Price, trade size, and information revelation in multi-period securities markets*

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Abstract. We study price formation in securities markets, using the sequential trade framework of Glosten and Milgrom (1985). This paper makes one basic methodological advance over previous research on sequential securities trading: we allow traders to choose from n trade sizes in a multi-period market, where n can be arbitrarily large. We examine how trade size multiplicity affects the intertemporal dynamics of trading strategies, bid-ask spreads, and information revelation. We show that price impact, as a function of trade size, is increasing and exhibits (discrete) concavity.

Key Words: Market microstructure; Glosten-Milgrom; Price formation; Sequential trade; Asymmetric information; Trade size; Bid-ask spreads.

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Market microstructure studies the price formation process, and how this process is affected by the organization of the market. The main objective of this paper is to understand how trade sizes affect the price formation process dynamically within an environment where traders can choose from multiple trade sizes.

There are two standard reference frameworks in the market microstructure theory. One is the continuous auction framework, first developed by Kyle (1985). The other is the sequential trade framework, introduced by Copeland and Galai (1983) and Glosten and Milgrom (1985). In the Kyle framework, the asset orders are submitted first then the asset prices are set and made public, whereas in the sequential trade framework the prices are announced before the orders are submitted. Both frameworks are sufficiently simple and well-behaved and they easily lend themselves to analysis of policy issues and empirical tests.[1] Although most markets are organized as in the sequential trade models, these models tend to be less tractable than the Kyle model, as Back and Baruch (2004) point out.[2]

In this paper, we adopt the sequential trade framework to study the relationship between price, trade size, and information. Sequential trade models consider markets where a risky asset is traded between a market-maker, strategic traders, and liquidity traders. First, the market-maker, who is not informed of the risky asset payoff, quotes the bid and ask price. Then either a strategic trader or a liquidity trader arrives at the market in a random manner. The liquidity trader's trading motive is not related to the risky asset payoff at all. Whereas the strategic trader has information on the risky asset payoff, hence her trades reveal information. In the model of Copeland and Galai (1983), the risky asset payoff becomes public information after each trade. In the Glosten and Milgrom (1985) model, trading goes on for many rounds before the risky asset payoff is made public. Therefore, the latter allows us to see how price compounds information over time. Glosten and Milgrom also show that the bid-ask spread declines in expectation, and that the spread eventually vanishes almost surely as the number of trading rounds tends to infinity.

One of the simplified assumptions in Glosten and Milgrom (1985) is that traders can only trade one

[1] See Madhavan (2000) and Biais, Glosten, and Spatt (2005) for extensive surveys of the literature.

[2] In a continuous time setup, Back and Baruch (2004) show that the equilibrium of the Glosten-Milgrom model is approximately the same as the equilibrium of the Kyle model, when the trade size is small and uninformed trades arrive frequently.

share at any given period. Easley and O'Hara (1987) extend the Glosten-Milgrom model by allowing for two trade sizes: one small and one large. By doing so, they theoretically justify the empirically observed phenomenon that block trades are made at "worse" prices than small trades. However, Easley and O'Hara (1987) mostly focus on the static characterization of equilibrium prices and spreads.[3]

Our analysis extends the analyses by Glosten and Milgrom (1985) and Easley and O'Hara (1987) in two directions: time and trade size. We extend Glosten and Milgrom's model by allowing for multiple trade sizes for traders to choose from. Also, in comparison to Easley and O'Hara, we are not confined in our analysis to two trade sizes, thus we focus more on the intertemporal equilibrium dynamics. In our model, both trade sizes and trading rounds can vary.

Our model generates several results related to how trade size affects the intertemporal dynamics of informed trading strategies, bid-ask spreads, and information revelation. First, consistent with empirical research (e.g., Hasbrouck, 1988, 1991; Algert, 1990; Madhavan and Smidt, 1991; and Easley, Keifer, and O'Hara, 1997), informed traders are more likely to submit large orders. In each period there is a positive cut-off trade size for the informed trader, who observes that the risky asset payoff is high. She assigns no probability to purchasing amounts below this trade size while assigning positive probability to each trade size above the cut-off. The situation is symmetric for the informed trader, who observes that the risky asset payoff is low: there is a positive least amount that she sells with positive probability, and she assigns positive probability to selling each allowed amount greater than this cut-off. The bid-ask spreads exist only in the trade sizes where informed trading is considered probable by the market-maker. The cut-off trade sizes for the informed traders can decrease over time, and small trade sizes initially with zero bid-ask spreads can later have positive spreads. Also, in accord with the asymptotic results of Glosten and Milgrom (1985), our model predicts that the market-maker learns the true risky asset payoff almost surely as the number of trading rounds tends to infinity, and that the bid-ask spreads vanish in the limit. Finally, our work yields results on the intertemporal dynamics of bid-ask spreads and transaction prices. If the probability of an informed trader arriving in the market is sufficiently high, then the bid-ask spreads

[3]Easley and O'Hara (1987) employ a model with a richer information structure (compared to the Glosten-Milgrom model and ours), which makes the analysis of intertemporal equilibrium dynamics more difficult.

tighten over time within the domain where both informed purchasing and informed selling are deemed probable by the market-maker. On the other hand, the bid-ask spreads temporarily widen within the same domain provided that the probability of an informed trader arriving in the market is sufficiently low and a large order execution has moved the market-maker's posterior belief on the asset payoff towards certainty in the previous period. We further show that the price impact,[4] as a function of trade size, is increasing and exhibits (discrete) concavity.[5] This non-linear, concave relationship between price impact and trade size has been documented in several empirical studies (see, e.g., Algert, 1990; Hasbrouck, 1991; Hausman, Lo and MacKinlay, 1992; Kempf and Korn, 1999).

Our model provides a richer framework for empirically studying the relationship between trade size and intertemporal price formation: the equilibrium is described in closed-form, and our theoretical results generate testable hypotheses concerning the dynamic impact of order flows on bid-ask spreads and transaction prices. However, our model also has limitations that should be addressed in future studies. In particular, as in Glosten and Milgrom (1985) and Easley and O'Hara (1987), this model assumes that traders re-trade with probability zero. If re-trading were allowed, then informed traders would presumably attempt to camouflage their information by spreading their trades over time. For instance, in a model where informed traders endogenously determine whether or not to re-trade, Back and Baruch (2007) show that informed traders randomize across all trade sizes. On the other hand, in an extended version of the Easley-O'Hara (1987) model with endogenous re-trading, Seppi (1990) shows that, even when a large order can be broken up into a sequence of small orders, informed traders may still optimally submit large orders. It needs to be seen to what extent the issue of re-trading affects the implications of our existing model concerning dynamic impact of order flows on spreads and transaction prices.

This paper presents a theoretical depiction of quote-driven markets with competitive market-making, as is the case in Glosten and Milgrom (1985). NASDAQ Stock Market and London Stock Exchange's

[4] Price impact is the absolute value of the price change caused by the latest trade.

[5] Using a financial market model based on Kyle (1985), Dridi and Germain (2004) show that the impact of order flow on prices is non-linear. This result is similar to ours, however we obtain the non-linearity of price impact with a very basic information structure, whereas Dridi and Germain (2004) need to employ a sophisticated structure where informed traders receive a signal that perfectly reveals the direction but not the exact amount of the risky asset payoff.

SEAQ[6] are quote-driven markets with competitive market-making. CBOE, one of the world's largest options markets, is also quote-driven and operates with competitive market-makers. Other quote-driven markets include the eSpeed government bond trading system and the Reuters 3000 foreign exchange trading system.[7]

It should be noted that, with the implementation of the new Order Handling Rules in 1997, NASDAQ has become more of an order-driven market like the NYSE. This limits the applicability of our results to NASDAQ data sets prior to 1997. On the other hand, stocks are still traded under a quote-driven system in London Stock Exchange's SEAQ, and the same is true for options in CBOE. Our theoretical findings can be tested in these markets.

The organization of our paper is as follows. Section 1 presents the model and the equilibrium concept. In Section 2, we present the equilibrium analysis and our results. Section 3 concludes. Most of the proofs are delegated to the Appendix.

1. The model

We consider a market in which potential buyers and sellers trade a risky asset with a competitive market-maker. The economy lasts for $T+2$ many periods.[8] The periods are indexed by $t = 0, 1, ..., T, T+1$. Trade takes place in periods $t = 1, ..., T$ and consumption of a single good in period $T + 1$. The risky asset pays off in period $T + 1$. The risky asset payoff $\tilde{v}$ takes values from the set $\{0, V\}$ with the prior probability $\Pr(\tilde{v} = 0) = \delta$. We assume that $V > 0$ and $0 < \delta < 1$.

There are three types of agents in the economy: informed traders, liquidity traders, and a competitive market-maker. Informed traders are risk neutral, and they try to maximize their expected profits by trading. Informed traders also know the realization of the risky asset payoff $\tilde{v}$. Liquidity traders trade according to

[6] The Stock Exchange Automated Quotation system (SEAQ) is a system for trading mid-cap LSE stocks. SEAQ consists of around 1,500 stocks. Stocks must have at least two market-makers to be eligible for trading via SEAQ. Other quote-driven systems in LSE include SEAQI, where blue chip international stocks are traded, and European Quoting Service (EQS), where liquid MiFID securities are traded. Sources: http://www.londonstockexchange.com and Levin (2003, p. 26).

[7] Source: Harris (2003, p. 93).

[8] The economy can last for infinitely many periods, i.e., T can be infinity.

their liquidity needs, which are exogenous to the model. The competitive market-maker supplies against the demands of informed traders and liquidity traders.

Traders can choose from multiple trade sizes when they are trading the risky asset. In particular, they can trade the risky asset in the trade sizes that are elements of the set $\Omega_n := \{-n, ..., -1, 0, 1, ..., n\}$. In our notation, k and $-k$ represent the purchase and the sale of k units of the risky asset, respectively. $\Omega_n^+ := \{1, ..., n\}$ denotes the set of possible purchase trade sizes, while $\Omega_n^- := \{-n, ..., -1\}$ denotes the set of possible sales trade sizes. 0 represents no trade.

The timing of events in our model is as follows:

1. In period 0, nature chooses the realization $v \in \{0, V\}$ of the risky asset payoff $\tilde{v}$. Informed traders observe v.

2. In successive periods, indexed by $t = 1, ..., T$, the events realize in the following order:

    · Having observed the realized trades in periods $1, ..., t-1$, the competitive market-maker posts a price for each trade size in Ω_n.

    · A new trader (either an informed trader or a liquidity trader) arrives at the market and learns market-maker's price quote for each trade size.

    · If the trader is informed, she takes the profit-maximizing quote. If the trader is a liquidity trader, she trades in the trade size determined by her liquidity needs.

