{y}) \in \mathcal{E}(b)$ with $\bar{x} < 1$ and $\bar{y} < 1$, such that $D_H(b,\bar{x},\bar{y}) = 0$ and $D_L(b,\bar{x},\bar{y}) = 0$. By Lemma 4 and Lemma 5, we have:

$$0 = D_L(b,\bar{x},\bar{y}) > D_L(b,\bar{x},1) > D_L(b,1,1). \tag{8}$$

By symmetry, we can also prove $0 > D_L(b,1,1)$. □
Proof of (b). By (a) of this proposition, Regime HL does not arise. Aiming at a contradiction, suppose that Regime H arises. Then there exists an $\bar{x} < 1$ to satisfy $D_H(b,\bar{x},1) = 0$. Then, by Lemma 5, we must have $D_H(b,1,1) < 0$, which contradicts our assumption. By symmetry, we can prove that Regime L does not arise. □
Proof of (c). First, as $D_L(b,1,1) < 0$ and $D_H(b,1,1) \geq 0$, Regime $\emptyset$ does not arise because taking an honest strategy is not optimal for the low type. Also, by (a) of this proposition, Regime HL does not arise. Now suppose that Regime H arises. Then there exists an $\bar{x} < 1$ to satisfy $D_H(b,\bar{x},1) = 0$. Then, by Lemma 5, we must have $D_H(b,1,1) < 0$, which contradicts our assumption.

Lemma 6 indicates that there is no $y < 1$ to satisfy $D_H(b,1,y) = 0$. As D_H is continuous in y, $D_H(b,1,y) \geq 0$ must hold for all $y \in [0,1]$. Now, as $D_L(b,1,1) < 0$ and $D_H(b,1,1) \geq 0$, (3) and Lemma 6 imply that there exists a $\bar{y} < 1$ to satisfy

$$D_L(b,1,\bar{y}) = 0 \quad \text{and} \quad D_H(b,1,\bar{y}) \geq 0.$$

Therefore, we can see that Regime L arises. In addition, by Lemma 5, there is only one $\bar{y}$ to satisfy $D_L(b,1,\bar{y}) = 0$. □

Proof of (d). Done symmetrically with (c) of this proposition. □
Proof of (e). Suppose that there is one element in $\mathcal{E}(b)$ that belongs to Regime H. Then, by Lemma 5, there is no other element in $\mathcal{E}(b)$ that also belongs to Regime H. Symmetrically the same holds for Regime L. □ □

Proposition 4 states that when being honest is profitable for each type of informed trader, he would not manipulate. Further, within each regime, an equilibrium is unique. This comes out from the fact that the payoff difference is monotonically decreasing with respect to each type's strategy in a certain range, as shown in Lemma 4.

As stated in (e) of Proposition 4, multiple equilibria are possible when $D_H(b,1,1) < 0$ and $D_L(b,1,1) < 0$ hold. Although within each regime, an equilibrium is unique, in the next subsection, we show that when the trading periods become large, there are multiple equilibria with manipulations. As stated in Section 4, our computer simulation also finds that when the trading periods become large, three regimes, Regime H, Regime L and Regime $\emptyset$ can arise at the same time.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The unique existence of equilibrium is a direct consequence of Proposition 4. We prove that bid and ask prices are monotonically increasing in market maker's prior beliefs. When nobody manipulates, by the Bayes' rule we can show that bid and ask prices decrease as market makers' belief b decreases. By (a) of Proposition 4, Regime HL does not arise. So, it remains to show that the result holds in Regime H and L. Since the argument is symmetric, we only prove the result for Regime L. Now, suppose that the type-L manipulates at b. Then, as the type-L's indifference condition for σ we have:

$$-\alpha_t(b) + V_{t+1}(\alpha_t(b)) = \beta_t(b) + V_{t+1}(\beta_t(b)). \tag{9}$$

Taking the first derivative we obtain:

$$\alpha_t'(b)(-1 + V_{t+1}'(\alpha_t(b))) = \beta_t'(b)(1 + V_{t+1}'(\beta_t(b))). \tag{10}$$

By Proposition 3, $-1+V_{t+1}'(\alpha_t(b)) > 0$. By condition $(\mathbf{M})$, $(1+V_{t+1}'(\beta_t(b))) > 0$. Therefore (10) indicates that: $\alpha_t'(b) > 0$ if and only if $\beta_t'(b) > 0$. Let $\Delta = (1-\mu)\gamma + \mu(1-\sigma_{Lt}(b)) + \sigma_{Lt}'(b)b(1-b)$. Then, we have

$$\alpha_t'(b) = \frac{((1-\mu)\gamma + \mu)\cdot\Delta}{(1-\mu)\gamma + \mu b + \mu(1-b)(1-\sigma_{Lt}(b))} \quad \text{and} \quad \beta_t'(b) = \frac{(1-\mu)(1-\gamma)\cdot(1-\Delta)}{(1-\mu)(1-\gamma) + \mu(1-b)\sigma_{Lt}(b)}. \tag{11}$$

Thus, if $\alpha_t'(b) \leq 0$, we must have $\Delta < 0$, which implies $\beta_t'(b) > 0$. We obtain a contradiction because (10) would not hold. □

3.2 Sufficiently Small Informed Trading Probability

Due to tractability, in the following two subsections, we consider two cases. In this subsection, we consider the case such that μ is sufficiently small, while in the next subsection, we consider the case where $T = 2$. To present the case where μ is sufficiently small, we split a unit interval into $\lfloor \mu^r \rfloor$ subintervals so that $T = \lfloor \frac{1}{\mu^r} \rfloor$.[8]

In this setting, we consider a class of Markov equilibria, which satisfies conditions $(\mathbf{C})$, $(\mathbf{M})$, $(\mathbf{SH})$ and $(\mathbf{SL})$ for Theorem 1. We call this class of equilibria a *tame* equilibrium, which is formally defined as follows.

Definition 3. *We say that a Markov equilibrium is* tame *if the value functions in every period t satisfy conditions $(\mathbf{C})$, $(\mathbf{M})$, $(\mathbf{SH})$ and $(\mathbf{SL})$.*

Theorem 2. *Let μ be sufficiently close to 0 and $T = \lfloor \frac{1}{\mu^r} \rfloor$ for some $r > 0$. The following holds:*

(a) Let $r \in (0, 1]$. Then, the tame equilibrium is the unique equilibrium. Moreover, there is no manipulation in equilibrium.

(b) Let $r \in (1, 2)$. Then, the tame equilibrium is the unique equilibrium. Moreover, in equilibrium, manipulation arises such that at most one type of trader manipulates at some belief b in some period t.

(c) Let $r \in (2, +\infty)$. Then, there are multiple equilibria, including multiple tame equilibria. Moreover, in equilibrium, manipulation arises such that both types of trader simultaneously manipulate at some belief b in some period t.

It may be easy to see that the monotonicity of $\tilde{y}_\theta$ by Lemma 6 yields a "single crossing property." By using this property, we obtain the following result.

Lemma 7. *At prior belief b in period t,*

Case I. *if $x^L < x^H < 1 \quad$ and $\quad 1 > y^L > y^H$, then only Regime HL arises;*

Case II. *if $1 > x^L > x^H \quad$ and $\quad y^L < y^H < 1$, then Regime H, Regime L and Regime HL arise.*

Proof. First, we show that Regime HL arises in both cases. By symmetry, assume

$$x^L < x^H < 1 \quad \text{and} \quad 1 > y^L > y^H. \tag{12}$$

Lemma 5 and (12) indicate that $D_H(b, 1, 1) < 0$ and $D_L(b, 1, 1) < 0$. By Lemma 5, $x^H < 1$. Because $\tilde{y}_L(x^L) = 1$, $x^L < x^H < 1$ and $\tilde{y}_L(1) = y^L$, $\tilde{y}_L(x^H) \in (y^L, \tilde{y}_L(x^L))$ by Lemma 6. Thus, we obtain

[8] When μ goes to 0, T goes to the infinity. At this limit, the setting may converge to the continuous-time setting.

$\tilde{y}_L(x^H) < 1 = \tilde{y}_L(x^L)$. Thus, by (12) we obtain that $\tilde{y}_L(x^H) < \tilde{y}_H(x^H) = 1$ and $y^L = \tilde{y}_L(1) > \tilde{y}_H(1) = y^H$. By applying the intermediate value theorem to $\tilde{y}_H$ and $\tilde{y}_L$, there exists an $x \in (x^H, 1)$ to satisfy $\tilde{y}_H(x) = \tilde{y}_L(x)$. Thus, by the definition of $\tilde{y}_H$ and $\tilde{y}_L$, we obtain x and $y = \tilde{y}_H(x)$ to satisfy both of the indifference conditions. Thus, Regime HL arises.

Now, if $y^L < y^H$ and $x^L > x^H$, Regimes H, L and HL arise by Lemma 7, because Lemma 5 indicates that $D_H(b, 1, y^L) > 0$ and $D_L(b, x^H, 1) > 0$. We can prove Case II symmetrically. □

Proposition 5. *Take $x \in (0, 1]$ and $y \in (0, 1]$ and suppose that $x > 1-y$. For $r < 1$, $\frac{A_\mu(b,x,y)-B_\mu(b,x,y)}{\mu^r}$ is sufficiently close 0. For $r > 1$, $\frac{A_\mu(b,x,y)-B_\mu(b,x,y)}{\mu^r} > M$ for M sufficiently large.*

Proof of Proposition 5. For future reference, we compute that

$$\begin{aligned} A_\mu(b,x,y) - b &= \tfrac{\mu b(1-b)(x-1+y)}{\mu[bx+(1-b)(1-y)]+(1-\mu)\gamma}; \\ b - B_\mu(b,x,y) &= \tfrac{\mu b(1-b)(x-1+y)}{\mu[b(1-x)+(1-b)y]+(1-\mu)(1-\gamma)}. \end{aligned}$$

As $x > 1-y$, for $r < 1$, $\frac{(A_\mu(b,x,y)-B_\mu(b,x,y))}{\mu^r}$ equals

$$b(1-b)(x-1+y)\cdot\mu^{1-r} \\ \times\left(\frac{1}{\mu[bx+(1-b)(1-y)]+(1-\mu)\gamma} + \frac{1}{[\mu b(1-x)+(1-b)y]+(1-\mu)(1-\gamma)}\right).$$

When μ is sufficiently small, μ^{1-r} is sufficiently large for $r > 1$, while it is sufficiently small for $r < 1$. Because $0 < \frac{1}{\mu[bx+(1-b)(1-y)]+(1-\mu)\gamma}, \frac{1}{[\mu b(1-x)+(1-b)y]+(1-\mu)(1-\gamma)} < +\infty$, when μ^{1-r}, we obtain the desired results. □

Proposition 4 and Proposition 5 yield the following result.

