The Black Market for Currencies: Theory and Evidence*

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Abstract

This paper proposes a tractable model of the black market for currencies where the black market premium on foreign currency arises endogenously and depends on the relative inflation rates of domestic and foreign currencies. Experience from countries like Argentina, Zimbabwe etc. suggests that higher domestic inflation is associated with higher rates of black market premium on foreign currency. I analyze data from Iran, Venezuela and four South Asian economies between 1981-1997 using a panel VAR and fixed effect panel regression and find a negative association between the two variables. The pattern is also noticed in case of India when studied separately. Using a New Monetarist framework, I offer a plausible explanation as to why this association could be negative. The black market is modeled as a market of currency exchange that can be used by buyers of one country to readjust their portfolio when access to the official market is infrequent and after the realization of a shock that forces them to either consume local or foreign goods. I show that after allowing for currency substitution in the portfolio choice problem of the agents, in the stationary monetary equilibrium

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the rate of black market premium could be decreasing in the domestic inflation rate if agents are not very risk averse. Else, if agents are sufficiently risk averse, then the black market premium is increasing in domestic inflation.

1 Introduction

This paper builds a model explaining the origins of black markets for currencies; the black market premium arises endogenously and depends on relative inflation rates of domestic and foreign currencies. In particular I analyze the case of Iran, Venezuela and the four South Asian economies of India, Pakistan, Nepal, Sri Lanka between 1981-1997. Conventional wisdom suggests that countries with higher relative domestic inflation rates will have higher rates of black market premium. However, after analyzing data from these six countries between 1981-1997, I find a negative association between black market premium rates and domestic inflation (relative to US inflation). This behavior is intriguing and might appear anomalous. Using the New Monetarist framework I offer a possible explanation as to why this association could be negative. Defining exchange rate as the price of foreign currency in local currency, the black market premium rate is percentage deviation of the black market exchange rate from the official market’s exchange rate.

The black market for currency exchange is another market of currency exchange that exists alongside the official, legally recognized currency exchange market which includes banks, licensed financial institutions, authorized money changers, and certain websites registered for this purpose. It is a monetary phenomenon observed more often in emerging market economies (EMEs) and less developed countries (LDCs). Nevertheless, developed countries, like Iceland during the crisis of 2008-2011, have witnessed rise in black market activities as well, albeit for a brief period. A key feature of the black market for currencies, like any other black market is that it is illegal. The market for currency exchange is part of a country’s financial system and dealing in foreign currencies often require sanction from the authorities who specify the channels, times, and platforms through which such trades can be carried out. This is especially true for EMEs and LDCs which have a greater degree of restrictions on the financial openness of their countries due to capital controls, stability concerns, concern about illegal economic activities, terrorism etc. This paper does not study the purpose of currency restrictions; instead it takes these as given and studies the properties of black market trade. Despite these restrictions, due to imperfect monitoring illegal trade in currencies thrive. Due to this not-so-legal status of the black market, it operates on a
relatively smaller scale than its official counterpart. It is often location specific, transactions are carried out mostly in cash, leave no paper trails and carries some degree of risk. The risk comes both from the possibility of confiscation by government authorities and from the possibility of being swindled or robbed. For example, since it is unregulated, in the black market, someone could be sold fake currency notes or get robbed in a narrow alleyway. In countries where a black market for currencies exists, such trades are ubiquitous in places like convenience stores, border crossings, it involves people with stacks of US dollars on the corner of a street, or perhaps a worker at an international airport who runs a side business of buying/ selling foreign currencies etc. In Libya, a country which has recently witnessed a rise in black market trade in currencies, it has been carried out mostly in the gold souks of the major cities. Until recently, in Argentina, the participants in black market for currencies included taxi drivers, small stores selling essential commodities, as well as the arbolitos, people who stand planted for hours in one spot and offer handfuls of US dollars to everyone who passes. In India, the black market has traditionally consisted of a network of unlicensed money changers, some businesses with foreign connections, foreign travelers and households who hold foreign currency\(^1\). This pattern contrasts sharply with the officially recognized currency exchange which consists of banks and legitimate financial intermediaries (including websites), that have greater visibility, advertise the rates they offer through various channels and are required to issue receipt and register every transaction they make.