3. In period $T+1$, the realization of $\tilde{v}$ is publicly disclosed, and consumption takes place.

The type of the trader arriving in period t is determined by the random variable $\tilde{\theta}_t$, which takes values from the set $\{i_v, l\}$. The letters i_v and l denote the informed type and the liquidity type, respectively. The random variables $\{\tilde{\theta}_t \colon t = 1, ...T\}$ are i.i.d. across the periods $1, ..., T$ and satisfy $\Pr(\tilde{\theta}_t = i_v) = \mu$. If the trader type in period t is l, then the demand at that period is determined by the random variable $\tilde{L}_t$, which takes values from Ω_n. The random variables $\{\tilde{L}_t \colon t = 1, ..., T\}$ are i.i.d. and satisfy $\Pr(\tilde{L}_t = q)$ $= \gamma(q) > 0$. For any given period t, the random variables $\tilde{\theta}_t$, $\tilde{L}_t$, $\tilde{v}$ are mutually independent. We assume that informed traders, who trade once, gets the chance to re-trade with probability 0. Thus, informed traders behave myopically and they (rationally) ignore the effect of their trades on future periods. The market-maker is risk-neutral and her price quotes make her zero expected profit in each period,[9] i.e., in period t, $t = 1, ..., T$, she chooses the price of each trade size $q \in \Omega_n$ equal to the expected value of the risky asset payoff conditional on her information at period t and the trade realization being equal to q. Informed traders and the market-maker correctly anticipate each other's trading and pricing strategies. The structure of the economy, described so far, is common knowledge.

Next we describe the details with regard to the market-maker's pricing strategy and informed traders' trading strategy. To that end, we first need to introduce some notation. Let q_t denote the trade size that the market-maker receives in period t, i.e., q_t is the realized trade size for period t. A *period-t history* $h_t := (q_1, ..., q_t)$ is the sequence of realized trade sizes for periods up until $t+1$. The space of all possible period-t histories, $t \geq 1$, is denoted by $\Omega_n^t := \prod_{\tau=1}^{t} \Omega_n$, and h_t is taken to be the generic element of Ω_n^t. $h_T \in \Omega_n^T$ is called a *complete history*. h_t is said to be *consistent* with $h_T = (q_1, ..., q_T) \in \Omega_n^T$ if $h_T = (h_t, q_{t+1}, ..., q_T)$. For notational convenience, we let $h_0 = \emptyset$. Also, we let $\pi_t \colon \Omega_n^{t-1} \times \Omega_n \to \mathbb{R}$ represent the market-maker's pricing strategy function (i.e., her price menu for all trade sizes), so that $\pi_t(h_{t-1}, q)$ is the market-maker's price quote for trade size q given history h_{t-1}.

Since there is a price quote for each trade size, it is possible for informed traders to obtain the same profit from two or more different trade sizes. In such cases, informed traders assign positive probabilities to those trade sizes that yield equal profit when they determine their demands. We formalize this as follows: In our model, a trading strategy is a probability function $\psi \colon \Omega_n \to [0,1]$ such that $\sum_{q \in \Omega_n} \psi(q) = 1$. The support of ψ is given by $\text{supp}(\psi) := \{q \in \Omega_n | \psi(q) \neq 0\}$. We let $\Delta(\Omega_n) := \{\psi \colon \Omega_n \to [0,1] \big| \sum_{q \in \Omega_n} \psi(q) = 1\}$ denote the set of all possible trading strategies. *Informed trader's trading strategy for price menu π_t prescribes a probability distribution $\psi_t(v, h_{t-1}, \pi_t) \in \Delta(\Omega_n)$ over trade sizes in Ω_n for each $v \in \{0, V\}$ and history $h_{t-1} \in \Omega_n^{t-1}$.* We let $\psi_t(q | v, h_{t-1}, \pi_t)$ denote the probability assigned to trade size q by the probability distribution $\psi_t(v, h_{t-1}, \pi_t)$. Among all trading strategies, the informed trader chooses the strategy that maximizes her expected profit given the market-

[9]A Bertrand competition among market-makers is the standard assumption to have zero expected profit for the (competitive) market-maker.

maker's price menu. Therefore, *informed trader's optimal trading strategy for price menu* π_t *prescribes the probability distribution* $\psi_t^*(v, h_{t-1}, \pi_t) \in \Delta(\Omega_n)$ *over trade sizes in* Ω_n *for each* $v \in \{0, V\}$ *and history* $h_{t-1} \in \Omega_n^{t-1}$ *such that*

$$\psi_t^*(v, h_{t-1}, \pi_t) \in \arg\max_{\psi \in \Delta(\Omega_n)} \sum_{q \in \Omega_n} \psi(q) \left[q \left(v - \pi_t(h_{t-1}, q) \right) \right].$$

The market-maker is Bayesian. She updates her belief about the risky asset payoff in each period after having observed the realized trade size for that period. Formally, $\delta_t(h_{t-1}, q)$ is the probability assigned by the market-maker to the risky payoff being equal to 0 given that realized history is $h_{t-1} \in \Omega_n^{t-1}$ and the realized trade size in period-t is going to be q. As a notational convenience, we let $\delta_0 = \delta$. Bayesian updating dictates

$$\begin{aligned} \delta_t(h_{t-1}, q) &:= \Pr(\tilde{v} = 0 | h_{t-1}, q) \\ &= \frac{\Pr(\tilde{v} = 0 | h_{t-1}) \; [\mu \, \psi_t(q | 0, h_{t-1}, \pi_t) + (1 - \mu) \, \gamma(q)]}{\sum_{v \in \{0, V\}} \Pr(\tilde{v} = v | h_{t-1}) \, \mu \, \psi_t(q | v, h_{t-1}, \pi_t) + (1 - \mu) \, \gamma(q)} \end{aligned} \tag{1}$$

if the market-maker believes that the informed trader is employing trading strategy ψ_t in period-t. As the market-maker makes zero profit from her price quotes, her price menu π_t satisfies

$$\pi_t(h_{t-1}, q) = (1 - \delta_t(h_{t-1}, q)) \, V, \quad \forall h_{t-1} \in \Omega_n^{t-1}, \; \forall q \in \Omega_n. \tag{2}$$

We say that $\pi_t(h_{t-1}, q)$ *satisfies the zero-profit condition if Equation (2) holds.*

Next we define the equilibrium for our economy:

Definition 1. *An equilibrium consists of the market-maker's price menus* $\{\pi_t^* \colon t = 1, ..., T\}$*, informed traders' trading strategies* $\{\psi_t^* \colon t = 1, 2, ..., T\}$*, and posterior beliefs* $\{\delta_t^* \colon t = 1, 2, ..., T\}$ *such that for all* $t \in \{1, \cdots, T\}$ *and for all* $h_{t-1} \in \Omega_n^{t-1}$

(P1) $\pi_t^*(h_{t-1}, q)$ *satisfies the zero-profit condition (2) given the posterior belief* $\delta_t^*(h_{t-1}, q)$ *for all* $q \in \Omega_n$*,*

(P2) $\psi_t^*(v, h_{t-1}, \pi_t^*)$ *is informed traders' optimal trading strategy for price menu* π_t^* *for all* $v \in \{0, V\}$*,*

(B) *for all* $q \in \Omega_n$*,* $\delta_t^*(h_{t-1}, q)$*,* $\pi_t^*(h_{t-1}, q)$ *and* $\{\psi_t^*(v, h_{t-1}, \pi_t^*) \colon v \in \{0, V\}\}$ *satisfy Equation (1).*

Condition (B) specifies the equilibrium belief. It essentially reflects two critical assumptions of our model: first, the market-maker is Bayesian; second, informed traders and the market-maker correctly anticipate each other's trading and pricing strategies.[10]

Finally, we define the bid-ask spread for trade size q: *the period-t bid-ask spread for history $h_{t-1} \in \Omega_n^{t-1}$ and trade size $q \in \{1, ..., n\}$ is given by*

$$S_t(h_{t-1}, q) := \pi_t(h_{t-1}, q) - \pi_t(h_{t-1}, -q), \tag{3}$$

where π_t is the market-maker's price menu.

2. Sequential trades with multiple trade sizes

Our analysis makes one basic methodological advance over previous research on sequential trade models: we let traders choose from multiple trade sizes. This allows us to see the impact of trade size on price menus, trading strategies, and information revelation in a multi-period economy.

2.1. Informed traders' equilibrium trading strategies

We first examine the impact of trade size on trading strategies. To that end, we analyze informed traders' equilibrium trading strategies. The following proposition lists some of the basic properties of the equilibrium trading strategies of informed traders:

Proposition 1. *If $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) \colon t = 1, ..., T\right\}$ is an equilibrium, then for all $h_{t-1} \in \Omega_n^{t-1}$ and $t \in \{1, ..., T\}$ informed traders' equilibrium trading strategy ψ_t^* satisfies the following:*

(a) $\psi_t^*(0|v, h_{t-1}, \pi_t^*) = 0$ *for all* $v \in \{0, V\}$,

(b) $\psi_t^*(q|V, h_{t-1}, \pi_t^*) = 0$ *for all* $q \in \Omega_n^-$,

(c) $\psi_t^*(q|0, h_{t-1}, \pi_t^*) = 0$ *for all* $q \in \Omega_n^+$.

[10] In other words, informed traders and the market-maker have rational expectations about each other's strategies.

Part (a) of Proposition 1 states that informed traders always trade in non-zero quantities. This is simply a consequence of the information asymmetry between informed traders and the market-maker. Since the market-maker can never fully infer the risky payoff realization v at any given period $t < \infty$ due to the presence of liquidity traders, informed traders, who know v, are better off trading non-zero quantities of the risky asset as they can make non-zero profits by doing so. Parts (b) and (c) say that informed traders sell when $v = 0$ and buy when $v = V$, respectively. Since the market-maker always quotes a price strictly between 0 and V (due to her uncertainty about the risky asset payoff), informed traders are better off selling when they know $v = 0$ and they are better off buying when they know $v = V$.

The next result shows that informed traders' equilibrium trading strategies satisfy a special condition: given any history and period, there is a cut-off trade size above which informed traders buy with positive probabilities if $v = V$ and another cut-off size above which informed traders sell with positive probabilities if $v = 0$. Formally:

Theorem 2. *If* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) \colon t = 1, ..., T\right\}$ *is an equilibrium, then for all* $h_{t-1} \in \Omega_n^{t-1}$ *and* $t \in \{1, ..., T\}$ *there exist cut-off trade sizes* $k_t^+(h_{t-1}) \geq 1$ *and* $k_t^-(h_{t-1}) \geq 1$ *such that*

$$\begin{aligned} \operatorname{supp}\{\psi_t^*(V, h_{t-1}, \pi_t^*)\} &= \{k_t^+(h_{t-1}), \cdots, n\}, \quad \textit{and} \\ \operatorname{supp}\{\psi_t^*(0, h_{t-1}, \pi_t^*)\} &= \{-n, \cdots, -k_t^-(h_{t-1})\}. \end{aligned}$$

Theorem 2 essentially says the following: if informed traders assign positive probability to trade size q in their equilibrium trading strategy, then they also assign positive probabilities to trade sizes larger than q. The intuition is as follows: Suppose to the contrary that in equilibrium there is a trade size $q' > q$ such that informed traders never trade in. Then, foreseeing the strategy employed by the informed traders, the market-maker would post a "better" price to the trade size q' compared to the trade size q. That implies the informed traders would make a larger profit by trading in size q' because not only the price is better but also the size is larger. However, this contradicts q being in the support of the equilibrium trading strategy as q should yield higher profits than q'.