Proposition 6. *Let $T = \lfloor \frac{1}{\mu^r} \rfloor$ for some $r > 0$ and a sufficiently small μ. Then,*

(a) if $r \in (0, 1]$, Regime $\emptyset$ arises and manipulation does not arise at any belief in period t;

(b) if $r \in (1, +\infty)$, Regime H arises at some belief b that is sufficiently close to 1, and Regime L arises at some belief b that is sufficiently close to 0 in period t;

(c) if $r \in (1, 2)$, Regime HL never arises at any belief in period t;

(d) if $r \in (2, +\infty)$, Regime H, Regime L and Regime HL arise at some belief b in period t.

Proof. We first show that when $r \leq 1$, Regime $\emptyset$ arises for a sufficiently small μ. Proposition 4's (a) indicates that if Regime $\emptyset$ does not arise, then an honest strategy is not optimal for at least one type. For notational simplicity, we write

$$\bar{A} := A(b, 1, 1) \quad \text{and} \quad \bar{B} := B(b, 1, 1).$$

Aiming to obtain a contradiction, by $(\mathbf{DL})$ suppose that there exists an arbitrarily small ϵ_A and ϵ_B for which the following holds:

$$\left(\mu\lfloor\tfrac{1}{\mu^r}\rfloor-\mu(t-1)\right)\left(\bar{A}-\bar{B}\right)+\epsilon_A^L\bar{A}-\epsilon_B^L\bar{B} \;\;>\;\; \left(\bar{A}+\bar{B}\right). \tag{13}$$

As $\mu\lfloor\frac{1}{\mu^r}\rfloor-\mu(t-1)\leq\mu\lfloor\frac{1}{\mu^r}\rfloor$,

$$\mu\lfloor\tfrac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right)+\epsilon_A^L\bar{A}-\epsilon_B^L\bar{B} \;\;>\;\; \left(\bar{A}+\bar{B}\right). \tag{14}$$

When $r\leq 1$, (14) does not hold as the left-hand side (LHS) is arbitrarily close to 0 by Proposition 5 and the right-hand side (RHS) is strictly greater than 0 for $b\in(0,1)$. This is a contradiction. By symmetry, we can also prove that $D_H(b,1,1)\geq 0$, and (a) in Proposition 4 completes the first claim.

Second, let $r\in(1,2)$. Then, let $b=\mu$. As μ is sufficiently small, similarly to (14), Regime L arises if, for an arbitrarily small $\epsilon_{\bar{A}}$, $\epsilon_{\bar{B}}$, ϵ_A^L and ϵ_B^L,

$$\mu\lfloor\frac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right)+\epsilon_A^L\mu(1+\epsilon_{\bar{A}})-\epsilon_B^L\mu(1+\epsilon_{\bar{B}})>\mu\cdot\left(2+\epsilon_{\bar{A}}+\epsilon_{\bar{B}}\right),$$

which indicates

$$\lfloor\frac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right)>2+\epsilon_{\bar{A}}+\epsilon_{\bar{B}}-\epsilon_A^L(1+\epsilon_{\bar{A}})+\epsilon_B^L(1+\epsilon_{\bar{B}}). \tag{15}$$

By Proposition 5, for $r>1$, the LHS is sufficiently large and the RHS is sufficiently close to 2. Therefore, the above holds. By the same argument, we can see that Regime H does not arise at $b=\mu$ by (c) of Proposition 4 because Proposition 5 indicates that $\mu\lfloor\frac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right)$ is sufficiently close to 0 for $r>1$ and so

$$\mu\lfloor\frac{1}{\mu^r}\rfloor\left(\bar{A}-\bar{B}\right)+\epsilon_B^H\mu(1+\epsilon_{\bar{B}})-\epsilon_A^H\mu(1+\epsilon_{\bar{A}})<2-\mu\cdot\left(2+\epsilon_{\bar{A}}+\epsilon_{\bar{B}}\right).$$

On the other hand, symmetrically we can prove that at $b=1-\mu$, Regime H arises and Regime L does not arise.

Third, let $r<2$. Seeking a contradiction, suppose that in period t, there exist $\bar{\alpha}$ and $\bar{\beta}$ satisfying the two indifference conditions. Then, by substituting $(\mathbf{DH})$ and $(\mathbf{DL})$ into the indifference conditions, for sufficiently small ε_L's and ε_H's,

$$\begin{aligned}\left(\mu\lfloor\tfrac{1}{\mu^r}\rfloor-\mu(t-1)\right)(\bar{\alpha}-\bar{\beta})+\varepsilon_L^{\bar{\alpha}}-\varepsilon_L^{\bar{\beta}} &= \bar{\alpha}+\bar{\beta};\\ -\left(\mu\lfloor\tfrac{1}{\mu^r}\rfloor-\mu(t-1)\right)(\bar{\alpha}-\bar{\beta})+\varepsilon_H^{\bar{\alpha}}-\varepsilon_H^{\bar{\beta}} &= \bar{\alpha}+\bar{\beta}-2.\end{aligned} \tag{16}$$

Then, notice that we must have $\bar{\alpha}+\bar{\beta}\approx 1$, because for a sufficiently small μ, combining the two in (16) yields $0\approx 2(\bar{\alpha}+\bar{\beta})-2$. Thus, we obtain:

$$\left(\mu\lfloor\frac{1}{\mu^r}\rfloor-\mu(t-1)\right)(\bar{\alpha}-\bar{\beta})\approx 1. \tag{17}$$

Note that as $t \in \{1, \cdots, \lfloor \frac{1}{\mu^r} \rfloor - 1\}$,

$$\mu \lfloor \frac{1}{\mu^r} \rfloor (\bar{\alpha} - \bar{\beta}) > \left(\mu \lfloor \frac{1}{\mu^r} \rfloor - \mu(t-1) \right) (\bar{\alpha} - \bar{\beta}) \geq (\bar{\alpha} - \bar{\beta}). \tag{18}$$

By applying the squeeze theorem to (18) and Proposition 5, the LHS of (17) is sufficiently close to 0, which contradicts (17).

Finally, it suffices to show that when $r > 2$, Regime H and Regime L simultaneously arise at some belief b, because by Lemma 7, Regime HL also arises. First, we show that $D_H(b, 1, 1) < 0$ and $D_L(b, 1, 1) < 0$ hold simultaneously at some belief b. Note that when Regime HL arises, $\bar{\alpha} + \bar{\beta} \approx 1$ holds, as indicated in the proof of the previous claim, and thus $\bar{A} + \bar{B} \approx 1$ holds because for any x and y, $A_\mu(b, x, y)$ and $B_\mu(b, x, y)$ both converge to b.

Property $(\mathbf{DL})$ indicates that:

$$\frac{V_{t+1}(\bar{A}) - V_{t+1}(\bar{B})}{\bar{A} - \bar{B}} \approx \mu \lfloor \frac{1}{\mu^r} \rfloor - \mu(t-1). \tag{19}$$

For a sufficiently small μ, as $\bar{A} + \bar{B} \approx 1$,

$$\frac{\bar{A} + \bar{B}}{\bar{A} - \bar{B}} \approx \frac{1}{\bar{A} - \bar{B}}. \tag{20}$$

Proposition 5 implies that $\mu \lfloor \frac{1}{\mu^r} \rfloor \cdot (\bar{A} - \bar{B})$ is sufficiently large and must be larger than 1. Thus, comparing (19) and (20) yields:

$$\frac{V_{t+1}(\bar{A}) - V_{t+1}(\bar{B})}{\bar{A} - \bar{B}} > \frac{\bar{A} + \bar{B}}{\bar{A} - \bar{B}}. \tag{21}$$

Therefore, we can conclude that $D_L(b, 1, 1) < 0$ holds, and by symmetry we can also prove that $D_H(b, 1, 1) < 0$ at the same time. Then, by (3) and Lemma 5 there exists a y_L such that $D_L(b, 1, y^L) = 0$. Let $\bar{\alpha}_0 = A(b, 1, y^L)$ and $\bar{\beta}_0 = B(b, 1, y^L)$. Then, we consider the type-H trader. As $\bar{\alpha}_0$ and $\bar{\beta}_0$ are sufficiently close to b, and $(\bar{\alpha}_0 - \bar{\beta}_0)\mu(T - t + 1)$ is sufficiently large due to Proposition 5,

$$\bar{\beta}_0 - 1 + \bar{\beta}_0 \mu(T - t + 1) < 1 - \bar{\alpha}_0 + \bar{\alpha}_0 \mu(T - t + 1), \tag{22}$$

and for any sufficiently small ε_H's,

$$\bar{\beta}_0 - 1 + \bar{\beta}_0 \mu(T - t + 1) + \varepsilon_H^\beta < 1 - \bar{\alpha}_0 + \bar{\alpha}_0 \mu(T - t + 1) + \varepsilon_H^\alpha. \tag{23}$$

Given that (23) and $(\mathbf{DH})$ indicate that $D_H(b, 1, y_L) > 0$, Regime L arises. Symmetrically, we can prove that Regime H arises. By Lemma 7, Regime HL also arises. Therefore, these three different regimes coexist at belief b and time t. This completes the proof. □

One implication of Proposition 6 is straightforward. When the number of trading periods grows more rapidly than the informed trading probability, Regime HL arises. On the other hand, if there are not enough trading periods, manipulation itself does not arise.