While its unregulated nature might attract individuals involved in criminal activities (like drugs, arms trafficking etc.) to exchange monies, it is not the only purpose the black market serves. In fact, currency circulating in the black market could have been acquired through official channels. Goldberg (1995) considers a model with leakage of foreign currency into the black market for the case of Russia. A more recent example will be that of Libya’s, where one can get a letter of authorization from the central bank to acquire foreign currency to purchase goods abroad with the intention of selling them domestically. Sellers of foreign currency, either posing as traders or through their connections in the central bank acquire such authorization and due to lack of proper verification, sell the foreign currency in the black market instead of buying goods from abroad. This story has its parallels in other parts of the world. In India, laws specify how much foreign currency a resident can buy legally for foreign trips and how much foreign currency can they bring in and retain after a foreign trip. However, due to imperfect monitoring there is leakage from the official market to the

\(^{1}\text{In India it is illegal to hold foreign currency without explicit authorization which can be obtained only in cases of foreign travel or by licensed exporters and importers}\)
black market. The association between inflation rates and the black market premium rate presented in this paper hints at a currency portfolio choice by economic agents. Indian rupee has been a weak currency for more than half a century. At the time of independence, in the absence of sufficient investment opportunities in interest bearing assets, it was not uncommon for the urban wealthier sections of South Asian society to hold their wealth in a combination of rupees, gold and the British pound. This sort of portfolio diversification was a response to the relatively higher cost of holding wealth only in rupees. Post-independence, following global trends the preference shifted from British pound to US dollars, but the portfolio diversification pattern has persisted among certain sections of Indian society. Although India and other South Asian countries have not witnessed dollarization in means of payment inside the country, the dollarization of portfolios is not uncommon. Although, in recent times, with increased access to capital markets and an improved economy, activity in the currency black market seems to have abated in India, if newspaper reports are to be believed it is still active in other South Asian countries like Pakistan and Nepal.

In this paper, I argue that one of the reasons behind the existence of black market is that, in presence of limited or infrequent access to official currency exchange facilities, the black market provides agents with an opportunity to sell their foreign currency to finance domestic consumption which must be paid in local currency units. It also provides the opportunity to exchange local currency for foreign currency when agents need to purchase goods from abroad, in the absence official channels. Agents tend to diversify their portfolio for two reasons: (i) to beat the inflation cost of a particular currency and (ii) in anticipation of the opportunity to consume foreign-produced goods or domestically produced goods. Once agents learn what sort of goods they are going to consume, they can access the black market.

\[2\text{Indian laws allow foreign currency worth USD 25,000 to be acquired legally from banks by furnishing proof of foreign business trips which cover a broad category of trips including attending international conference, seminar, specialized training, study tour, apprentice training etc. Indian laws also allow up to USD 100,000 worth of foreign currency to be bought by resident Indians for medical treatment abroad on self declaration basis of essential details, without insisting on any estimate from a medical service provider in India or abroad. However, how much of this is actually taken out of the country and how much left behind is not stringently monitored nor do the banks strictly verify the validity of such business/ medical trips. On the other hand, for travelers returning to India after a foreign trip there is no upper limit to the amount of foreign currency they can bring into the country as long as they file a customs declaration form. On return from a foreign trip travelers are required to surrender unspent foreign exchange held in the form of currency notes within 90 days of return. However, they are free to retain foreign exchange up to USD 2,000 for future use. While most people spend the amount they acquire from the authorized dealers or surrender excess foreign currency they bring with them, there are instances of leakage into the black market. On a personal note, the author has spent significant time in India and has had the opportunity to come across certain agents who work as intermediaries in the black market for foreign currency. Information about the sources of foreign currency comes from informal talks with them.} \]
in order to convert their local currency into foreign currency or vice versa. The black market for currency exchange is not “parallel” in the sense that it functions simultaneously alongside the official exchange market and agents make a choice to visit one or the other. Instead, as discussed in the examples, these two markets are sequential and agents access both. The black market provides liquidity in presence of frictions such as the timing of the official market and the time at which consumption (foreign or domestic) shocks are realized. The timing issue of the official exchange market is valid for many EMEs and LDCs where due to foreign exchange arrangements and restrictions, money changing services offered through official outlets like banks and other authorized financial intermediaries are not available at all times.