Also, Theorem 2 lets us use a simple classification system for informed traders' equilibrium trading strategies. Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) \colon t = 1, ..., T\right\}$ be an equilibrium. The equilibrium trading strategy of informed traders, ψ_t^*, is said to be:[11]

- *k partially pooling on the long side for history h_{t-1}* if $0 < \psi_t^*(q|V, h_{t-1}, \pi_t^*) \leq 1$ for all $q \in \{k, k+1, ..., n\}$ and $\psi_t^*(q|V, h_{t-1}, \pi_t^*) = 0$ for all $q \in \{0, ..., k-1\}$,

- *k partially pooling on the short side for history h_{t-1}* if $0 < \psi_t^*(q|0, h_{t-1}, \pi_t^*) \leq 1$ for all $q \in \{-n, -n-1, ..., -k\}$ and $\psi_t^*(q|0, h_{t-1}, \pi_t^*) = 0$ for all $q \in \{0, ..., -k+1\}$.

According to this simple classification, Theorem 2 implies that there exist k_t^+ and k_t^- such that informed traders' equilibrium trading strategy is k_t^+ partially pooling on the long side and k_t^- partially pooling on the short side. For convenience, we also employ the following terminology: we say ψ_t^* is:

- *separating on the long side (short side) for history h_{t-1}* if ψ_t^* is n partially pooling on the long side (short side) for history h_{t-1},

- *completely pooling on the long side (short side) for history h_{t-1}* if ψ_t^* is 1 partially pooling on the long side (short side) for history h_{t-1}.

We now turn our attention to the necessary and sufficient conditions for informed traders' equilibrium trading strategies to be k partially pooling, $1 \leq k \leq n$.

Proposition 3. *Let $\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\}$ be an equilibrium. The equilibrium trading strategy of informed traders, ψ_t^*, is:*

(a) k_t^+ partially pooling on the long side for history h_{t-1} if and only if

$$(1-\mu) \sum_{i=k_t^+}^{n} \left(1 - \frac{i}{k_t^+}\right) \gamma(i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu \quad > \quad 0, \quad \textit{and} \tag{4a}$$

$$(1-\mu) \sum_{i=k_t^+-1}^{n} \left(1 - \frac{i}{k_t^+ - 1}\right) \gamma(i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu \quad \leq \quad 0, \tag{4b}$$

[11] Note that we use the terms "separating" and "pooling" in a non-standard way since we follow the terminology used by Easley and O'Hara (1987). For example, by "k partially pooling" we mean that the informed traders have a mixed strategy above the cut-off size k. Informed and liquidity traders pool above the cut-off, but they also pool in the case the informed traders employ the "separating" equilibrium trading strategy.

(b) k_t^- partially pooling on the short side for history h_{t-1} if and only if

$$(1-\mu)\sum_{i=k_t^-}^{n}\left(1-\frac{i}{k_t^-}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\mu \quad > \quad 0, \quad \textit{and} \tag{4c}$$

$$(1-\mu)\sum_{i=k_t^- -1}^{n}\left(1-\frac{i}{k_t^- -1}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\,\mu \quad \leq \quad 0. \tag{4d}$$

To better understand the implications of Proposition 3, we examine the necessary and sufficient conditions for the two special cases of k partially pooling trading strategies: separating and completely pooling. We have following result for separating strategies:

Corollary 4. *Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) \colon t = 1, ..., T\right\}$ be an equilibrium. The equilibrium trading strategy of informed traders, ψ_t^*, is:*

(a) separating on the long side for history h_{t-1} if and only if

$$\frac{n}{n-1} \geq 1+\frac{(1-\delta_{t-1}^*(h_{t-1}))\mu}{\gamma(n)(1-\mu)}, \tag{5a}$$

(b) separating on the short side for history h_{t-1} if and only if

$$\frac{n}{n-1} \geq 1+\frac{\delta_{t-1}^*(h_{t-1})\,\mu}{\gamma(-n)(1-\mu)}. \tag{5b}$$

In the case of separating trading strategies, traders trade only in the largest trade size, namely n. Observe that, for separating trading strategies, conditions (4a) and (4c) in Proposition 3 become redundant as, for $k_t^+ = k_t^- = n$, these conditions reduce to $(1-\delta_{t-1}^*(h_{t-1}))\,\mu \,>\, 0$ and $\delta_{t-1}^*(h_{t-1})\,\mu \,>\, 0$, respectively, which necessarily hold since the market-maker can never fully infer the risky payoff realization v at any given period t or history h_{t-1} due to the presence of liquidity traders (meaning that $0 < \delta_{t-1}^*(h_{t-1}) < 1$). This observation and straightforward manipulations on (4b) and (4d) prove Corollary 4.

Now let us focus on the implication of this result. Corollary 4 implies that as the probability μ of informed trading goes up an equilibrium trading strategy becomes less likely to be separating. The intuition is straightforward: Following Theorem 2, informed traders always assign positive probability to the largest trade size n in their equilibrium trading strategies. If the probability of informed trading is high, then the market-maker posts a large bid-ask spread for trade size n to avoid loss inflicted by informed traders. This makes trading in the largest trade size less attractive for informed traders, hence they decrease their likelihood of trading in size n by assigning positive probabilities to smaller trade sizes in their trading strategies. Therefore, an equilibrium trading strategy is less likely to be separating if the probability of informed trading is high.

The following result provides the necessary and sufficient conditions for equilibrium trading strategies to be completely pooling:

Corollary 5. *Let $\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\}$ be an equilibrium. The equilibrium trading strategy of informed traders, ψ_t^*, is:*

(a) completely pooling on the long side for history h_{t-1} if and only if

$$(1-\mu)\sum_{i=1}^{n}(1-i)\gamma(i) + (1-\delta_{t-1}^*(h_{t-1}))\mu \quad > \quad 0, \tag{6a}$$

(b) completely pooling on the short side for history h_{t-1} if and only if

$$(1-\mu)\sum_{i=1}^{n}(1-i)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\,\mu \quad > \quad 0. \tag{6b}$$

In the case of completely pooling trading strategies, informed traders trade in all possible trade sizes with positive probabilities. Therefore, conditions (4b) and (4d) in Proposition 3 are redundant, and the result above is obtained as a straightforward corollary of Proposition 3.

Note the following implication of Corollary 5: as the probability of informed trading μ increases, informed traders' equilibrium trading strategy becomes more likely to be completely pooling. The intuition behind this result is very much in line with the intuition we gave for Corollary 4. Following Theorem 2, the market-maker knows that informed traders are more likely to trade in large trade sizes and as a consequence she posts large bid-ask spreads for these large sizes. This makes informed traders to assign positive probabilities to smaller trade sizes so that they can enjoy "better" price quotes. Essentially, increased pooling gives informed traders increased coverage by liquidity traders against the market-maker. Therefore, if the probability of informed trading is sufficiently high, the equilibrium trading strategy of informed traders is completely pooling.

2.2. Existence and uniqueness of equilibrium

In the standard sequential trade models, traders, who trade once, get the chance to re-trade with probability zero. This is also the case in our model. Therefore, informed traders' time scope for portfolio decisions is confined to one period. As a consequence, the only link between consecutive periods is the market-maker's belief on the risky asset payoff. That is, δ^*_{t-1} is the only parameter from period-$(t-1)$, which the period-t equilibrium strategies π^*_t, ψ^*_t and the period-t equilibrium belief δ^*_t depend on. Let us demonstrate this in detail:

Take a complete history $h_T \in \Omega_n^T$, and let $\{h_t\colon t=1,...,T\}$ be the sequence of histories consistent with h_T. Fix period $t \in \{1,...,T\}$ and history h_{t-1}. Recall from equilibrium condition (B) that

$$\delta^*_t(h_{t-1},q) \;=\; \frac{\delta^*_{t-1}(h_{t-1})\,[\mu\,\psi^*_t(q|0,h_{t-1},\pi^*_t)+(1-\mu)\,\gamma(q)]}{\delta^*_{t-1}(h_{t-1})\,\mu\,\psi^*_t(q|0,h_{t-1},\pi^*_t)+(1-\delta^*_{t-1}(h_{t-1}))\,\mu\,\psi^*_t(q|V,h_{t-1},\pi^*_t)+(1-\mu)\,\gamma(q)}. \tag{7}$$

Equation (7) shows the functional relation between the period-t equilibrium belief $\delta^*_t(h_{t-1},\cdot)\colon \Omega_n \to (0,1)$ and the period-$(t-1)$ equilibrium belief $\delta^*_{t-1}(h_{t-1})$. Also, the zero-profit condition of the equilibrium, namely (P1), dictates that the market-maker's price menu is of the form

$$\begin{aligned}\pi^*_t(h_{t-1},q) \;&=\; (1-\delta^*_t(h_{t-1},q))\,V \\ &=\; \frac{(1-\delta^*_{t-1}(h_{t-1}))\,[\mu\,\psi^*_t(q|V,h_{t-1},\pi^*_t)+(1-\mu)\,\gamma(q)]\,V}{\delta^*_{t-1}(h_{t-1})\,\mu\,\psi^*_t(q|0,h_{t-1},\pi^*_t)+(1-\delta^*_{t-1}(h_{t-1}))\,\mu\,\psi^*_t(q|V,h_{t-1},\pi^*_t)+(1-\mu)\,\gamma(q)}.\end{aligned} \tag{8}$$

Equation (8) shows the functional relation between the period-t price menu $\pi^*_t(h_{t-1},\cdot)\colon \Omega_n \to \mathbb{R}$ and $\delta^*_{t-1}(h_{t-1})$. Finally, the functional relation between informed traders' period-t equilibrium trading strategy $\psi^*_t(\cdot,h_{t-1},\pi^*_t) : \{0,V\} \to \prod_{q=1}^n [0,1]$ and $\delta^*_{t-1}(h_{t-1})$ is derived in Lemma 3.[12] It states that if ψ^*_t is k^+_t partially pooling on the long side and k^-_t partially pooling on the short side for history h_{t-1}, then

$$\psi^*_t(q|V,h_{t-1},\pi^*_t) = \begin{cases} 0 & : \quad q \in \{-n,\cdots,k^+_t-1\} \\ \dfrac{(1-\mu)\sum_{i=k^+_t}^n \left(1-\frac{i}{q}\right)\gamma(i)+\left(1-\delta^*_{t-1}(h_{t-1})\right)\mu}{\left(1-\delta^*_{t-1}(h_{t-1})\right)\mu\sum_{i=k^+_t}^n \frac{i\,\gamma(i)}{q\,\gamma(q)}} & : \quad q \in \{k^+_t,\cdots,n\}; \end{cases} \tag{9a}$$

[12]See the Appendix.

$$\psi_t^*(q|0,h_{t-1},\pi_t^*) = \begin{cases} 0 & : \quad q \in \{-k_t^- + 1, \cdots, n\} \\ \frac{(1-\mu)\sum_{i=k_t^-}^{n}\left(1-\frac{i}{|q|}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\mu}{\delta_{t-1}^*(h_{t-1})\mu\sum_{i=k_t^-}^{n}\frac{i\,\gamma(-i)}{|q|\,\gamma(q)}} & : \quad q \in \{-n, \cdots, -k_t^-\}. \end{cases} \tag{9b}$$

We derived period-t equilibrium belief δ_t^* and period-t equilibrium strategies π_t^* and ψ_t^* as functions of $\left(\delta_{t-1}^*(h_{t-1}), k_t^+, k_t^-\right) \in (0,1) \times \Omega_n^+ \times \Omega_n^-$ in Equations (7), (8), and (9a)-(9b), respectively. For existence of equilibrium in period-t, it remains to show that, given $\delta_{t-1}^*(h_{t-1})$, there indeed exist some trade sizes $k_t^+ \in \Omega_n^+$ and $k_t^- \in \Omega_n^-$ such that informed traders' period-t equilibrium trading strategy ψ_t^* is k_t^+ partially pooling on the long side and k_t^- partially pooling on the short side. It follows from Proposition 3 that this is equivalent to showing the following:

Lemma 1. *Given $\delta_{t-1}^*(h_{t-1}) \in (0,1)$, there exist $k_t^+ \in \Omega_n^+$ and $k_t^- \in \Omega_n^-$ that satisfy the inequalities (4a)-(4b) and (4c)-(4d), respectively.*

We showed that given $\delta_{t-1}^*(h_{t-1}) \in (0,1)$, there exist period-$t$ equilibrium belief and strategies, and that they are prescribed by (7), (8), and (9a)-(9b). Since these belief and strategies are functions of $\delta_{t-1}^*(h_{t-1})$, k_t^+ and k_t^-, uniqueness of period-t equilibrium for given $\delta_{t-1}^*(h_{t-1})$ is equivalent to the uniqueness of the endogenous parameters k_t^+ and k_t^- described in Lemma 1.