Proposition 6 also indicates that when μ is sufficiently small, Regime L arises around $b = \mu$, which is also sufficiently small. Proposition 10 in Subsection 3.4 computes the slopes of the value functions at $b = 0$ and $b = 1$, which also shows how the slopes at the edges grow over time. Manipulation arises when the market maker is almost correct or very wrong. For a sufficiently small μ, Proposition 6 implies that manipulation arises when the market maker is almost correct. The important factor is $\bar{A} + \bar{B}$ or $2 - (\bar{A} + \bar{B})$. This can be thought of as the difference of costs that the type-L or the type-H trader has to incur in order to manipulate. When the value functions are almost linear, the effect of manipulation, which can be measured by the slopes of the value functions, is almost constant everywhere in the region. When this constant effect is not so large, the informed trader would only manipulate when the cost of manipulating is small. As such, Regime HL does not arise as the two regions where the cost of manipulation is small for each type do not overlap.

To prove the second theorem, we use Proposition 4 recursively by applying backwards induction. An intuition behind the second theorem is that when $r \leq 1$, the equilibrium is still unique but there is no manipulation as there are too few trading rounds for manipulation to arise in equilibrium. Conversely, when $r > 2$, there would be too many chances to trade and Regime HL may arise. Indeed, μ^r for $r \in (1, 2)$ is the interval of trading periods for which the equilibrium is unique and Regime HL does not arise.

Proof of Theorem 2. To prove (a), by Proposition 6, in equilibrium manipulation does not arise when $r \leq 1$ for a sufficiently small μ. Take a supremum of such a μ and call it μ_0. Similarly, we can prove (b) and (c). $\square$

One may wonder if the result holds for $r = 2$. When $r = 2$, following the proof of Proposition 5, we obtain:

$$\frac{(\alpha_{t,\mu}(b) - \beta_{t,\mu}(b))}{\mu^{r-1}} \approx b(1-b)(\sigma_{H,\mu} - 1 + \sigma_{L,\mu}) \left(\frac{1}{\gamma} + \frac{1}{1-\gamma} \right).$$

By substituting the above into (18), we can see that whether a pair of bid and ask prices to satisfy (17) exists depends on γ.

3.3 The two-period model

As mentioned earlier, we consider the simplest case of the model where $T = 2$. In this case, we can show that the conditions for Theorem 1 hold and an equilibrium exists uniquely. To see this, note that in the last period $t = 2$, there is no chance to re-trade for the informed traders. Therefore, there is no manipulation and only Regime $\emptyset$ arises. Thus, in period $t = 1$, the future value functions that each type of informed traders face are $V_2(b) = B(b, 1, 1)$ and $J_2(b) = 1 - A(b, 1, 1)$. By using Bayes' rule, we can show that the conditions for Theorem 1 are met, because the bid price is strictly convex and the ask price is strictly concave. First, we show this result and then the informed trader's manipulation rate in period $t = 1$.

Theorem 3. *The equilibrium exists uniquely when $T = 2$. Also, Regime HL does not arise in equilibrium.*

Proof. When $t = 2$, there is no manipulation because there is no period left to recoup their loss. Thus, the informed traders trade honestly in the last period, and the equilibrium strategy in this period is unique. Now let's consider $t = 1$. If Regime HL arises, the following two equations simultaneously hold:

$$\begin{aligned} \beta_1(b) + \mu B(\beta_1(b), 1, 1) &= -\alpha_1(b) + \mu B(\alpha_1(b), 1, 1) \\ \beta_1(b) - 1 + \mu\left(1 - A(\beta_1(b), 1, 1)\right) &= 1 - \alpha_1(b) + \mu\left(1 - A(\alpha_1(b), 1, 1)\right). \end{aligned} \tag{24}$$

Combining them yields:

$$2 = \mu\left([B(\alpha_1(b), 1, 1) - (1 - A(\alpha_1(b), 1, 1))] - [B(\beta_1(b), 1, 1) - (1 - A(\beta_1(b), 1, 1))]\right). \tag{25}$$

Note that $1 \geq B(\alpha_1(b), 1, 1) - B(\beta_1(b), 1, 1)$ and $1 \geq A(\alpha_1(b), 1, 1) - A(\beta_1(b), 1, 1)$ by Bayes' rule. Therefore, the right-hand-side of (25) is

$$\mu\left([B(\alpha_1(b), 1, 1) - B(\beta_1(b), 1, 1)] + [A(\alpha_1(b), 1, 1) - A(\beta_1(b), 1, 1)] - 2\right) \leq 0.$$

Thus (25) is impossible and there is no $\alpha_1(b)$ and $\beta_1(b)$ to satisfy (25). So, Regime HL is impossible to arise. By Proposition 1, we obtain the desired result. □

Theorem 3 allows us to conduct a comparative statics, and we study how the informed trader's manipulation rate changes as the market maker prior belief changes. To keep calculations less messy, define the functions $h(b)$ and $l(b)$ such that for $\sigma \in \mathcal{E}(b)$, $h(b) = (1-\mu)\gamma + \mu\sigma_{H1}(b)$, $l(b) = (1-\mu)\gamma + \mu(1 - \sigma_{L1}(b))$. In addition, define $P(b) = h(b) \times b + l(b) \times (1 - b)$. In words, P is the market maker's expectation that buy order comes.

Proposition 7.

- *Bid price is monotonically increasing.*
- *Ask price is monotonically increasing.*
- $\frac{1}{b(1-b)} > \frac{l'(b)}{l(b)}$ *and* $\frac{1}{b(1-b)} > \frac{-h'(b)}{1-h(b)}$.

By taking the first derivative of prices and (11), we can compute that in Regime L,

$$\alpha'(b) \cdot b = \alpha(b) \cdot \frac{F_l(b)}{P(b)} \text{ and } \beta'(b) \cdot b = \beta(b) \cdot \frac{1 - F_l(b)}{1 - P(b)}. \tag{26}$$

and in Regime H,

$$\alpha'(b) \cdot b = \alpha(b) \cdot \frac{\bar{l}}{h(b)} \cdot \frac{F_h(b)}{P(b)} \text{ and } \beta'(b) \cdot b = \beta(b) \cdot \frac{1 - \bar{l}}{1 - h(b)} \cdot \frac{1 - F_h(b)}{1 - P(b)}. \tag{27}$$

By symmetry, consider Regime L. Then the following holds

$$\beta(b) + \mu B(\beta(b), 1, 1) = -\alpha(b) + \mu B(\alpha(b), 1, 1). \tag{28}$$

Then

$$\beta'(b)(1 + \mu B'(\beta(b), 1, 1)) = \alpha'(b)(-1 + \mu B'(\alpha(b), 1, 1)). \tag{29}$$

By Proposition 3, $-1 + \mu B'(\alpha(b), 1, 1) > 0$. Then $\beta'(b)$ and $\alpha'(b)$ have the same sign. As we saw in the proof of Theorem 1, $\alpha'(b) \leq 0$ yields a contradiction because by (26), $F_l(b) \leq 0$ and then $\beta_t'(b) > 0$, which indicate that $\beta'(b)$ and $\alpha'(b)$ have the different signs. In this way, we can obtain the first and second statements of 7. Formally, Proposition 7 is a direct consequence of the following lemma. We keep the proof of this lemma in the second appendix (proofs) as it involves some lengthy calculations.

Lemma 8.

- *Let* $F_l(b) = l(b) - l'(b)b(1-b)$. *In Regime L, we have* $F_l(b) < P(b)$ *and* $P'(b) > 0$.
- *Let* $F_h(b) = h(b) + h'(b)b(1-b)$. *In Regime H, we have* $F_h(b) > P(b)$ *and* $P'(b) > 0$.

Proof of Proposition 7. By Theorem 3, the equilibrium strategy is unique and differentiable except for the set of points where regimes change as the market maker's belief changes. Therefore, the informed trading rate for each type is bounded except for a set of countable points where regimes change with respect to the market maker's belief. However, by Theorem 3, Regime HL does not arise and in Regime $\emptyset$, the change of the informed trader's manipulation rate with respect to the changes in the market maker's belief is zero. Therefore, the informed trading rate for each type is bounded in the interval $(0, 1)$.

By Lemma 8 and (26) or (27), the monotonicity of prices are direct. By (11), we obtain $1 - F_l(b) > 0$ and $F_l(b) > 0$. Therefore, $1 > F_l(b) = l(b) - l'(b)b(1-b) > 0$. By modifications, we obtain $\frac{1}{b(1-b)} > \frac{l'(b)}{l(b)}$. The last claim $\frac{1}{b(1-b)} > \frac{-h'(b)}{1-h(b)}$ is proved symmetrically. □

Consider a monotone function f defined $[0, 1]$. We define the following term:

Definition 4. *A monotone function f is* arc-convex with respect to $\bar{x}$ *if* $\frac{f(x)-f(\bar{x})}{x-\bar{x}}$ *is increasing. A monotone function f is* arc-concave with respect to $\bar{x}$ *if* $\frac{f(x)-f(\bar{x})}{x-\bar{x}}$ *is decreasing.*

Proposition 8.

- *In Regime $\emptyset$, ask price $\alpha(b)$ is strictly concave and bid price $\beta(b)$ is strictly convex.*
- *In Regime L, ask price $\alpha(b)$ is arc-concave with respect to 0 and bid price $\beta(b)$ is arc-convex with respect to 0.*
- *In Regime H, ask price $\alpha(b)$ is arc-concave with respect to 1 and bid price $\beta(b)$ is arc-convex with respect to 1.*

Note that strictly convex functions are arc-convex and strictly concave functions are arc-concave. When nobody manipulates in equilibrium (that is, in regime $\emptyset$), it is easy to verify Proposition 8 by taking the second derivatives of the two functions. Now we prove the result for the other two regimes, as by Theorem 3, Regime HL does not arise.