In this paper, I use a two-country, two-currency version of the Lagos and Wright (2005) model where, following the traditional international macroeconomics literature as well as Zhang (2014), I model the official currency exchange market as a frictionless spot Walrasian market in the second subperiod. The black market for currency exchange is another spot Walrasian market embedded in the first subperiod which is accessible by agents from one of the countries. The first subperiod also consists of two decentralized markets for special goods – one for each country. In the second subperiod, all agents produce and consume a consumption good and choose their portfolio of the two monies for the next time period. At the beginning of the first subperiod there are a set of agents called ‘buyers’ who only consume, but do not produce in this period, receive an idiosyncratic shock that matches them with a foreign seller or a domestic seller. ‘Sellers’ are another set of agents who produce a special good, but does not consume in the first subperiod. The labels ‘buyers’ and ‘sellers remain unchanged over periods. Buyers make take-it-or-leave-it (TIOLI) offers to the seller in the special goods market.

In the second subperiod buyers make an ex ante portfolio choice and choose an optimal real portfolio consisting of both monies. When the shock is realized in the next period, a buyers who can access the black market would want to convert their portfolio into sellers currency as the seller only accepts that currency. In the stationary monetary equilibrium buyers’ portfolio would consist more of the currency that has lower inflation. In the black market the relative supply of the two monies determine what the black market premium rate would be. Since the focus is on stationary monetary equilibrium, inflation rates are constant and agents would build this into their portfolio choice problem. If the cost of holding the domestic currency is higher than that of the foreign currency, buyers will dollarize their portfolios and as a result there will be more foreign currency available in the black market.
than domestic currency leading to a decline in the premium rate. This could be a probable explanation for the negative association between relative domestic inflation and rate of black market premium noticed in the data for certain countries.

One of the earliest works on the topic of currency black market is Dornbusch et al. (1983) for the Brazilian black market. In their model Dornbusch et al. (1983) propose a partial-equilibrium model of the black market using a stock and flow portfolio balance approach where demand for dollars depends positively on their relative yield and on wealth. Despite its simplicity and elegance, this partial equilibrium model assumes that the amount of wealth held in local currency is exogenous, which is unlikely as agents would diversify their portfolio between dollars and local currency depending upon their relative returns, wealth as well as the possibility of buying domestic or foreign goods. The black market premium depends on relative supply of both local currency and foreign currency. Therefore, the equilibrium results of Dornbusch et al. (1983) on the black market premium are not generally robust to the possibility of currency substitution. Other models of the black market include de Macedo (1982), de de Macedo (1987) which model dollar holdings by individuals as a means of diversifying a ‘portfolio’ of assets held to maximize expected returns on invested wealth while minimizing the variance of these returns.

The literature on the black market broadly consists of three different strands. The first consists of exchange rate reforms (or unification) and policy reforms in the presence of black markets, these include Goldberg (1995), Goldberg and Karimov (1997), Phylaktis and Girardin (2001) etc. Some studies like Kharas and Pinto (1989), Pinto (1991) also focus on inflationary implication of unification of black market rates and official exchange rates. A second strand of the literature which include Gupta (1981), Booth and Mustafa (1991) and Huett et al. (2014) study whether currency black markets efficiently process information about the state of the economy. Finally there are some papers that study real effect spillovers in presence of black market. For example Greenwood and Kimbrough (1987), study foreign exchange controls in an economy with black market and its impact on imports and welfare. Kamin (1995) studies how official devaluation in presence of black markets may lead to shrinking in aggregate output. In contrast to these papers, the current paper focuses on the black market premium and its association with relative inflation rates. In particular, this paper contributes to the empirical literature on black market by identifying a negative association between the black market premium rate and the relative domestic inflation. On the theoretical front it provides a model which explains this feature of the black market premium. It differs from other theoretical partial equilibrium models like Dornbusch et al.
(1983), Goldberg and Karimov (1997) by providing a setup where black market exists as a result of optimal decision making by economic agents. Therefore, the model presented here justifies the existence of black market. Furthermore, the focus is on stationary monetary equilibrium wherein agents can vary the amount of both monies they hold in response to relative rates of inflation and the possibility of domestic/ foreign consumption.