Lemma 2. *Given $\delta_{t-1}^*(h_{t-1}) \in (0,1)$, if there exist $k_t^+ \in \Omega_n^+$ and $k_t^- \in \Omega_n^-$ that satisfy the inequalities (4a)-(4b) and (4c)-(4d), respectively, then they must be unique.*

Our analysis so far implies that given $\delta_{t-1}^*(h_{t-1})$, there uniquely exist period-t equilibrium belief δ_t^* and period-t equilibrium strategies π_t^* and ψ_t^*. Since $\delta_0 = \delta$ is an exogenous parameter of the model, the overall equilibrium can be uniquely derived using Equations (7), (8), and (9a)-(9b) in a recursive fashion. In summary, we proved the following result:

Proposition 6. *There exists a unique equilibrium $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$. That is, given a complete history $h_T \in \Omega_n^T$, the sequence of histories $\{h_t \colon t = 1, ..., T\}$ consistent with h_T, and $t \in \{1, ..., T\}$, the equilibrium price menu $\pi_t^*(h_{t-1}, \cdot) \colon \Omega_n \to \mathbb{R}$, the equilibrium trading strategy of informed traders*

$\psi_t^*(\cdot, h_{t-1}, \pi_t^*)\colon \{0, V\} \to \mathbb{R}^{|\Omega_n|}$, *and the equilibrium posterior belief* $\delta_t^*(h_{t-1}, \cdot)\colon \Omega_n \to \mathbb{R}$ *uniquely exist.*

It should be noted that the assumption of exogenous and price inelastic liquidity trading is crucial for the uniqueness of equilibrium. Absent this assumption, multiple equilibria may occur: Dow (2004) extends Glosten and Milgrom's (1985) model by endogenizing liquidity trading. In particular, Dow models liquidity traders as hedgers[13] with an initial risk exposure that is correlated with the risky asset payoff and, by doing so, shows that multiple equilibria may arise due to coordination failure.

2.3. Equilibrium dynamics

In this section we turn our attention to the equilibrium dynamics. First, we examine the dynamics of the equilibrium trading strategies of informed traders. As trades unfold over time, the market-maker updates her belief on the risky asset payoff. Consequently, she also updates the price menu, and in turn informed traders revise their trading strategies. Given a k partially pooling trading strategy in period-t, the revision of the trading strategy in period-$(t+1)$ can take place in two ways: (1) informed traders can maintain k as the cut-off size and just change the probabilities they assign to trade sizes over k, or (2) they can alter the cut-off size, hence change the support of their trading strategy. Naturally, the latter implies a significant change in the trading behavior of informed traders, and that is what we are after: we would like to see if informed traders ever change the support of their trading strategies over time. The following result sheds light on this.[14]

Proposition 7. *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ *be an equilibrium and* $h_t = (h_{t-1}, q_t) \in \Omega_n^t$*. Let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*. Also, let*

[13] Such modelling of liquidity trading, which makes liquidity demand price elastic, was introduced by Glosten (1989) and Spiegel and Subrahmanyam (1992).

[14] Note the following weaker version of Proposition 7: (a) if ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1} and $q_t > 0$, then ψ_{t+1}^* is k_{t+1}^+ partially pooling on the long side for history (h_{t-1}, q_t) with $k_{t+1}^+ \le k_t^+$, (b) if ψ_t^* is k_t^+ partially pooling on the short side for history h_{t-1} and $q_t < 0$, then ψ_{t+1}^* is k_{t+1}^- partially pooling on the short side for history (h_{t-1}, q_t) with $k_{t+1}^- \le k_t^-$. This result follows in a fairly straightforward manner from the updating rule (1) and Proposition 3.

ψ^*_{t+1} be k^+_{t+1} partially pooling on the long side and k^-_{t+1} partially pooling on the short side for history h_t. Then the following hold:

(a) $k^+_{t+1} < k^+_t$ *if* $q_t \in \{k^+_t, ..., n\}$ *and*

$$\delta^*_{t-1}\left(\frac{(1-\mu)\sum_{i=k^+_t}^{n}\left(1-\frac{i}{q_t}\right)\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\mu}{(1-\mu)\sum_{i=k^+_t}^{n}\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\mu}\right) \geq \frac{(1-\mu)}{\mu\, k^+_t(k^+_t-1)}\sum_{i=k^+_t}^{n} i\,\gamma(i). \tag{10a}$$

(b) $k^-_{t+1} < k^-_t$ *if* $q_t \in \{-n, ..., -k^-_t\}$ *and*

$$\left(1-\delta^*_{t-1}\right)\left(\frac{(1-\mu)\sum_{i=k^-_t}^{n}\left(1-\frac{i}{|q_t|}\right)\gamma(-i)+\delta^*_{t-1}(h_{t-1})\,\mu}{(1-\mu)\sum_{i=k^-_t}^{n}\gamma(-i)+\delta^*_{t-1}(h_{t-1})\,\mu}\right) \geq \frac{(1-\mu)}{\mu\, k^-_t(k^-_t-1)}\sum_{i=k^-_t}^{n} i\,\gamma(-i). \tag{10b}$$

For the sake of exposition, we call $\{k^+, ..., n\}$ *the domain of informed purchasing*, $\{-n, ..., -k^-\}$ *the domain of informed selling*, and $\{-n, ..., -k^-\} \cup \{k^+, ..., n\}$ *the domain of informed trading* if the equilibrium trading strategy is k^+ partially pooling on the long side and k^- partially pooling on the short side for the given history and period. Proposition 7 reveals the following: (a) the domain of informed purchasing gets bigger in period-$(t+1)$ provided that the probability μ of informed trading is sufficiently high, a trade from the period-t domain of informed purchasing has occurred, and the market-maker believed that the risky asset payoff was highly likely to be 0 before the purchase realization; (b) the domain of informed selling gets bigger in period-$(t+1)$ provided that the probability μ of informed trading is sufficiently high, a trade from the period-t domain of informed selling has occurred, and the market-maker believed that the risky asset payoff was highly likely to be V before the sale realization.

Part (a) and part (b) of Proposition 7 can be motivated in similar fashions. Let us consider part (a). Suppose that the probability μ of informed trading is sufficiently high, a trade from the period-t domain of informed purchasing has realized, and the market-maker believed that the risky asset payoff was highly likely to be 0 before the purchase realization. An informed purchase would take place only if the risky asset payoff were V. Since the probability of informed trading is high and the market-maker previously believed that the risky asset payoff was highly likely to be 0, the realized purchase leads to a significant change in her belief. If informed trading strategy were not to be revised in period-$(t+1)$, the market-maker would substantially increase prices for the period-t domain of informed purchasing. Consequently, the period-t domain of informed purchasing would yield lower profits in period-$(t + 1)$. Therefore, if the true risky asset payoff is indeed V, informed traders revise their trading strategy by decreasing the cut-off size. By doing so, they increase the probability of liquidity trading within the domain of informed purchasing and this allows higher probability of profit-making for the market-maker. As the market-maker is bound to make zero expected profit in equilibrium, the enlarged domain of informed purchasing yields more favorable price quotes for the informed traders.

As illustrated by the argument above, the dynamics of informed trading strategies and the market-maker's learning process are closely related. We next tackle whether the market-maker's belief on the risky asset payoff converges to the truth as the number of trading periods tends to infinity. Glosten and Milgrom (1985) show that such convergence is obtained almost surely if the only available trade size is the unit trade size. In our generalized framework, where multiple trade sizes are available, the asymptotic result of Glosten and Milgrom (1985) still holds.

Theorem 8. *Suppose* $T = \infty$. *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., \infty\right\}$ *be an equilibrium. Given a complete history* $h_\infty \in \Omega_n^\infty$, *let* $(h_t : t = 1, ..., \infty)$ *be the sequence of histories consistent with* h_∞.

(a) $\delta_t^*(h_t)$ *converges to* 0 *almost surely as* t *tends to infinity if* $v = V$,

(b) $\delta_t^*(h_t)$ *converges to* 1 *almost surely as* t *tends to infinity if* $v = 0$.

This result is driven by the fact that transaction prices (i.e., the prices of trade sizes that have been acted upon) form a martingale. The martingale property of prices guarantees their convergence. Of course, even if the beliefs converge, they need not converge to the truth. However, as in Glosten and Milgrom (1985), after sufficiently high number of periods, the market-maker observes a sufficient number of informed trades, and these trades reveal the truth in the limit.

2.4. Bid-ask spreads

In this section, we investigate the equilibrium bid-ask spreads. The bid-ask spreads compensate the market-maker for the risk of doing business with informed traders. Therefore, the equilibrium bid-ask spreads depend on informed traders' equilibrium trading strategies: positive bid-ask spreads are observed only in the trade sizes that belong to the domain of informed trading. The following proposition formally states this result:

Proposition 9. *Let* $\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\}$ *be an equilibrium. Also, let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*. The equilibrium bid-ask spread* $S_t^*(h_{t-1}, \cdot) : \Omega_n^+ \to \mathbb{R}$ *satisfies the following:*

(a) $S_t^*(h_{t-1}, q) > 0$ *if and only if* $\min\{k_t^+, k_t^-\} \leq q \leq n$*,*

(b) $S_t^*(h_{t-1}, q) = 0$ *if and only if* $1 \leq q < \min\{k_t^+, k_t^-\}$*.*

Notice that small trade sizes initially with zero bid-ask spreads can later have positive spreads as the trades unfold over time. This actually follows from Proposition 9 and our discussions in Section 2.3. Section 2.3 has revealed that informed traders can enlarge the domain of informed trading over time as a remedy against the market-maker getting close to the truth. In light of Proposition 9, this means that the domain of positive bid-ask spreads will get bigger over time if the domain of informed trading is indeed enlarged. Formally, we have the following result as a direct corollary of Proposition 7 and Proposition 9:

Remark 10. *Let* $\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\}$ *be an equilibrium. Also, let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*.*

(a) If $k_t^+ \leq k_t^-$*,* $q_t \geq k_t^+$*, and* $\delta_t^*(h_{t-1}, q_t)$ *satisfies (10a), then*

$$S_t^*(h_{t-1}, k_t^+ - 1) = 0 \quad \textit{while} \quad S_t^*(h_t, k_t^+ - 1) > 0.$$

(b) If $k_t^- \leq k_t^+$*,* $q_t \leq -k_t^-$*, and* $\delta_t^*(h_{t-1}, q_t)$ *satisfies (10b), then*

$$S_t^*(h_{t-1}, k_t^- - 1) = 0 \quad \textit{while} \quad S_t^*(h_t, k_t^- - 1) > 0.$$

The bid-ask spreads exist due to the asymmetric information between the market-maker and informed traders. If the market-maker were to learn the truth about the risky asset payoff, there would be no spreads as the price of the risky asset would be set equal to its payoff. So, following Theorem 8, we know that the bid-ask spreads vanish almost surely as the number of trading periods tends to infinity.