Proof of Proposition 8. By Lemma 8, for $m > 1$,

$$\frac{\alpha(b)}{b} = \frac{\bar{h}}{P(b)} > \frac{\bar{h}}{P(mb)} = \frac{\alpha(mb)}{mb} \text{ and } \frac{\beta(b)}{b} = \frac{1-\bar{h}}{1-P(b)} < \frac{1-\bar{h}}{1-P(mb)} = \frac{\beta(mb)}{mb}. \tag{30}$$

Therefore we obtain the desired result. Similarly we can obtain the required result, because

$$\frac{1-\alpha(b)}{1-b} = \frac{\bar{l}}{P(b)} > \frac{\bar{l}}{P(mb)} = 1-\frac{\alpha(mb)}{1-mb} \text{ and } \frac{1-\beta(b)}{1-b} = \frac{1-\bar{l}}{1-P(b)} < \frac{1-\bar{l}}{1-P(mb)} = \frac{1-\beta(mb)}{1-mb}.$$

□

In general, the value functions are not strictly convex. As we show in the next section, the slopes of the value functions geometrically increase at the edges of market makers' prior 0 or 1. This increment is large compared to the one in the middle, which prevents the value functions from being strictly convex. However, Proposition 8 shows that bid and ask prices satisfy a certain type of convexity and concavity, as we defined as arc-convex and arc-concave. Roughly speaking, by Proposition 3, we know that when manipulations arise, the value functions are steep. At the edges where manipulation could arise, Proposition 8 shows that bid and ask prices satisfy arc-convexity and arc-concavity, and when manipulations do not happen in the middle of the interval for the market maker's belief, $[0, 1]$, bid and ask prices are simply strictly convex and concave. This indicates that the bid-ask spreads are the largest in the middle of the interval. In the model, the market maker imposes bid-ask spreads to hedge a risk of trading with the informed trader. In other words, the bid-ask spreads exist due to the asymmetric information between the market maker and informed traders. If the market maker is most uncertain about the risky payoffs, the spreads should be largest. Proposition 8 proves this intuition.

The last notion to be examined in our equilibrium analysis is the price impact. Price impact measures the absolute impact of trade on the risky asset price. This is the direct consequence of Proposition 8.

Proposition 9.

- *In Regime L, the following holds:*
  - *price impact of buy order* $\frac{\alpha(b)}{b}$ *is a decreasing function of belief.*
  - *price impact of sell order* $\frac{\beta(b)}{b}$ *is an increasing function of belief.*
- *In Regime H, the following holds:*
  - *price impact of buy order* $\frac{1-\alpha(b)}{1-b}$ *is an increasing function of belief.*
  - *price impact of sell order* $\frac{1-\beta(b)}{1-b}$ *is a decreasing function of belief.*

3.4 The Slopes of the Value Functions at the Edges

As we see from the analysis in the previous section, analytically solving for an equilibrium strategy in the general T period model is not tractable. Having said that, we can still compute the slopes at the edges for the general T-period model. This result is useful when we interpret our simulation result, particularly in understanding how the informed trader's payoffs evolves. The next proposition shows that even when the market maker knows the true state, the slopes of the value functions geometrically increase. As we see in our computer simulation in the fifth section, the type-H or the type-L trader starts to manipulate as the number of remaining trading periods increases, because the informed trader manipulates when the slopes of the value functions are steep. We keep the proof in the second appendix.

Proposition 10. *For all* $t \leq T$,

$$V_t'(0) = \frac{(T-t+1)\mu(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)} \text{ and } J_t'(1) = \frac{-(T-t+1)\mu(1-\mu)\gamma}{\mu+(1-\mu)\gamma}.$$

4 The Computation Procedure and Simulation Results

In this section, we lay out the procedure to compute the equilibrium in our model. The following summarizes our procedure.

1. We start with the terminal period T. In period T, nobody manipulates and thus we can compute the period-T value functions by Bayes' rule.

2. Given J_T and V_T, we find whether a set of prices to satisfy Bayes' rule as well as an indifference condition for each type. In other words, we check which regime stands in each belief.

3. We find a pair of equilibrium prices for each belief.

4. We compute the period-$T-1$ value functions. Then, we repeat the same procedure from Step 2.

In the appendix, we demonstrate the more detailed computation procedure including equations that in each regime, the equilibrium strategy has to satisfy in the simulation.

4.1 Simulation Results

This section explains our calibration results for the equilibrium. The Glosten–Milgrom framework in Back and Baruch (2004) is a continuous-time stationary case and their program attempts to find the value functions as a fixed point. To do this, they use an extrapolation method that requires calculating the slopes of the value functions. Because of this problem, Back and Baruch (2004) wrote that even though all the equilibrium conditions hold with a high degree of accuracy, the strategies were not estimated very accurately when manipulation arises. To avoid this problem, we use a linear interpolation method.

We approximate a continuous value function by linear segments and then solve the equilibrium. Given that no type of trader manipulates in the last period of the game, we can calculate the value functions in the last period along with the bid and ask prices. We then split the whole interval $[0, 1]$ into n segments and linearly interpolate the value function for each type of trader in each interval. We then attempt to find whether a pair of ask and bid prices exists such that each type of informed trader becomes indifferent between buy and sell orders or both types become indifferent between the two orders in each interval of the market maker's belief. Using the bid and ask prices we obtain using this procedure, we calculate the current-period value functions and repeat the procedure in the following periods.

Characteristics of Equilibrium

We first consider $\gamma = \frac{1}{2}$, so that the equilibrium is symmetric. Figure 1 exhibits the equilibrium bid and ask prices with respect to the market maker's prior belief for the periods from 201 to 400. The solid curves that present the highest and lowest points represent the ask and bid prices for the case where there is no manipulation. In the bid and ask price figures, there is a region of beliefs in which the bid or ask prices differ between the periods. It is in this region of beliefs that manipulation arises in equilibrium. As the informed trader's strategy differs between periods because the manipulation rate is time dependent, the bid and ask prices also differ between periods. The thick curves in the middle are indeed a stack of 200 lines, and as manipulation arises, these curves do not coincide with the single line for the no-manipulation case. Although it is obvious from Bayes' rule, from this figure we can also see that manipulation indeed decreases the bid–ask spread in a given interval of prior beliefs.

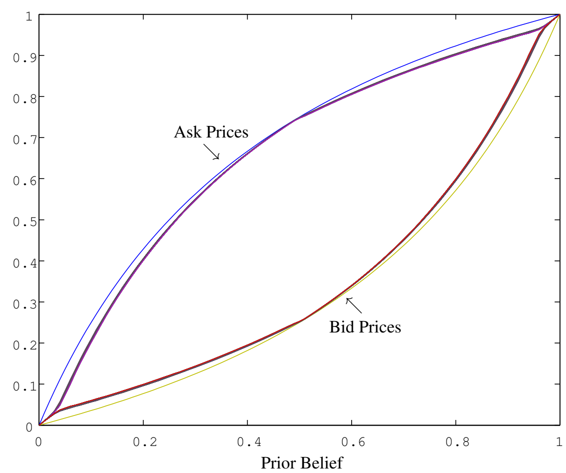

Figure 1: Bid and Ask Prices when $\mu = 0.5, \gamma = 0.5, t \in \{201, 400\}$

As shown in Calcagno and Lovo (2006), along with some empirical and experimental evidence (see Koski and Michaely (2000), Krinsky and Lee (1996) and Venkatesh and Chiang (1986)), our simulation also shows that the bid–ask spread is largest in the last period. In our analysis, this is a direct consequence of the fact that there is no manipulation in the last period. In the Calcagno and Lovo (2006) analysis, we observe this because the winner's curse increases when the terminal period comes near. Although our mechanisms differ, we observe a similar result here as well.

In equilibrium, the type-H trader does not sell with probability one and the type-L trader does not buy with probability one. This means that the informed trader either trades on his information or assigns a positive probability to both buy and sell orders. In the latter case, the informed trader is indifferent between buy and sell orders. This motivates consideration of the slopes of the value functions.

Manipulation

The results of the simulation also show that the type-H trader manipulates in a region of beliefs close to 0 and the type-L trader manipulates in a region of beliefs close to 1. This result is somewhat counterintuitive because, for example, if the type-H trader manipulates in a region of beliefs close to 0, the bid price will be very low and the trader can only obtain a little money. However, to affect the future payoffs through the updating of the market maker's beliefs, they will manipulate when the bid–ask spread is small and the slope of the next-period value function is steep.

To take a more careful look at a manipulative strategy, Figure 3 shows the equilibrium strategy for the type-H and the type-L trader in the $[0, 1]$ interval of prior beliefs. Interestingly, we can see that when the prior belief is close to 1, the type-H trader also manipulates; similarly, the type-L trader manipulates when the belief is close to 0. Indeed, this result is consistent with our calculation in Proposition 10. When the belief is very close to 0 or 1, the market maker almost knows the value of the asset. As shown in Proposition 10, the slopes of the value functions geometrically increase. As a result, the type-H or the type-L trader starts to manipulate as the number of remaining trading periods increases.