Proposition 11. *Suppose* $T = \infty$*. Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., \infty\right\}$ *be an equilibrium. Given a complete history* $h_\infty \in \Omega_n^\infty$*, let* $(h_t : t = 1, ..., \infty)$ *be the sequence of histories consistent with* h_∞*. The equilibrium bid-ask spread* $S_t^*(h_t)$ *converges to* 0 *almost surely as* t *tends to infinity.*

Next we examine the functional relation between bid-ask spreads and trade sizes. To that end, we first make the following mathematical definition. Let $X \in \mathbb{Z}$. We say $f \colon X \to \mathbb{R}$ exhibits *discrete concavity* if, for any $x, x-1, x+1 \in X$, it holds that

$$f(x+1) - f(x) \leq f(x) - f(x-1). \tag{11}$$

f is said to exhibit *strict discrete concavity* if (11) holds with strict inequality. Notice that our definition for discrete concavity essentially provides an extension of the concavity definition of continuous functions to discrete spaces. Recall that a differentiable function is concave if and only if its first-order derivative is a decreasing function. In a discrete space, this corresponds to first-order difference being a decreasing function, as in (11). The following proposition sheds light on the functional relation between the equilibrium bid-ask spreads and trade sizes.

Proposition 12. *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) \colon t = 1, ..., T\right\}$ *be an equilibrium. Also, let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*. The equilibrium bid-ask spread* $S_t^*(h_{t-1}, q)$*, as a function of* $q > 0$*, is strictly increasing and exhibits strict discrete concavity in the domain* $\{\max\{k_t^-, k_t^+\}, ..., n\}$*.*

Proposition 12 reveals that the equilibrium bid-ask spread, as a function of trade size, is strictly increasing and exhibits strict discrete concavity within the domain of trade sizes where both informed purchasing and informed selling are deemed probable by the market-maker. The significance of (discrete) concavity of the bid-ask spread will be made clear in Section 2.5, when we elaborate on the price impacts of different trade sizes. For now, let us focus on the monotonicity of the bid-ask spread. One should be careful in the interpretation of this result. The fact that the largest trade size has the highest bid-ask spread does not necessarily imply that informed traders are most likely to trade in the largest trade size.

Remark 13. *Let* $\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\}$ *be an equilibrium. Also, let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}.

(a) Given $q, \hat{q} \in \{k_t^+, ..., n\}$ *and* $q > \hat{q}$*, if* $\gamma(q)$ *is sufficiently small, then*

$$\psi_t^*(q|V, h_{t-1}, \pi_t^*) < \psi_t^*(\hat{q}|V, h_{t-1}, \pi_t^*).$$

(b) Given $q, \hat{q} \in \{-n, ..., -k_t^-\}$ *and* $|q| > |\hat{q}|$*, if* $\gamma(q)$ *is sufficiently small, then*

$$\psi_t^*(q|0, h_{t-1}, \pi_t^*) < \psi_t^*(\hat{q}|0, h_{t-1}, \pi_t^*).$$

Remark 13 shows that informed traders are more likely to trade in small trade sizes provided that the probability of liquidity trading in relatively larger sizes is sufficiently small. The intuition for this is quite straightforward: informed traders need liquidity traders to camouflage their trades against the market-maker, and if liquidity trading at a given trade size is unlikely to happen, then informed traders prefer trading in other trade sizes. This may at first seem to contradict the monotonicity of bid-ask spreads in trade sizes, as proved in Proposition 12. After all, the bid-ask spreads exist to compensate the market-maker of doing business with informed traders. However, note that the market-maker's compensation not only depends on the probability of facing an informed trader but also the trade size itself: the larger the trade size, the higher the informed traders' total expected profit, hence the market-maker needs a larger spread. Therefore, large spreads associated with large trade sizes should not be necessarily interpreted as an indication of high probability of informed trading in those large sizes.

Another issue of interest regarding bid-ask spreads is their dynamic behavior. In particular, we are interested in the conditions under which the spreads tighten and the conditions under which they widen from one period to another. Proposition 14 lays out a sufficient condition for spreads' tightening over time.

Proposition 14. *Let* $\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\}$ *be an equilibrium and* $h_t = (h_{t-1}, q_t) \in \Omega_n^t$*. Let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*. Also, let* ψ_{t+1}^* *be* k_{t+1}^+ *partially pooling on the long side and* k_{t+1}^- *partially pooling on the short side for history* h_t*. Given* $q \in \{\max\{k_t^-, k_t^+, k_{t+1}^-, k_{t+1}^+\}, ..., n\}$*, if* $q_t \in \{-n, ..., -k_t^-\} \cup \{k_t^+, ..., n\}$ *and* μ *is sufficiently* *large, then*

$$S^*_{t+1}(h_{t-1}, q_t, q) < S^*_t(h_{t-1}, q).$$

Proposition 14 analyzes the spreads for trade sizes where both informed purchasing and informed selling are deemed probable by the market-maker in periods t and $t+1$. It shows that such spreads tighten from one period to another provided that the probability of an informed trader arriving in the market is sufficiently high and an order of a size within the domain of informed trading has been executed in the previous period. The intuition is as follows: From the perspective of the market-maker, a high probability of facing an informed trader and a previous order of a size within the domain of informed trading taken together imply that the previous order most likely comes from an informed trader. This leads the market-maker to substantially revise her beliefs. That, in turn, implies a decrease in the degree of asymmetric information between the market-maker and informed traders. Moreover, the degree of asymmetric information is bound to decrease even further in the upcoming period given the high likelihood of an informed trader arriving in the market once again. Hence the market-maker needs smaller spreads to protect herself against the informed traders, i.e., the spreads tighten.

Proposition 15 provides a sufficient condition under which the spreads widen from one period to another.

Proposition 15. *Let* $\left\{(\pi^*_t, \psi^*_t, \delta^*_t) : t = 1, ..., T\right\}$ *be an equilibrium and* $h_t = (h_{t-1}, q_t) \in \Omega^t_n$. *Let* ψ^*_t *be* k^+_t *partially pooling on the long side and* k^-_t *partially pooling on the short side for history* h_{t-1}. *Also, let* ψ^*_{t+1} *be* k^+_{t+1} *partially pooling on the long side and* k^-_{t+1} *partially pooling on the short side for history* h_t. *Define*

$$A^-_t := \sum_{i=k^-_t}^{n} \gamma(-i), \quad B^-_t := \sum_{i=k^-_t}^{n} i\,\gamma(-i), \quad A^+_t := \sum_{i=k^+_t}^{n} \gamma(i), \quad B^+_t := \sum_{i=k^+_t}^{n} i\,\gamma(i), \quad t = 1, ..., T.$$

Given $q \in \left\{\max\{k^-_t, k^+_t, k^-_{t+1}, k^+_{t+1}\}, ..., n\right\}$,

(a) *if* μ *is sufficiently small,* $q_t \in \{-n, ..., -k^-_t\}$, $\delta^*_{t-1}(h_{t-1}) \geq \frac{A^+_t B^+_{t+1}}{A^+_{t+1} B^+_t}$, *and*

$$|q_t| > \frac{\left(1 - \delta^*_{t-1}(h_{t-1})\right) A^+_t B^-_t \left(A^+_{t+1} B^-_{t+1} - A^-_{t+1} B^+_{t+1}\right)}{A^-_{t+1}\left(\left(1 - \delta^*_{t-1}(h_{t-1})\right) B^-_t A^+_t A^+_{t+1} + A^-_t \left(\delta^*_{t-1}(h_{t-1}) A^+_{t+1} B^+_t - A^+_t B^+_{t+1}\right)\right)}, \quad \text{(12a)}$$

then

$$S^*_{t+1}(h_{t-1}, q_t, q) > S^*_t(h_{t-1}, q);$$

(b) if μ is sufficiently small, $q_t \in \{k^+_t, ..., n\}$, $1 - \delta^*_{t-1}(h_{t-1}) \geq \frac{A^-_t B^-_{t+1}}{A^-_{t+1} B^-_t}$, and

$$q_t > \frac{\delta^*_{t-1}(h_{t-1}) A^-_t B^+_t \left(A^-_{t+1} B^+_{t+1} - B^-_{t+1} A^+_{t+1} \right)}{A^+_{t+1} \left(A^+_t \left((1 - \delta^*_{t-1}(h_{t-1})) A^-_{t+1} B^-_t - A^-_t B^-_{t+1} \right) + \delta^*_{t-1}(h_{t-1}) A^-_t A^-_{t+1} B^+_t \right)}, \quad \text{(12b)}$$

then

$$S^*_{t+1}(h_{t-1}, q_t, q) > S^*_t(h_{t-1}, q).$$

As is the case in Proposition 14, Proposition 15 analyzes the spreads for trade sizes where both informed purchasing and informed selling are deemed probable by the market-maker in periods t and $t + 1$. Observe from Proposition 15 that such bid-ask spreads widen provided that (i) the probability of an informed trader arriving in the market is sufficiently small, (ii) the market-maker assigns a high probability to the risky asset payoff being of low value (high value) in period $t - 1$, and (iii) a sufficiently large sale (purchase) order is executed in period t. The economic intuition behind this observation is as follows: Given that the probability of an informed trader arriving in the market is small, informed traders are better camouflaged by frequent liquidity trading, hence the informed traders are much more likely to submit large orders. Therefore, the market-maker assigning a high probability to the asset payoff being of low value in period $t - 1$ and a large sale order execution in period t taken together imply that the market-maker is almost certain that the payoff is of low value by the beginning of period $t+1$. However, then any period $t+1$ purchase order within the domain of informed trading would lead the market-maker to update her beliefs away from certainty, meaning that the degree of asymmetric information would substantially increase. The expectation of such likely increase in the degree of asymmetric information makes the market-maker post relatively larger spreads in period $t + 1$. In other words, the spreads widen.

In Hasbrouck's (1991) empirical study, it is shown that large trades cause bid-ask spreads to widen. According to our model, large orders do not categorically cause spreads to widen (e.g., spreads may tighten even after large trades if the probability of an informed trader arriving in the market is sufficiently high), however we now have a better understanding of the conditions under which the large orders would trigger spreads to widen.[15] Furthermore, Hasbrouck (1991) notes that the widening in the spreads after a large trade is temporary. This finding can be justified by our model. As large trades are likely to be in the domain of informed trading, they lead to the market-maker updating her posterior belief. Theorem 8 shows that the market-maker gets close to the truth regarding the risky asset payoff after a sufficiently high number of periods. Hence, bid-ask spreads are eventually bound to vanish, as indicated by Proposition 11. Consequently, the widening in the spreads can only be temporary.

2.5. Price impact

The last notion to be examined in our equilibrium analysis is the price impact. Price impact measures the absolute impact of trade size on the risky asset price. Formally, *the period-t price impact of trade $q \in \Omega_n$ for history $h_{t-1} \in \Omega_n^{t-1}$ is given by:*

$$I_t(h_{t-1}, q) \;=\; \left|\pi_t(h_{t-1}, q) - \pi_{t-1}(h_{t-1})\right|. \tag{13}$$

Hasbrouck's (1991) estimates for a sample of NYSE firms suggest that price impact, as a function of trade size, is increasing and concave. Using different datasets and estimation techniques, Algert (1990), Hausman, Lo, and MacKinlay (1992), and Kempf and Korn (1999) also find evidence of a non-linear, concave relationship between price changes and trade size. Their studies suggest that large orders lead to relatively small incremental price changes while small orders lead to relatively large incremental price changes. We provide a theoretical justification for these empirical findings in the following proposition:

Proposition 16. *Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ be an equilibrium. Also, let ψ_t^* be k_t^+ partially pooling on the long side and k_t^- partially pooling on the short side for history h_{t-1}.*

(a) The equilibrium price impact $I_t^*(h_{t-1}, q)$, as a function of trade size $|q|$, is increasing.

(b) The equilibrium price impact $I_t^*(h_{t-1}, q)$, as a function of trade size $|q|$, exhibits strict discrete concavity in the domain $\{-n, ..., -k_t^-\} \cup \{k_t^+, ..., n\}$.

[15] Of course, in a model with re-trading, different justifications can be developed for Hasbrouck's (1991) empirical finding.

This result follows from two basic equilibrium properties: (1) only a transaction in the domain of informed trading leads to a change in the market-maker's posterior belief, hence a change in her price menu, and (2) given the market-maker's price menu, all transactions in the domain of informed trading yield the same expected profit (if a transaction yielded a lower expected profit compared to others, informed traders would not have made that transaction in equilibrium). The first equilibrium property implies that the equilibrium price impact of trade q equals zero if q is outside the domain of informed trading. The second equilibrium property implies that informed traders' expected profit, $(v - \pi_t^*(h_{t-1}, q))\, q$, is same over all trades, q, within the domain of informed trading. Consequently, period-t equilibrium price $\pi_t^*(h_{t-1}, q)$ is proportional to $-\frac{1}{q}$ within the domain of informed trading. This in turn implies that the equilibrium price impact, as a function of trade size $|q|$, is increasing and exhibits discrete concavity.

We should note that the assumption that traders re-trade with probability zero is critical in addressing the implications of price impact concavity. When re-trading is allowed, manipulative trading cannot be ruled out with a concave price impact function. In particular, Huberman and Stanzl (2004) show that, in the presence of re-trading and a non-linear price impact function, a trader can manipulate prices by buying and then selling the same asset, with the expectation of earning a positive profit from such a manipulation. If such manipulation occurs, this seriously puts in question the viability of securities markets since traders can take infinite positions to earn manipulation profits. Of course, markets will stay viable in our model as long as traders are volume-constrained.

3. Concluding remarks

To quote an old Wall Street adage, "It takes volume to move prices." This paper investigates the relationship between trade sizes and the dynamic process of price formation. Following Glosten and Milgrom (1985) and Easley and O'Hara (1987), we assume that the response of asset prices to trading activity is a consequence of asymmetric information. Our theoretical study reveals the following:

1. In each period there is a positive cut-off trade size for the informed trader who observes that the risky asset payoff is of high value. She assigns no probability to purchasing amounts below this trade size

because, even at the price induced by the market-maker's priors, such trades cannot capture her equilibrium information rents. She assigns positive probability to purchasing the cut-off trade size, by definition. In equilibrium any positive trade size that she assigns zero probability to is priced according to the market-maker's priors, so she must assign positive probability to each trade size above the cut-off because otherwise purchasing the cut-off trade size would be suboptimal. The situation is symmetric for the informed trader who observes that the risky asset payoff is of low value. There is a positive least amount that she sells with positive probability, and she assigns positive probability to selling each allowed amount greater than this cut-off.

2. Bid-ask spreads exist only in the trade sizes where informed trading is deemed probable by the market-maker.

3. The cut-off trade sizes decrease following a trade provided that the trade leads to a substantial change in the market-maker's belief. Consequently, the domain of trade sizes, where informed trading is deemed probable by the market-maker, can get bigger over time. Therefore, small trade sizes initially with zero bid-ask spreads can later have positive spreads.

4. The market-maker learns the true risky asset payoff almost surely as the number of trading rounds tends to infinity. Hence, the bid-ask spreads are eventually bound to vanish.

5. The bid-ask spread, as a function of trade size, is strictly increasing and exhibits strict (discrete) concavity within the domain where both informed purchasing and informed selling are deemed probable by the market-maker. Also, the price impact, as a function of trade size, is increasing and exhibits (discrete) concavity. The empirical findings of Algert (1990), Hasbrouck (1991), Hausman, Lo, and MacKinlay (1992), and Kempf and Korn (1999) lend support to the latter result.

6. If the probability of an informed trader arriving in the market is sufficiently high, then the bid-ask spreads tighten over time within the domain where both informed purchasing and informed selling are deemed probable by the market-maker. On the other hand, the bid-ask spreads temporarily widen within the same domain provided that the probability of an informed trader arriving in the market

is sufficiently low and a large order execution has moved the market-maker's posterior belief on the asset payoff towards certainty in the previous period.

There are a number of directions in which our theoretical study can be furthered. One of them is to introduce price discreteness. Numerous empirical studies tackle the dynamics of discrete bid and ask quotes and investigate the impact of tick size reduction (i.e., price decimalization) on market quality.[16] Another possible extension is to allow for re-trading: as in Glosten and Milgrom (1985), traders re-trade with probability zero in our current model. Numerous studies show that there is room for price manipulation when re-trading is allowed (see, e.g., Allen and Gale, 1992; Chakraborty and Yilmaz, 2004; Huberman and Stanzl, 2004).

[16]See, e.g., Goldstein and Kavajecz, 2000; Harris, 1991, 1994; Hasbrouck, 1999.

Appendix

Proof of Proposition 1. (a) Suppose $\psi_t^*(0|V, h_{t-1}, \pi_t^*) > 0$ for some $t \geq 1$ and $h_{t-1} \in \Omega_n^{t-1}$. Then we have:

$$0 \;\geq\; (\,V - \pi_t^*(h_{t-1}, q))\, q \;=\; \delta_t^*(h_{t-1}, q)\, V\, q, \quad \forall q \in \Omega_n. \tag{14}$$

Following (1), $0 < \delta_t^*(h_{t-1}, q) < 1$ as the probability of liquidity trading is positive over all trade sizes. Therefore inequality (14) fails to hold when $q \in \Omega_n^+$. This proves that $\psi_t^*(0|V, h_{t-1}, \pi_t^*) = 0$ for all $t \geq 1$ and $h_{t-1} \in \Omega_n^{t-1}$. In a similar fashion, it can be easily proved that $\psi_t^*(0|0, h_{t-1}, \pi_t^*) = 0$ for all $t \geq 1$ and $h_{t-1} \in \Omega_n^{t-1}$.

(b) Suppose there exist $t \geq 1$ and $h_{t-1} \in \Omega_n^{t-1}$ such that $\psi_t^*(q|V, h_{t-1}, \pi_t^*) > 0$ for some $q \in \Omega_n^-$. Then it must hold that:

$$(V - \pi_t^*(h_{t-1}, q))\, q \geq 0. \tag{15}$$

However, since $0 < \delta_t^*(h_{t-1}, q) < 1$ and $q \in \Omega_n^-$,

$$(V - \pi_t^*(h_{t-1}, q))\, q \;=\; \delta_t^*(h_{t-1}, q)\, V\, q \;<\; 0.$$

This contradicts with (15).

(c) The proof is similar to that of (b). □

Proof of Theorem 2. Suppose there exists an equilibrium $\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\}$ such that for some $t \geq 1$, $h_{t-1} \in \Omega_n^{t-1}$, and $i, j \in \Omega_n^+$ with $i > j$

$$\psi_t^*(j|V, h_{t-1}, \pi_t^*) > 0, \qquad \psi_t^*(i|V, h_{t-1}, \pi_t^*) = 0.$$

Then it must hold that:

$$(V - \pi_t^*(h_{t-1}, j))\, j \;\geq\; (V - \pi_t^*(h_{t-1}, i))\, i.$$

This in turn implies together with (1), (2), and Proposition 1 that:

$$\frac{j}{i} \;\geq\; \frac{V - \pi_t^*(h_{t-1}, i)}{V - \pi_t^*(h_{t-1}, j)} \;=\; 1 + \frac{(1 - \delta_t^*(h_{t-1}, j))\, \psi_t^*(j|V, h_{t-1}, \pi_t^*)\, \mu}{(1 - \mu)\gamma(j)}. \tag{16}$$

Since $\frac{j}{i} < 1$ and $\psi_t^*(j|V, h_{t-1}, \pi_t^*) > 0$ by assumption, (16) fails to hold. This proves that there exists $k_t^+(h_{t-1}) \in \Omega_n^+$ such that $\mathrm{supp}\{\psi_t^*(V, h_{t-1}, \pi_t^*)\} = \{k_t^+(h_{t-1}), \cdots, n\}$. In a similar fashion, it can be proved that $\mathrm{supp}\{\psi_t^*(0, h_{t-1}, \pi_t^*)\} = \{-n, \cdots, -k_t^-(h_{t-1})\}$ for some $k_t^-(h_{t-1}) \in \Omega_n^-$. $\square$

Some of the proofs in the rest of the Appendix are based on the following lemma.

Lemma 3. *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ *be an equilibrium. Also, let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*. Then*

$$\psi_t^*(q|V, h_{t-1}, \pi_t^*) = \begin{cases} 0 & : \quad q \in \{1, \cdots, k_t^+ - 1\} \\ \dfrac{(1-\mu)\sum_{i=k_t^+}^n \left(1-\frac{i}{q}\right)\gamma(i) + \left(1-\delta_{t-1}^*(h_{t-1})\right)\mu}{\left(1-\delta_{t-1}^*(h_{t-1})\right)\mu \sum_{i=k_t^+}^n \frac{i\gamma(i)}{q\gamma(q)}} & : \quad q \in \{k_t^+, \cdots, n\} \end{cases} \tag{17a}$$

and

$$\psi_t^*(q|0, h_{t-1}, \pi_t^*) = \begin{cases} 0 & : \quad q \in \{-k_t^- + 1, \cdots, -1\} \\ \dfrac{(1-\mu)\sum_{i=k_t^-}^n \left(1-\frac{i}{|q|}\right)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu}{\delta_{t-1}^*(h_{t-1})\mu \sum_{i=k_t^-}^n \frac{i\,\gamma(-i)}{|q|\,\gamma(q)}} & : \quad q \in \{-n, \cdots, k_t^-\}. \end{cases} \tag{17b}$$

Proof of Lemma 3. As ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1}, $\psi_t^*(q|V, h_{t-1}, \pi_t^*) = 0$ for $q \in \{1, \cdots, k_t^+ - 1\}$. Now let $q \in \{k_t^+, ..., n\}$. The equilibrium definition imposes that:

$$q[V - \pi_t^*(h_{t-1}, q)] \;=\; i[V - \pi_t^*(h_{t-1}, i)], \quad \forall i \in \{k_t^+, ..., n\}, \tag{18a}$$

$$q[V - \pi_t^*(h_{t-1}, q)] \;\geq\; i[V - \pi_t^*(h_{t-1}, i)], \quad \forall i \in \{0, 1, \cdots, k_t^+ - 1\}. \tag{18b}$$