Table 1 describes how manipulation starts to arise. Manipulation starts to arise when the market maker is almost correct or very wrong. In a sense, there are two types of manipulation. Manipulation for $r \in (1, 2)$ corresponds to that which arises when the market maker is almost correct. Our simulation shows that as the slope becomes steeper, manipulation arises only when the market maker is very wrong. In other words, the other type of manipulation disappears and the remaining type of

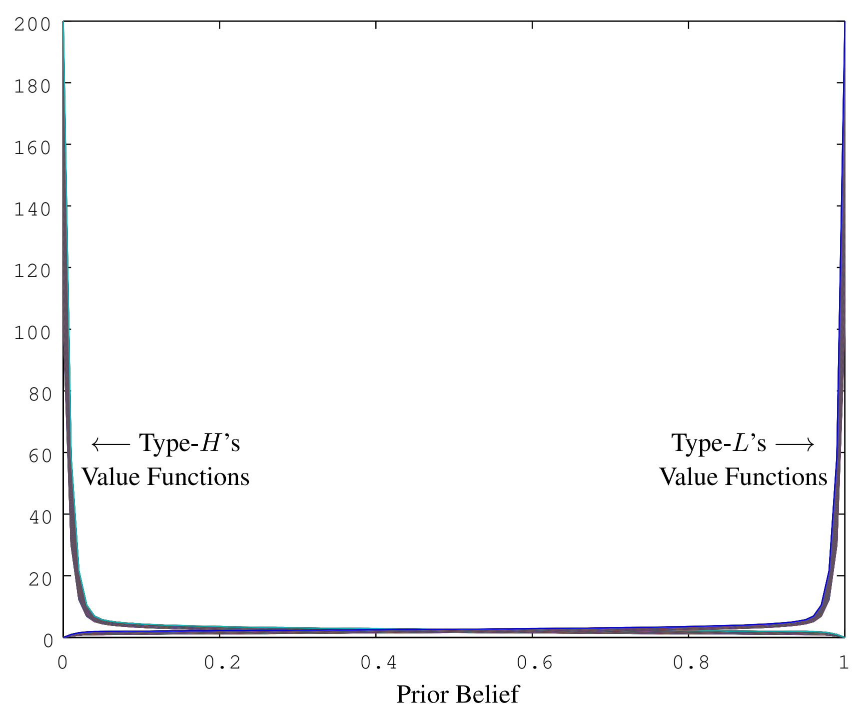

Figure 2: Bid and Ask Prices when $\mu = 0.5, \gamma = 0.5, t \in \{201, 400\}$

manipulation expands. This conveys the intuition that the slopes of the value functions are indeed an incentive for the informed trader, and as they increase, manipulation begins to take place over a wider range.

	$b = 0.01$	$b = 0.02$	$\cdots$	$b = 0.98$	$b = 0.99$
$t = 11$	L	$\emptyset$	$\cdots$	$\emptyset$	H
$t = 12$	L	H	$\cdots$	L	H
$t = 13$	H	H	$\cdots$	L	L

Table 1: Regimes and Beliefs for $t = 11, 12, 13$

In this way, Regime HL starts to arise in our simulation as each region that each type of trader manipulates overlaps with the other. In our simulation, in period $t = 152$, Regime HL starts to arise at belief 0.5, and it similarly starts to arise at beliefs $b = 0.01$ or 0.99 in period $t = 271$.[9] As the market maker is completely wrong at $b = 0$ or 1, the informed trader does not manipulate because the prices are favourable for the type-H ($b = 0$) or the type-L ($b = 1$) trader. On the other hand, in the region close to $b = 0$ or 1, as the value function is steep, the type-H trader manipulates around $b = 0$ or the type-L trader manipulates around $b = 1$. Therefore, there is a spike in the rate of informed trading near the edges. As we can see from Figure 3, there appears to be a discontinuity in the informed strategy when the market maker is very wrong. The important idea in our theory of a tame equilibrium is to make manipulation arise near the other edge (that is, when the market maker is almost correct) and not near this edge (when the market maker is very wrong).

As we can see in Figure 2, the value functions are not globally convex. Near $b = 0$ or 1, some parts appear to be concave. Thus, manipulation arises as the slope near $b = 0$ or 1 becomes sufficiently steep. This manipulation remains in the unique equilibrium when μ becomes sufficiently small and T is sufficiently large. In other words, when the value functions become sufficiently "linear," the moment when the market maker begins to make a mistake is the only opportunity to manipulate.

Multiple Equilibria

In our simulation results, we also observe multiple equilibria. As we have shown in Theorem 2, a unique equilibrium without manipulation arises in the earlier periods, then a unique equilibrium with manipulation starts to arise. Finally, multiple equilibria arise in the later periods. For example, when $\mu = 0.2$, a unique equilibrium with manipulation starts to arise in period $t = 22$, and then multiple

[9] From Figure 1, it may be difficult to see that at $b = 0.5$, the bid and ask prices in Regime $\emptyset$ differ from the simulated prices for $t = 201, \cdots, 400$, because the manipulation rates at $b = 0.5$ are quite small. Indeed, the bid price without manipulation is 0.25 and the ask price without manipulation is 0.75, while the simulated equilibrium bid prices for these periods range from 0.2511 to 0.2528 and the ask prices for these periods range from 0.7489 to 0.7472.

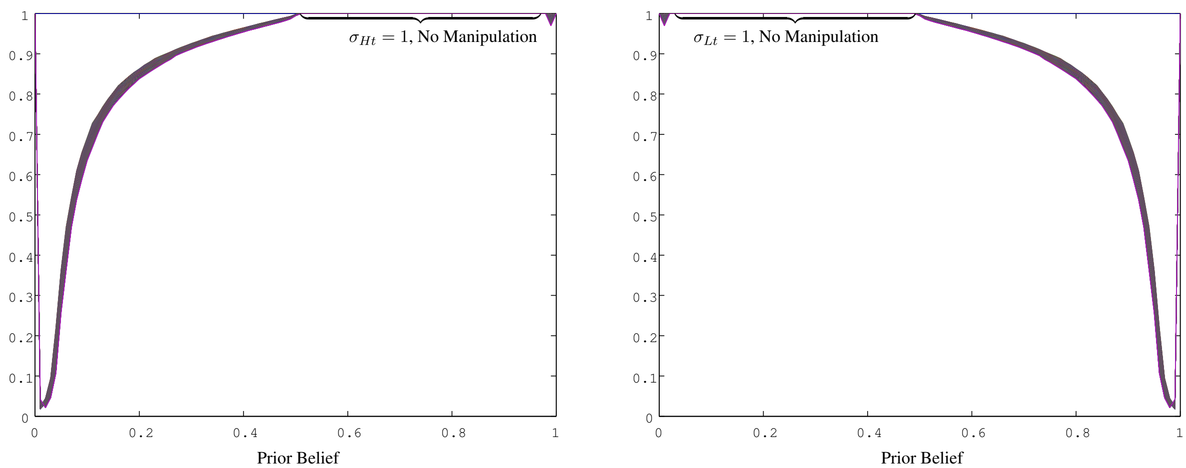

Figure 3: Manipulation Rates σ_{Ht} and σ_{Lt} when $\mu = 0.5, \gamma = 0.5, t \in \{201, 400\}$

equilibria arise in period $t = 52$. In this case, for $r = 1.9$, $\lfloor \frac{1}{\mu^r} \rfloor = 21$ and for $r = 2.5$, $\lfloor \frac{1}{\mu^r} \rfloor = 55$. We also run a similar exercise for the values $\mu = 0.1$ and $\mu = 0.3$. This observation for multiple equilibria is consistent for these different values of μ's.

Effects of an Asymmetric Liquidity Distribution

To this point, we have considered the symmetric case in the sense that the liquidity for a buy is equally likely as the liquidity for a sell. Here we consider how an asymmetric liquidity distribution affects bid and ask prices and who manipulates in equilibrium. The following four figures in Figure 4 and Figure 5 show the bid and ask prices, as well as value functions, when $\gamma = 0.2$ and $\gamma = 0.8$. Comparing Figure 4 and 5, we can see that the type-H trader at belief b is a mirror image of the type-L trader at belief $1 - b$. We also observe that the informed trader manipulates when the slope is steep. As we can predict from Bayes' rule, as γ decreases, the type-H trader's value functions are positioned closer to 0. As such, the region where its value functions are steep becomes smaller and the region in which the type-H trader manipulates become more restricted. The reverse holds for the type-L trader.

5 Concluding Remarks

In this paper, we developed a model of dynamic informed trading from a canonical framework in the market microstructure literature. We make a fundamental contribution to the literature by providing a theorem describing conditions under which a unique equilibrium arises. We also provided a computational method to find equilibria.

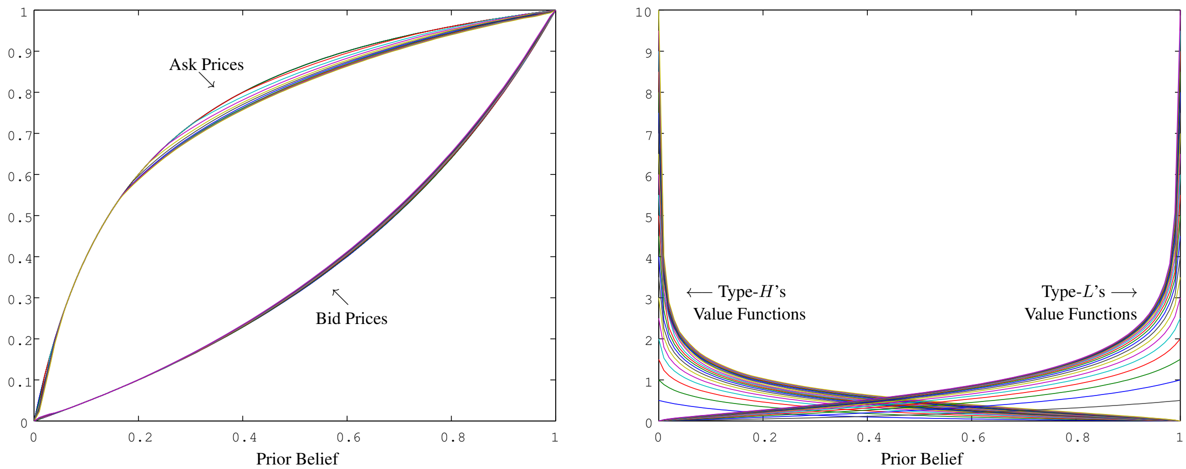

Figure 4: Bid and Ask Prices and Value Functions when $\mu = 0.5, \gamma = 0.2, t \in \{1, 20\}$