Following (2) and (18a),

$$\frac{i}{q} \;=\; \frac{\delta_t^*(h_{t-1}, q)}{\delta_t^*(h_{t-1}, i)}, \quad \forall i \in \{k_t^+, ..., n\}.$$

The equation above, (1), and Proposition 1 together imply that for $i \in \{k_t^+, ..., n\}$

$$\left(1 - \delta_{t-1}^*(h_{t-1})\right)\mu \left[\psi_t^*(q|V, h_{t-1}, \pi_t^*)\frac{i\gamma(i)}{q\gamma(q)} - \psi_t^*(i|V, h_{t-1}, \pi_t^*)\right] = (1-\mu)\left(1 - \frac{i}{q}\right)\gamma(i). \tag{19}$$

Summing the left- and right- hand sides of (19) over $i \in \{k_t^+, ..., n\} \setminus \{q\}$, we obtain:

$$\begin{aligned} &\left(1 - \delta_{t-1}^*(h_{t-1})\right)\mu \left[\psi_t^*(q|V, h_{t-1}, \pi_t^*) \sum_{i=k_t^+, i\neq q}^n \frac{i\gamma(i)}{q\gamma(q)} - \sum_{i=k_t^+, i\neq q}^n \psi_t^*(i|V, h_{t-1}, \pi_t^*)\right] \\ =\;\; &(1-\mu) \sum_{i=k_t^+, i\neq q}^n (1 - \frac{i}{q})\gamma(i). \end{aligned} \tag{20}$$

Replacing $\sum_{i=k,i\neq q}^{n}\psi_t^*(i|V,h_{t-1},\pi_t^*)$ with $1-\psi_t^*(q|V,h_{t-1},\pi_t^*)$ and rearranging the terms in (20) yields:

$$\psi_t^*(q|V,h_{t-1},\pi_t^*) \;=\; \frac{(1-\mu)\displaystyle\sum_{i=k_t^+}^{n}\left(1-\frac{i}{q}\right)\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu}{(1-\delta_{t-1}^*(h_{t-1}))\mu\displaystyle\sum_{i=k_t^+}^{n}\frac{i\gamma(i)}{q\gamma(q)}}.$$

Hence, Equation (17a) is obtained. Equation (17b) can be obtained in a similar fashion. □

Proof of Proposition 3. (a) Suppose ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1}. This means $\psi_t^*(q|V,h_{t-1},\pi_t^*)>0$ for $q\in\{k_t^+,...,n\}$ and $\psi_t^*(q|V,h_{t-1},\pi_t^*)=0$ for $q\notin\{k_t^+,...,n\}$. Hence, following Lemma 3, the inequalities (4a) and (4b) must hold.

Now suppose the inequalities (4a) and (4b) hold. Following Theorem 2, there exists some K such that ψ_t^* is K partially pooling on the long side for history h_{t-1}. From Lemma 3, we have:

$$\psi_t^*(q|V,h_{t-1},\pi_t^*)=\begin{cases}0 & : \quad q\in\{1,\cdots,K-1\}\\ \dfrac{(1-\mu)\sum_{i=K}^{n}\left(1-\frac{i}{q}\right)\gamma(i)+\left(1-\delta_{t-1}^*(h_{t-1})\right)\mu}{\left(1-\delta_{t-1}^*(h_{t-1})\right)\mu\sum_{i=K}^{n}\frac{i\gamma(i)}{q\gamma(q)}} & : \quad q\in\{K,\cdots,n\}\end{cases}$$

As ψ_t^* is K partially pooling on the long side for h_{t-1}, it must be true that:

$$\begin{aligned}(1-\mu)\sum_{i=K}^{n}\left(1-\frac{i}{K}\right)\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu \quad &> \quad 0,\\ (1-\mu)\sum_{i=K-1}^{n}\left(1-\frac{i}{K-1}\right)\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu \quad &\leq \quad 0.\end{aligned}$$

Since the inequalities (4a) and (4b) hold, we have $k_t^+=K$. This proves that ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1}.

(b) The proof is similar to that of (a). □

Proof of Lemma 1. Suppose to the contrary that there exists no $k_t^+\in\Omega_n^+$ that satisfies (4a)-(4b). Then either

$$(1-\mu)\sum_{i=q}^{n}\left(1-\frac{i}{q}\right)\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu \;\leq\; 0, \quad \forall q\in\{1,...,n\}, \tag{21}$$

or

$$(1-\mu)\sum_{i=q-1}^{n}\left(1-\frac{i}{q-1}\right)\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu \;>\; 0, \quad \forall q\in\{1,...,n\}. \tag{22}$$

Inequality (21) cannot hold since:

$$(1-\mu)\sum_{i=n}^{n}\left(1-\frac{i}{q}\right)\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\mu \;=\; (1-\delta^*_{t-1}(h_{t-1}))\mu \;>\; 0.$$

Also,

$$(1-\mu)\sum_{i=0}^{n}\left(1-\frac{i}{0}\right)\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\mu \;=\; -\infty,$$

and hence inequality (22) cannot hold, either. Therefore inequalities (4a)-(4b) must be satisfied by some $k_t^+ \in \Omega_n^+$.

Similar arguments imply that inequalities (4c)-(4d) are satisfied by some $k_t^- \in \Omega_n^+$. □

Proof of Lemma 2. Suppose to the contrary that inequalities (4a)-(4b) are satisfied by both $q \in \Omega_n^+$ and $q' \in \Omega_n^+$. Without loss of generality, assume that $q > q'$. Then:

$$\begin{aligned}
& (1-\mu)\sum_{i=q'}^{n}\left(1-\frac{i}{q'}\right)\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\mu \\
\le\; & (1-\mu)\sum_{i=q-1}^{n}\left(1-\frac{i}{q'}\right)\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\mu \\
\le\; & (1-\mu)\sum_{i=q-1}^{n}\left(1-\frac{i}{q-1}\right)\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\mu \\
\le\; & 0,
\end{aligned} \tag{23}$$

where the last inequality follows from the assumption that q satisfies (4b). However (23) violates the assumption that q' satisfies (4a). This proves that inequalities (4a)-(4b) must be satisfied, if at all, by a unique $k_t^+ \in \Omega_n^+$.

Similar arguments imply that inequalities (4c)-(4d) are satisfied, if at all, by a unique $k_t^- \in \Omega_n^+$. □

Proof of Proposition 7. (a) Let (10a) hold and q_t be from the domain $\{k_t^+, ..., n\}$. Assume to the contrary that $k_{t+1} \ge k_t$. Since ψ^*_{t+1} is k_{t+1} partially pooling on the long side for history $h_t = (h_{t-1}, q_t)$, from Proposition 3 we have:

$$(1-\mu)\sum_{i=k^+_{t+1}-1}^{n}\left(1-\frac{i}{k^+_{t+1}-1}\right)\gamma(i)+(1-\delta^*_{t}(h_{t-1},q_t))\mu \;\le\; 0. \tag{24}$$

As $k_{t+1} \geq k_t$, from (24) we obtain:

$$(1-\mu)\sum_{i=k_t^+-1}^{n}\left(1-\frac{i}{k_t^+-1}\right)\gamma(i)+(1-\delta_t^*(h_{t-1},q_t))\mu \ \leq \ 0. \tag{25}$$

Since ψ_t^* is k_t^+ partially pooling on the long side for h_{t-1}, by Proposition 3, the inequalities (4a)- (4b) also hold. (4a) and (25) together yield:

$$(1-\mu)\left[\sum_{i=k_t^+}^{n}\left(1-\frac{i}{k_t^+}\right)\gamma(i)-\sum_{i=k_t^+-1}^{n}\left(1-\frac{i}{k_t^+-1}\right)\gamma(i)\right] \ > \ (\delta_{t-1}^*(h_{t-1})-\delta_t^*(h_{t-1},q_t))\mu.$$

Given that $q_t \in \{k_t^+, ..., n\}$, The inequality above, (1), and Proposition 1 imply:

$$\begin{aligned} \frac{(1-\mu)}{\mu\, k_t^+(k_t^+-1)}\sum_{i=k_t^+}^{n} i\,\gamma(i) \quad &> \quad (\delta_{t-1}^*(h_{t-1})-\delta_t^*(h_{t-1},q_t)) \\ &= \quad \delta_{t-1}^*\left(\frac{(1-\mu)\sum_{i=k_t^+}^{n}\left(1-\frac{i}{q_t}\right)\gamma(i)+(1-\delta_{t-1}(h_{t-1}))\mu}{(1-\mu)\sum_{i=k_t^+}^{n}\gamma(i)+(1-\delta_{t-1}(h_{t-1}))\mu}\right). \end{aligned} \tag{26}$$

(26) contradicts (10a). Therefore, it must hold that $k_{t+1} < k_t$.

(b) The proof is similar to that of (a). □

Proof of Theorem 8. Let $\mathcal{H}_\infty$ denote the sigma field generated by all the possible histories h_∞. We consider the market-maker's equilibrium belief as a stochastic process, which we denote by $\{\delta_t : t = 1, ..., T\}$. Following a theorem in Fristedt and Gray (1996, Theorem 3, p. 432), there exists a unique distribution over δ_t conditional on $(\delta_0, \cdots, \delta_{t-1})$. Therefore, the collection of probability distributions $\{\delta_t : t = 1, ..., T\}$ and the probability space $(\Omega_n^\infty, \mathcal{H}_\infty, \mathcal{P})$ are well-defined. Notice that δ_t is a martingale with respect to the market-maker's information set. By the martingale convergence theorem, δ_t converges almost surely to a random variable $\hat{\delta}$. Next we prove that $\hat{\delta} = 0$ if $v = V$ and $\hat{\delta} = 1$ if $v = 0$. Let $v = V$ and suppose to the contrary that there exists a period τ and histories h'_t such that for all $t \geq \tau$

$$\Pr(h'_t : |\delta_t(h'_t) - p| > \epsilon) \ = \ 0 \tag{27}$$

for some $p \in (0, 1]$ and arbitrary small ϵ. Following Theorem 2, for all $t \geq \tau$ there exists some k_t^+ such that informed traders' equilibrium trading strategy ψ_t^* is k_t^+ partially pooling on the long side for history

h'_{t-1}. By (1), if a trade larger than $k^+_{\tau+1}(h'_\tau)$ realizes in period-$(\tau+1)$, the market-maker's belief $\delta_{\tau+1}$ deviates from the interval of $[p-\epsilon, p+\epsilon]$. Formally,

$$\Pr(h'_{\tau+1} : |\delta_{\tau+1}(h'_{\tau+1}) - p| > \epsilon) \;=\; \Pr(h'_\tau : |\delta_\tau(h'_\tau) - p| \leq \epsilon) \left(\mu + \sum_{i=k^+_\tau(h'_\tau)}^{n} \gamma(i)\right) \;>\; 0,$$

which contradicts with (27). Therefore, it must hold that $\hat{\delta} = 0$. It can be similarly shown that $\hat{\delta} = 1$ if $v = 0$. □