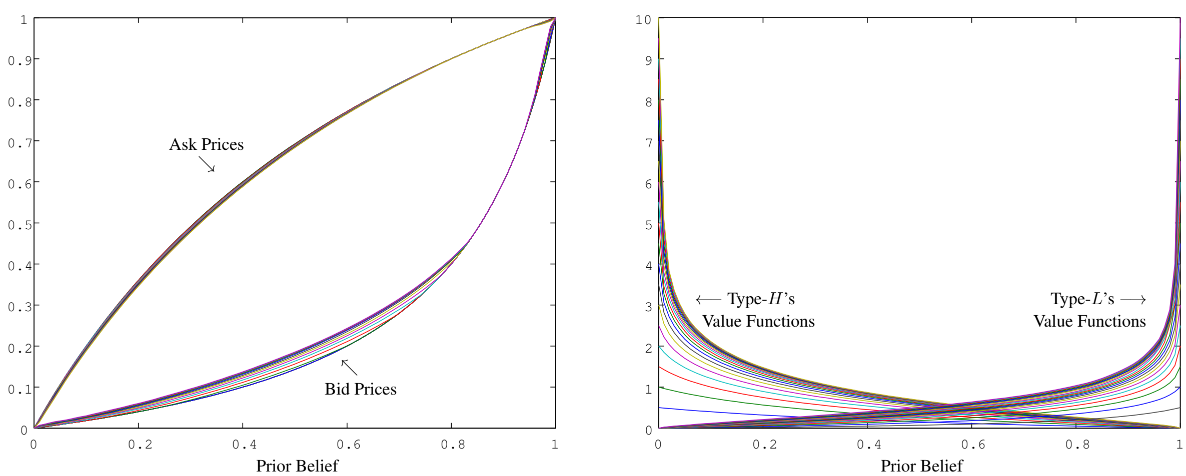

Figure 5: Bid and Ask Prices and Value Functions when $\mu = 0.5, \gamma = 0.8, t \in \{1, 20\}$

From our analysis, several important research questions arise. While our model uses a discrete setting, as the limit of our setting with sufficiently small μ's in the discrete-time setting could converge to a continuous-time setting, our paper can add some insights to the literature on continuous-time models. One fundamental problem in financial econometrics is to accurately identify the stock price dynamics and analyze how a different market structure affects these dynamics. Many more studies address the relationship between prices and dynamic trading in a continuous-time setting.[10] De Meyer (2010) studies an n-times repeated zero-sum game of incomplete information and shows that the asymptotics of the equilibrium price process converge to a Continuous Martingale of Maximal Variation (hereafter CMMV). As De Meyer (2010) points out, this CMMV class could provide natural dynamics that may be useful in financial econometrics, although it remains an open question as to whether the equilibrium dynamics in a non-zero-sum game still belong to the CMMV class.

First, given the association with De Meyer (2010), our paper provides a fundamental framework for a non-zero-sum trading game. Adding a time discount factor to the informed trader's profit to bring our analysis into a continuous-time setting is an obvious extension. Extending our analysis for the case of sufficiently small μ to see how our results are related to those works should be an interesting research agenda.

Second, the existence of a unique equilibrium in the Kyle model remains an open question in the literature. As shown in Back and Baruch (2004), the equilibrium in the Glosten–Milgrom model converges to that in the Kyle model. Our analysis shows that there is a possibility of multiple equilibria. We could use our analysis to understand how a unique equilibrium in the dynamic Glosten–Milgrom model converges to the equilibrium in the Kyle model.

Third, one may question whether the market maker's belief concerning the risky asset's payoff converges to the truth as the number of trading periods tends to infinity. Recently, there has been renewed interest in private information and learning. Examples include Golosov *et al.* (2009) and Loertscher and McLennan (2013). Although their settings are quite different from ours, both question whether uninformed agents learn private information. In the market microstructure literature, Glosten and Milgrom (1985) originally show that such convergence is obtained almost surely if the only available trade size is the unit trade size and an informed trader can trade only once. Ozsoylev and Takayama (2010) show a similar result where the informed trader can trade only once but in multiple sizes. We expect that this result will also hold in our framework, as an intuition similar to the Martingale convergence theorem holds when the equilibrium is unique. However, as there is a

[10]Brunnermeier and Pedersen (2005) consider the dynamic strategic behaviour of large traders and show that "overshooting" occurs in equilibrium, while Back and Baruch (2007) analyze different market systems by allowing the informed traders to trade continuously within the Glosten–Milgrom framework. Lastly, within an extended Kyle framework, Collin-Dufresne and Fos (2012) study insider trading where the liquidity provided by noise traders follows a general stochastic process, and show that even though the level of noise trading volatility is observable, in equilibrium the measured price impact is stochastic.

possibility that multiple equilibria will arise, and especially that both types of trader will manipulate at the same time, it would be interesting to see how this type of manipulation affects the market's learning.

Appendix I: The Detailed Computation Procedure

As we make use of an approximation, we set out a different notation for the purpose of calibration. Bold-faced letters denote approximated variables in our simulation. For example, in the calibration, we denote the probability that the type-H trader buys in the high state at period t by $\mathbf{h}_t$ and the probability that the type-L trader sells in the low state by $\mathbf{l}_t$. Moreover, let

$$\mathbf{H}_t = (1-\mu)\gamma + \mu \mathbf{h}_t \quad \text{and} \quad \mathbf{L}_t = (1-\mu)(1-\gamma) + \mu \mathbf{l}_t.$$

Then, $\mathbf{H}_t$ is the probability that a buy occurs in the high state in period t and $\mathbf{L}_t$ is the probability that a sell occurs in the low state. We can write:

$$\boldsymbol{\alpha}_t = \frac{\mathbf{H}_t b}{\mathbf{H}_t b + (1-\mathbf{L}_t)(1-b)} \quad \text{and} \quad \boldsymbol{\beta}_t = \frac{(1-\mathbf{H}_t)b}{(1-\mathbf{H}_t)b + \mathbf{L}_t(1-b)}. \tag{31}$$

When the type-L trader manipulates, we write the bid price as a function of the ask price and the probability that a buy will occur in the high state. Then, we obtain:

$$\boldsymbol{\beta}_t = \frac{\boldsymbol{\alpha}_t b(1-\mathbf{H}_t)}{(\boldsymbol{\alpha}_t - b\mathbf{H}_t)}. \tag{32}$$

In the computer program, we inspect each interval of b to check whether there is a pair of ask and bid prices that satisfies the following indifference condition for the type-L:

$$-\boldsymbol{\alpha}_t + \mathbf{V}_{t+1}(\boldsymbol{\alpha}_t) = \boldsymbol{\beta}_t + \mathbf{V}_{t+1}(\boldsymbol{\beta}_t), \tag{33}$$

where $\boldsymbol{\beta}_t$ satisfies (32). In our procedure, the new function $\mathbf{V}_{t+1}$ is constructed through a linear interpolation from V_{t+1}, which is: for $\boldsymbol{\alpha}_t \in [b_k, b_{k+1}]$,

$$\mathbf{V}_{t+1}(\boldsymbol{\alpha}_t) = (\boldsymbol{\alpha}_t - b_k)\frac{V_{t+1}(b_{k+1}) - V_{t+1}(b_k)}{(b_{k+1} - b_k)} + V_{t+1}(b_k), \tag{34}$$

and for $\boldsymbol{\beta}_t \in [b_j, b_{j+1}]$,

$$\mathbf{V}_{t+1}(\boldsymbol{\beta}) = (\boldsymbol{\beta}_t - b_j)\frac{V_{t+1}(b_{j+1}) - V_{t+1}(b_j)}{(b_{j+1} - b_j)} + V_{t+1}(b_j). \tag{35}$$

Similarly, when the type-H trader manipulates, we write the ask price as a function of the bid price and the probability that a buy will occur in the low state. First, we solve $\mathbf{H}_t$ as a function of the ask price $\boldsymbol{\alpha}_t$. Then we have:

$$\mathbf{H}_t = \frac{\boldsymbol{\alpha}_t(1-b)(1-\mathbf{L}_t)}{(1-\boldsymbol{\alpha}_t)b}. \tag{36}$$

Then, we substitute $\mathbf{H}$ into the bid price. Then, we obtain:

$$\boldsymbol{\beta}_t = \frac{(b-\boldsymbol{\alpha}_t) + \boldsymbol{\alpha}_t \mathbf{L}_t(1-b)}{(b-\boldsymbol{\alpha}_t) + \mathbf{L}_t(1-b)}. \tag{37}$$

m_k^θ	$:=$	$\frac{F_{t+1}^\theta(b_{k+1})-F_{t+1}^\theta(b_k)}{b_{k+1}-b_k}$
m_j^θ	$:=$	$\frac{F_{t+1}^\theta(b_{j+1})-F_{t+1}^\theta(b_j)}{(b_{j+1}-b_j)}$
A_k^θ	$:=$	$m_k^\theta - 1$
B_j^θ	$:=$	$m_j^\theta + 1$
C^L	$:=$	$(b_j m_j^L - \mathbf{V}_{t+1}(b_j)) - (b_k m_k^L - \mathbf{V}_{t+1}(b_k))$
C^H	$:=$	$(b_j m_j^H - \mathbf{J}_{t+1}(b_j)) - (b_k m_k^H - \mathbf{J}_{t+1}(b_k)) + 2$
$K(\theta)$	$:=$	$-A_k^\theta + B_j^\theta - C^\theta$
$G(\theta, \mathbf{L}_t, b)$	$:=$	$B_j^\theta\left[(1-\mathbf{L}_t)(1-b)-b\right] - A_k^\theta\left[(1-\mathbf{L}_t)(1-b)-1\right] + C^\theta\left[1-2(1-\mathbf{L}_t)(1-b)\right]$
$N(\theta, \mathbf{L}_t, b)$	$:=$	$B_j^\theta b - C^\theta\left[1-(1-\mathbf{L}_t)(1-b)\right]$
T_θ	$:=$	$\mathbf{H}_t b\left(B_j^\theta - A_k^\theta - 2C^\theta\right) + \left(-B_j^\theta b + C^\theta\right)$
M_θ	$:=$	$\mathbf{H}_t b\left(-bB_j^\theta + A_k^\theta + C^\theta\right)$

Table 2: Summary of Abbreviated Notations

* Each θ belongs to $\{H, L\}$ and for each F^θ, $F^H = \mathbf{J}$ and $F^L = \mathbf{V}$.