Proof of Proposition 9. From (1), (2), and Lemma 3, we have:

$$\begin{aligned} S^*_t(h_{t-1}, q) \;&=\; \qquad (28) \\ =\;& (\delta_t(h_{t-1}, -q) - \delta_t(h_{t-1}, q))\, V \\ =\;& \delta^*_{t-1}(h_{t-1})\, V \\ & \left[\frac{\mu\, \psi^*_t(-q|0,h_{t-1},\pi^*_t)+(1-\mu)\gamma(-q)}{\delta^*_{t-1}(h_{t-1})\, \mu\, \psi^*_t(-q|0,h_{t-1},\pi^*_t)+(1-\mu)\gamma(-q)} - \frac{(1-\mu)\gamma(q)}{\mu\,(1-\delta^*_{t-1}(h_{t-1}))\, \psi^*_t(q|V,h_{t-1},\pi^*_t)+(1-\mu)\gamma(q)}\right] \\ =\;& \delta^*_{t-1}(h_{t-1})\, (1-\delta^*_{t-1}(h_{t-1}))\, \mu\, V \\ & \frac{(1-\mu)[\psi^*_t(q|V,h_{t-1},\pi^*_t)\gamma(-q)+\psi^*_t(-q|0,h_{t-1},\pi^*_t)\gamma(q)]+\mu\, \psi^*_t(q|V,h_{t-1},\pi^*_t)\, \psi^*_t(-q|0,h_{t-1},\pi^*_t)}{\big((1-\delta^*_{t-1}(h_{t-1}))\, \mu\, \psi^*_t(q|V,h_{t-1},\pi^*_t)+(1-\mu)\, \gamma(q)\big)\big(\delta^*_{t-1}(h_{t-1})\, \mu\, \psi^*_t(-q|0,h_{t-1},\pi^*_t)+(1-\mu)\, \gamma(-q)\big)}. \end{aligned}$$

If $1 \leq q < \min\{k^-_t, k^+_t\}$, then $\psi^*_t(q|V, h_{t-1}, \pi^*_t) = \psi^*_t(-q|0, h_{t-1}, \pi^*_t) = 0$ and consequently (28) yields $S^*_t(h_{t-1}, q) = 0$. If $\min\{k^-_t, k^+_t\} \leq q \leq n$, then $\max\{\psi^*_t(q|V, h_{t-1}, \pi^*_t), \psi^*_t(-q|0, h_{t-1}, \pi^*_t)\} > 0$; hence (28) yields $S^*_t(h_{t-1}, q) > 0$.

On the other hand, if $S^*_t(h_{t-1}, q) = 0$, then following (28) it must hold that $\psi^*_t(q|V, h_{t-1}, \pi^*_t) = \psi^*_t(-q|0, h_{t-1}, \pi^*_t) = 0$ and this together with Lemma 3 implies $1 \leq q < \min\{k^-_t, k^+_t\}$. If $S^*_t(h_{t-1}, q) > 0$, then following (28) it must hold that $\max\{\psi^*_t(q|V, h_{t-1}, \pi^*_t), \psi^*_t(-q|0, h_{t-1}, \pi^*_t)\} > 0$, which together with Lemma 3 implies $\min\{k^-_t, k^+_t\} \leq q \leq n$. □

Proof of Proposition 11. The result immediately follows from (28) and Theorem 8. □

Proof of Proposition 12. From Lemma 3 and (28), we have:

$$S^*_t(h_{t-1}, q) \;=\; \delta^*_{t-1}(h_{t-1})\, V \left[\frac{1-\delta^*_{t-1}(h_{t-1})}{\delta^*_{t-1}(h_{t-1})}\, \frac{(1-\mu)\sum_{i=k^-_t}^{n}\left(1-\frac{i}{q}\right)\gamma(-i)+\delta^*_{t-1}(h_{t-1})\,\mu}{(1-\mu)\sum_{i=k^-_t}^{n}\gamma(-i)+\delta^*_{t-1}(h_{t-1})\,\mu} + \frac{(1-\mu)\sum_{i=k^+_t}^{n}\left(1-\frac{i}{q}\right)\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\,\mu}{(1-\mu)\sum_{i=k^+_t}^{n}\gamma(i)+(1-\delta^*_{t-1}(h_{t-1}))\,\mu}\right] \qquad (29)$$

whenever $q \geq \max\{k_t^-, k_t^+\}$. The result immediately follows from (29). □

Proof of Remark 13. From Lemma (3), we have:

$$\lim_{\gamma(q)\to 0} \psi_t^*(q|V, h_{t-1}, \pi_t^*) = 0 \quad \forall q \in \{k_t^+, ..., n\}, \tag{30a}$$

$$\lim_{\gamma(q)\to 0} \psi_t^*(q|0, h_{t-1}, \pi_t^*) = 0 \quad \forall q \in \{-n, ..., -k_t^-\}. \tag{30b}$$

Results listed in (a) and (b) of the remark immediately follow from (30a) and (30b), respectively. □

Proof of Proposition 14. Define

$$A_t^- := \sum_{i=k_t^-}^{n} \gamma(-i), \quad B_t^- := \sum_{i=k_t^-}^{n} i\,\gamma(-i), \quad A_t^+ := \sum_{i=k_t^+}^{n} \gamma(i), \quad B_t^+ := \sum_{i=k_t^+}^{n} i\,\gamma(i), \quad t = 1, ..., T.$$

Let $q \in \big\{\max\{k_t^-, k_t^+, k_{t+1}^-, k_{t+1}^+\}, ..., n\big\}$. Also, let $q_t \in \{-n, ..., -k_t^-\}$. It follows from (29), (1), and Lemma 3 that:

$$S_t^*(h_{t-1}, q) = V\Bigg((1-\delta_{t-1}^*(h_{t-1}))\,\frac{(1-\mu)A_t^- - \frac{(1-\mu)B_t^-}{q} + \delta_{t-1}^*(h_{t-1})\mu}{(1-\mu)A_t^- + \delta_{t-1}^*(h_{t-1})\mu} + \delta_{t-1}^*(h_{t-1})\,\frac{(1-\mu)A_t^+ - \frac{(1-\mu)B_t^+}{q} + (1-\delta_{t-1}^*(h_{t-1}))\mu}{(1-\mu)A_t^+ + (1-\delta_{t-1}^*(h_{t-1}))\mu}\Bigg), \tag{31a}$$

$$\begin{aligned} S_{t+1}^*(h_{t-1}, q_t, q) = V \Bigg(& \frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)B_t^-}{|q_t|((1-\mu)A_t^- + \delta_{t-1}^*(h_{t-1})\mu)}\, \frac{(1-\mu)A_{t+1}^- - \frac{(1-\mu)B_{t+1}^-}{q} + \left(1 - \frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)B_t^-}{|q_t|((1-\mu)A_t^- + \delta_{t-1}^*(h_{t-1})\mu)}\right)\mu}{(1-\mu)A_{t+1}^- + \left(1 - \frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)B_t^-}{|q_t|((1-\mu)A_t^- + \delta_{t-1}^*(h_{t-1})\mu)}\right)\mu} \\ & + \left(1 - \frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)B_t^-}{|q_t|((1-\mu)A_t^- + \delta_{t-1}^*(h_{t-1})\mu)}\right) \frac{(1-\mu)A_{t+1}^+ - \frac{(1-\mu)B_{t+1}^+}{q} + \left(\frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)B_t^-}{|q_t|((1-\mu)A_t^- + \delta_{t-1}^*(h_{t-1})\mu)}\right)\mu}{(1-\mu)A_{t+1}^+ + \left(\frac{(1-\delta_{t-1}^*(h_{t-1}))(1-\mu)B_t^-}{|q_t|((1-\mu)A_t^- + \delta_{t-1}^*(h_{t-1})\mu)}\right)\mu}\Bigg). \end{aligned} \tag{31b}$$

Using (31a)-(31b) and L'Hôpital's Rule, we obtain:

$$\lim_{\mu\to 1} \big(S_{t+1}^*(h_{t-1}, q_t, q) - S_t^*(h_{t-1}, q)\big) = -V\,\frac{\frac{B_{t+1}^+}{q}}{A_{t+1}^+ + \frac{(1-\delta_{t-1}^*(h_{t-1}))B_t^-}{|q_t|\delta_{t-1}^*(h_{t-1})}} < 0.$$

Therefore $S_{t+1}^*(h_{t-1}, q_t, q) < S_t^*(h_{t-1}, q)$ for sufficiently large μ.

The proof is similar for $q_t \in \{k_t^+, ..., n\}$. □

Proof of Proposition 15. Let $q \in \left\{\max\{k_t^-, k_t^+, k_{t+1}^-, k_{t+1}^+\}, ..., n\right\}$. Also, let $q_t \in \{-n, ..., -k_t^-\}$. Equations (31a) and (31b) imply that:

$$\lim_{\mu \to 0} \left(S_{t+1}^*(h_{t-1}, q_t, q) - S_t^*(h_{t-1}, q)\right) =$$
$$= V \, \frac{\left(1-\delta_{t-1}^*(h_{t-1})\right) A_t^+ B_t^- \left(A_{t+1}^- B_{t+1}^+ - A_{t+1}^+ B_{t+1}^-\right) + |q_t| \left(\left(1-\delta_{t-1}^*(h_{t-1})\right) A_{t+1}^- B_t^- A_t^+ A_{t+1}^+ + A_t^- A_{t+1}^- \left(\delta_{t-1}^*(h_{t-1}) A_{t+1}^+ B_t^+ - A_t^+ B_{t+1}^+\right)\right)}{q |q_t| A_t^- A_{t+1}^- A_t^+ A_{t+1}^+}.$$

Therefore $S_{t+1}^*(h_{t-1}, q_t, q) > S_t^*(h_{t-1}, q)$ for sufficiently small μ provided that $\delta_{t-1}^*(h_{t-1}) \geq \frac{A_t^+ B_{t+1}^+}{A_{t+1}^+ B_t^+}$ and (12a) holds. This proves (a).

The proof for (b) is similar to that of (a). □

Proof of Proposition 16. Using (1), (2), and Lemma 3, we derive the following:

· if $q \notin \{-n, ..., -k_t^-\} \cup \{k_t^+, ..., n\}$, then:

$$I_t^*(h_{t-1}, q) = 0; \tag{32a}$$

· if $q \in \{k_t^+, ..., n\}$, then:

$$I_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1}) \, V \left(1 - \frac{(1-\mu) \sum_{i=k_t^+}^n i \, \gamma(i)}{q \left[(1-\mu) \sum_{i=k_t^+}^n \gamma(i) + (1-\delta_{t-1}^*(h_{t-1}))\mu\right]}\right); \tag{32b}$$

· if $q \in \{-n, ..., -k_t^-\}$, then:

$$I_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1}) \, V \tag{32c}$$
$$\left(\frac{(1-\mu)\left((1-\delta_{t-1}(h_{t-1})) \sum_{i=k_t^-}^n \left(1-\frac{i}{|q|}\right)\gamma(-i) + \delta_{t-1}(h_{t-1}) \sum_{i=k_t^-}^n \gamma(-i)\right) + \delta_{t-1}(h_{t-1}) \, \mu}{\delta_{t-1}(h_{t-1}) \left((1-\mu) \sum_{i=k_t^-}^n \gamma(-i) + \delta_{t-1}(h_{t-1}) \, \mu\right)} - 1\right).$$

The results immediately follow from (32a), (32b), and (32c). □

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