We inspect each interval of b to check whether there is a pair of ask and bid prices that satisfies the following indifference condition for the type-H:

$$1 - \boldsymbol{\alpha}_t + \mathbf{J}_{t+1}(\boldsymbol{\alpha}_t) = \boldsymbol{\beta}_t - 1 + \mathbf{J}_{t+1}(\boldsymbol{\beta}_t). \tag{38}$$

By applying the method of linear interpolation, we construct a function $\mathbf{J}_{t+1}$ that approximates J_{t+1} such that for $\boldsymbol{\alpha}_t \in [b_k, b_{k+1}]$,

$$\mathbf{J}_{t+1}(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}_t - b_k)\frac{J_{t+1}(b_{k+1}) - J_{t+1}(b_k)}{(b_{k+1} - b_k)} + J_{t+1}(b_k), \tag{39}$$

and for $\boldsymbol{\beta} \in [b_j, b_{j+1}]$,

$$\mathbf{J}_{t+1}(\boldsymbol{\beta}) = (\boldsymbol{\beta}_t - b_j)\frac{J_{t+1}(b_{j+1}) - J_{t+1}(b_j)}{(b_{j+1} - b_j)} + J_{t+1}(b_j). \tag{40}$$

Finally, we obtain the following propositions. To keep the notation simple, we use some abbreviations; these are summarised in Table 2.

Proposition 11. *When the type-L trader manipulates, an equilibrium ask price $\boldsymbol{\alpha}_t$ solves:*

$$\boldsymbol{\alpha}_t^2 A_k^L + \boldsymbol{\alpha}_t\left(-b\mathbf{H}_t A_k^L + b(\mathbf{H}_t - 1)B_j^L + C^L\right) - C^L b\mathbf{H}_t = 0,$$

subject to $\mathbf{H}_t = (1-\mu)\gamma + \mu$ and

$$\mathbf{L}_t = \frac{\boldsymbol{\alpha}_t(1-b) - b(1-\boldsymbol{\alpha}_t)\mathbf{H}_t}{\boldsymbol{\alpha}_t(1-b)} \leq (1-\mu)(1-\gamma) + \mu.$$

Proof of Proposition 11. From (33),

$$-\boldsymbol{\alpha}_t + (\boldsymbol{\alpha}_t - b_k)m_k^L + \mathbf{V}_{t+1}(b_k) = \frac{\boldsymbol{\alpha}_t b(-1+\mathbf{H}_t)}{(-\boldsymbol{\alpha}_t + b\mathbf{H}_t)} + (\frac{\boldsymbol{\alpha}_t b(-1+\mathbf{H}_t)}{(-\boldsymbol{\alpha}_t + b\mathbf{H}_t)} - b_j)m_j^L + \mathbf{V}_{t+1}(b_j).$$

Reorganizing terms, we can obtain the desired equation. □

Proposition 12. *When the type-H trader manipulates, then an equilibrium ask price $\boldsymbol{\alpha}_t$ solves*

$$\boldsymbol{\alpha}_t^2 A_k^H + X\boldsymbol{\alpha}_t + Y = 0,$$

where

$$\begin{aligned} X &= 1 + b + 2\mathbf{L}_t(1-b) - [b_k + b + \mathbf{L}_t(1-b)]\, m_k^H + [b_j - 1 + \mathbf{L}_t(1-b)]\, m_j^H + \\ &\quad + \mathbf{J}_{t+1}(b_k) - \mathbf{J}_{t+1}(b_j); \\ Y &= -\mathbf{L}(1-b) + [b + \mathbf{L}_t(1-b)]\left[b_k m_k^H - 1 + \mathbf{J}_{t+1}(b_j) - \mathbf{J}_{t+1}(b_k)\right] + \\ &\quad + [b(1-b_j) - b_j \mathbf{L}_t(1-b)]\, m_j^H, \end{aligned}$$

subject to $\mathbf{L}_t = (1-\mu)(1-\gamma) + \mu$ and

$$\mathbf{H}_t = \frac{\boldsymbol{\alpha}_t(1-\mathbf{L}_t)(1-b)}{b(1-\boldsymbol{\alpha}_t)} \le (1-\mu)\gamma + \mu.$$

Proof of Proposition 12. From (38),

$$\begin{aligned} &1 - \boldsymbol{\alpha}_t + (\boldsymbol{\alpha}_t - b_k)m_k^H + \mathbf{J}_{t+1}(b_k) = \\ &\qquad = \tfrac{(b-\boldsymbol{\alpha}_t) + \boldsymbol{\alpha}_t \mathbf{L}_t(1-b)}{(b-\boldsymbol{\alpha}_t) + \mathbf{L}_t(1-b)} - 1 + (\tfrac{(b-\boldsymbol{\alpha}_t) + \boldsymbol{\alpha}_t \mathbf{L}_t(1-b)}{(b-\boldsymbol{\alpha}_t) + \mathbf{L}_t(1-b)} - b_j)m_j^H + \mathbf{J}_{t+1}(b_j). \end{aligned}$$

Similarly to Proposition 11, we can obtain the desired equation by calculation. □

Lastly, we consider the case where both types manipulate. To compute the equilibrium in this case, we simultaneously solve the two equations (33) and (38). Then, (33) and (38) can be re-written as: for each $\theta \in \{H, L\}$, Let $\mathbf{x}_H = \mathbf{H}_t b$ and $\mathbf{x}_L = (1-\mathbf{L}_t)(1-b)$. Then, for each $\theta \in \{H, L\}$, the indifference conditions can be rewritten as:

$$\begin{aligned} &\left[-A_k^\theta + B_j^\theta - C^\theta\right]\mathbf{x}_H^2 + \mathbf{x}_H\left(-bB_j^\theta + A_k^\theta + C^\theta\right) + \\ &\qquad + \mathbf{x}_L\left[\mathbf{x}_H\left(B_j^\theta - A_k^\theta - 2C^\theta\right) + \left(-B_j^\theta b + C^\theta\right)\right] - C^\theta \mathbf{x}_L^2 = 0. \end{aligned} \tag{41}$$

Proposition 13. *When both types manipulate, $\mathbf{x}_H = \mathbf{H}_t b$ and $\mathbf{x}_L = (1-\mathbf{L}_t)(1-b)$ satisfy* (41) *for each $\theta \in \{H, L\}$. Moreover, $\mathbf{H}_t$ and $\mathbf{L}_t$ must satisfy:*

$$(1-\mu)\gamma < \mathbf{H}_t < (1-\mu)\gamma + \mu\gamma \quad \textit{and} \quad (1-\mu)(1-\gamma) < \mathbf{L}_t < (1-\mu)(1-\gamma) + \mu. \tag{42}$$

Simultaneously solving for $\mathbf{H}_t$ and $\mathbf{L}_t$ is somehow tricky as the procedure has to find a two-dimensional fixed point. First, we identify a pair of strategies that derives bid and ask prices so as to make both types indifferent for each belief. Given $\mathbf{H}_t$, we can find an interval for $\mathbf{L}_t$ that (38) holds and given $\mathbf{L}_t$, (33) holds. By using this point as an initial point, we use the Newton–Raphson method to obtain the solution to the above equations. We denote the LHS of (41) by $f_\theta(\mathbf{x}_H, \mathbf{x}_L)$. Then, keeping all of the coefficients fixed, we obtain:

$$\begin{aligned}\frac{\mathrm{d}f_\theta}{\mathbf{x}_H} &= \left[-A_k^\theta + B_j^\theta - C^\theta\right] 2\mathbf{x}_H + \left(-bB_j^\theta + A_k^\theta + C^\theta\right) + \mathbf{x}_L\left(B_j^\theta - A_k^\theta - 2C^\theta\right)\\ \frac{\mathrm{d}f_\theta}{\mathbf{x}_L} &= \left[\mathbf{x}_H\left(B_j^\theta - A_k^\theta - 2C^\theta\right) + \left(-B_j^\theta b + C^\theta\right)\right] - 2C^\theta \mathbf{x}_L.\end{aligned}$$

Let $\mathbf{x} = \begin{pmatrix} \mathbf{x}_H \\ \mathbf{x}_L \end{pmatrix}$, $f(\mathbf{x}) = \begin{pmatrix} f_H(\mathbf{x}) \\ f_L(\mathbf{x}) \end{pmatrix}$ and $\mathcal{J} = \begin{pmatrix} \frac{\mathrm{d}f_H}{\mathrm{d}\mathbf{x}_H} & \frac{\mathrm{d}f_H}{\mathrm{d}\mathbf{x}_L} \\ \frac{\mathrm{d}f_L}{\mathrm{d}\mathbf{x}_H} & \frac{\mathrm{d}f_L}{\mathrm{d}\mathbf{x}_L} \end{pmatrix}$. By the Newton–Raphson method,

$$f(\mathbf{x} + \delta\mathbf{x}) = f(\mathbf{x}) + \mathcal{J}\delta\mathbf{x}.$$

Assuming $f(\mathbf{x} + \delta\mathbf{x}) \approx 0$ yields:

$$\delta\mathbf{x} = -\mathcal{J}^{-1} f(\mathbf{x}). \tag{43}$$

We obtain a convergent point $\mathbf{x}^*$ by using (43).

Proof of Proposition 13. Let $\mathbf{H}_t b + (1 - \mathbf{L}_t)(1 - b) = \mathbf{P}$. Then, from (33),

$$\begin{aligned}&(m_k^L - 1)(1 - \mathbf{P})\mathbf{H}_t b - b_k \mathbf{P}(1 - \mathbf{P}) m_k^L + \mathbf{P}(1 - \mathbf{P})\mathbf{V}_{t+1}(b_k)\\ &\qquad = (m_j^L + 1)\mathbf{P}(1 - \mathbf{H}_t) b - \mathbf{P}(1 - \mathbf{P}) b_j m_j^L + \mathbf{P}(1 - \mathbf{P})\mathbf{V}_{t+1}(b_j).\end{aligned}$$

Then,

$$\left(A_k^L \mathbf{H} - B_j^L \mathbf{P}\right) b - \mathbf{P}\mathbf{H}_t b (A_k^L - B_j^L) + \mathbf{P}(1 - \mathbf{P}) C^L = 0.$$

Reorganizing terms, resubstituting $\mathbf{P} = [\mathbf{H}_t b + (1 - \mathbf{L}_t)(1 - b)]$, and again reorganizing terms, we obtain:

$$\begin{aligned}&\left[-A_k^L + B_j^L - C^L\right] \mathbf{H}_t^2 b^2\\ &+\mathbf{H}_t b\left[B_j^L\left[(1 - \mathbf{L}_t)(1 - b) - b\right] - A_k^L\left[(1 - \mathbf{L}_t)(1 - b) - 1\right] + C^L\left[1 - 2(1 - \mathbf{L}_t)(1 - b)\right]\right]\\ &-(1 - \mathbf{L}_t)(1 - b)\left[B_j^L b - C^L\left[1 - (1 - \mathbf{L}_t)(1 - b)\right]\right] = 0.\end{aligned} \tag{44}$$

By symmetry, we obtain:

$$\begin{aligned}&\left[-A_k^H + B_j^H - C^H\right] \mathbf{H}_t^2 b^2\\ &+\mathbf{H}_t b\left[B_j^H\left[(1 - \mathbf{L}_t)(1 - b) - b\right] - A_k^H\left[(1 - \mathbf{L}_t)(1 - b) - 1\right] + C^H\left[1 - 2(1 - \mathbf{L}_t)(1 - b)\right]\right]\\ &-(1 - \mathbf{L}_t)(1 - b)\left[B_j^H b - C^H\left[1 - (1 - \mathbf{L}_t)(1 - b)\right]\right] = 0.\end{aligned} \tag{45}$$

□

Appendix II: Proofs

Proof or Lemma 8. Suppose that the low type manipulates. Then we have:

$$\alpha(b)+\beta(b)=B(\alpha(b),1,1)-B(\beta(b),1,1).$$

Note that $\alpha(b)-\beta(b)$ is non-zero. By dividing both sides by $\alpha(b)-\beta(b)$, we obtain:

$$\frac{\alpha(b)+\beta(b)}{\alpha(b)-\beta(b)}=\frac{B(\alpha(b),1,1)-B(\beta(b),1,1)}{\alpha(b)-\beta(b)}. \tag{46}$$

Since α and β are increasing and V_L is strictly convex, we must have:

$$\frac{\mathrm{d}\frac{\alpha(b)+\beta(b)}{\alpha(b)-\beta(b)}}{\mathrm{d}b}=2\cdot\frac{\alpha(b)\beta'(b)-\alpha'(b)\beta(b)}{(\alpha(b)-\beta(b))^2}>0. \tag{47}$$

Therefore we conclude:

$$\frac{\beta'(b)}{\beta(b)}>\frac{\alpha'(b)}{\alpha(b)}. \tag{48}$$

By (26) and (48), we obtain $\frac{1-F_l(b)}{1-P(b)}>\frac{F_l(b)}{P(b)}$. Since $F_l(b)>0$ and $P(b)>0$, we obtain: $F_l(b)<P(b)$. We can rewrite this as:

$$l(b)-l'(b)b(1-b)<\bar{h}\cdot b+l(b)\cdot(1-b). \tag{49}$$

Thus we conclude: $0<\bar{h}\cdot b-l(b)\cdot b+l'(b)b(1-b)=P'(b)\cdot b$.

The second statement can be proved symmetrically with the first statement. Suppose that the type-H manipulates. Similarly to (46), we obtain:

$$\frac{\alpha(b)+\beta(b)-2}{\alpha(b)-\beta(b)}=\frac{A(\alpha(b),1,1)-A(\beta(b),1,1)}{\alpha(b)-\beta(b)}. \tag{50}$$

Since α and β are increasing and the last period value function is strictly convex, we must have

$$\frac{\mathrm{d}\frac{\alpha(b)+\beta(b)-2}{\alpha(b)-\beta(b)}}{\mathrm{d}b}=2\cdot\frac{\alpha(b)\beta'(b)-\alpha'(b)\beta(b)+(\alpha'(b)-\beta'(b))}{(\alpha(b)-\beta(b))^2}>0. \tag{51}$$

Then, we conclude

$$\frac{\beta'(b)}{1-\beta(b)}<\frac{\alpha'(b)}{1-\alpha(b)}. \tag{52}$$

By (27) and (52) we obtain

$$\frac{\beta(b)}{\alpha(b)}\cdot\frac{1-\alpha(b)}{1-\beta(b)}\cdot\frac{1-\bar{l}}{\bar{l}}\cdot\frac{h(b)}{1-h(b)}\leq\frac{1-P(b)}{1-F_h(b)}\cdot\frac{F_h(b)}{P(b)}. \tag{53}$$

By the Bayes rule, we have: $\frac{\beta(b)}{\alpha(b)} = \frac{1-h(b)}{h(b)} \cdot \frac{P(b)}{1-P(b)}$ and $\frac{1-\alpha(b)}{1-\beta(b)} = \frac{\frac{\bar{l}\cdot(1-b)}{P(b)}}{\frac{(1-\bar{l})(1-b)}{1-P(b)}}$. Thus we obtain:

$$\begin{aligned}\frac{\beta(b)}{\alpha(b)} \cdot \frac{1-\alpha(b)}{1-\beta(b)} \cdot \frac{1-\bar{l}}{\bar{l}} \cdot \frac{h(b)}{1-h(b)} &= \frac{1-h(b)}{h(b)} \cdot \frac{P(b)}{1-P(b)} \cdot \frac{\frac{\bar{l}\cdot(1-b)}{P(b)}}{\frac{(1-\bar{l})(1-b)}{1-P(b)}} \cdot \frac{1-\bar{l}}{\bar{l}} \cdot \frac{h(b)}{1-h(b)} \\ &> \frac{1-h(b)}{h(b)} \cdot \frac{P(b)}{1-P(b)} \cdot \frac{\bar{l}}{1-\bar{l}} \cdot \frac{1-P(b)}{P(b)} \cdot \frac{1-\bar{l}}{\bar{l}} \cdot \frac{h(b)}{1-h(b)} = 1.\end{aligned}$$

Hence we obtain $\frac{1-P(b)}{1-F_h(b)} \cdot \frac{F_h(b)}{P(b)} > 1$. Finally we obtain: $F_h(b) > P(b)$. We can rewrite this as

$$h(b) + h'(b)b(1-b) > h(b) \cdot b + \bar{l} \cdot (1-b). \tag{54}$$

Thus we conclude: $0 < h(b) \cdot (1-b) - \bar{l} \cdot (1-b) + h'(b)b(1-b) = P'(b) \cdot (1-b)$. □

Proof of Proposition 10. When b is equal to zero there will be no manipulation. Consequently, Bayes' rule gives

$$\alpha_t(b) = \frac{b(\mu + (1-\mu)\gamma)}{b\mu + (1-\mu)\gamma} \quad \text{and} \quad \beta_t(b) = \frac{b(1-\mu)(1-\gamma)}{\mu(1-b) + (1-\mu)(1-\gamma)}.$$

The derivatives of these functions at zero are

$$\alpha_t'(0) = \frac{\mu + (1-\mu)\gamma}{(1-\mu)\gamma} \quad \text{and} \quad \beta_t'(0) = \frac{(1-\mu)(1-\gamma)}{\mu + (1-\mu)(1-\gamma)}.$$

We compute that

$$\begin{aligned} V_t'(0) &= \mu(\beta_t'(0) + V_{t+1}'(0)\beta_t'(0)) + (1-\mu)(\gamma V_{t+1}'(0)\alpha_t'(0) + (1-\gamma)V_{t+1}'(0)\beta'(0)) \\ &= \mu\beta'(0) + V_{t+1}'(0)[(1-\mu)\gamma\alpha'(0) + ((1-\mu)(1-\gamma)+\mu)\beta'(0)] \\ &= \mu\tfrac{(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)} + V_{t+1}'(0)[(\mu + (1-\mu)\gamma) + (1-\mu)(1-\gamma)] \\ &= \tfrac{\mu(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)} + V_{t+1}'(0) = \tfrac{(T-t+1)\mu(1-\mu)(1-\gamma)}{\mu+(1-\mu)(1-\gamma)}, \end{aligned} \tag{55}$$

where the last equality is by induction.

Similarly, when b is very close to one, there will be no manipulation. Therefore, Bayes' rule gives

$$\alpha_t(b) = \frac{b(\mu + (1-\mu)\gamma)}{b\mu + (1-\mu)\gamma} \quad \text{and} \quad \beta_t(b) = \frac{b(1-\mu)(1-\gamma)}{\mu(1-b) + (1-\mu)(1-\gamma)}.$$

The derivatives of these functions at zero are

$$\alpha_t'(1) = \frac{(1-\mu)\gamma}{\mu + (1-\mu)\gamma} \quad \text{and} \quad \beta_t'(1) = \frac{\mu + (1-\mu)(1-\gamma)}{(1-\mu)(1-\gamma)}.$$

We compute that

$$\begin{aligned} J_t'(1) &= \mu(-\alpha_t'(1) + J_{t+1}'(1)\alpha_t'(1)) + (1-\mu)(\gamma J_{t+1}'(1)\alpha_t'(1) + (1-\gamma)J_{t+1}'(1)\beta'(1)) \\ &= \tfrac{-\mu(1-\mu)\gamma}{\mu+(1-\mu)\gamma} + J_{t+1}'(1)\left((1-\mu)\gamma + \mu + (1-\mu)(1-\gamma)\right). \end{aligned} \tag{56}$$

By using (56) recursively, we obtain the desired result. □

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