The rest of this paper is organized as follows. Section 2 reviews the evidence between relative inflation rates and the black market premium rate. Section 3 describes the physical environment of the model. Section 4 describes value functions and the optimal behavior by economic agents in different markets. Section 5 defines the equilibrium and discusses equilibrium portfolio choice and welfare consequences. Finally, section 7 concludes.

2 Evidence on Black Market Premium Rate and Inflation

In this section, I analyze the relationship between black market premium rates and inflation - in domestic currency and in U.S. dollars first for a set of twenty countries and then for a subset of countries. The black market I focus on is the black market for US dollars and any exchange rate is defined as the number of local currency units that need to be paid in order to acquire one US dollar. For the purpose of this study, I consider following countries: Argentina, Brazil, Chile, Colombia, Egypt, India, Indonesia, Iran, Kenya, Malaysia, Mexico, Nepal, Nigeria, Pakistan, South Africa, Sri Lanka, Tanzania, Thailand, Uganda and Venezuela for the years 1981-1997. First I use a panel vectorautoregression (VAR) approach with these twenty countries using as variables the log values of ratio of the black market exchange rate to the official exchange rate, the log values of the ratio of domestic inflation to US inflation and the index of financial openness developed by Chinn and Ito (2006). Using orthogonalized cumulative impulse responses based on Cholesky decomposition, I find that a shock which raises domestic inflation relative to US inflation raises the log ratio of the black market to the official exchange rate only slightly. Since the black market premium varies one-to-one with ratio of black market rate to official exchange rate, it suggests that with a sudden increase in the domestic inflation relative to the US inflation, the black market premium will rise in the short run by a small amount.

After analyzing the full sample, I choose a subset of six countries, namely, Iran, Venezuela

\[ \text{black market premium} = \frac{\text{local currency}/ \text{US dollar rate in black market}}{\text{local currency}/ \text{US dollar rate in official market}} - 1 \]
and the four South Asian countries of India, Pakistan, Nepal and Sri Lanka. As earlier, I adopt a panel VAR approach for these six countries. Using orthogonalized cumulative impulse response, for this subset of countries, I find a positive shock to the log ratio of domestic inflation to US inflation is associated with a decline in the log ratio of the black market rates to the official rates. This suggests a negative relationship between the black market premium and domestic inflation rate vis-a-vis to US inflation. i.e. as domestic inflation gets bigger relative to US inflation, we would notice a decline in black market premium over time. Thereafter I run a fixed effect panel regression with data from these six countries. The results suggest a negative effect of relative domestic inflation rate on the black market premium. Finally, I focus on one country, India for which data is available at monthly frequency. For this country, I find a negative effect of relative domestic inflation rates on the black market premium. In the next paragraph I describe the sources of data and the challenges it presents.

Data and challenges: The years 1981-1997 were chosen firstly because data on black market premium for all 12 months is available only until 1997. Secondly, for the years before 1981, reliable data on inflation is not available for Brazil. Brazil is a major country that has witnessed black market trade in US dollars in the 1980s and 1990s. Therefore, it should not be excluded from the study. Monthly data on black market premium comes from the dataset constructed by Reinhart and Rogoff (2009) by culling information from annual issues of Pick’s Currency Yearbook, Pick’s World Currency Report, Pick’s Black Market Yearbook. To the best of my knowledge, Pick’s books are the only source of data on black market premia and black market exchange rates. Unlike the official forex market, where exchange rates are easily visible through different media sources, black market exchange rates are not as conspicuous. Therefore, the figures on black market premia and black market exchange rates reported in Pick’s books are the average figures from surveys of black markets in different countries. These books were published by the International Currency Analysis, Inc. owned by Franz Pick, a New York-based currency analyst. After 1998 the publication was discontinued which explains the unavailability of data for recent years.

Data on inflation rates comes from the World Bank. It would be ideal to use monthly forecasts of annual inflation and a rational choice for that would be percentage changes in CPI from same month previous year. However, inflation forecasts or for that matter, CPI measures are not available at this level of frequency for every country for every year in my dataset. These include countries like Argentina, Brazil, Nepal. Due to these limitations in
the data, for the cross-country analyses I use annual inflation rates and average annual black market premium rates.

Finally for financial openness I use the Index of Financial Openness (KAOPEN) by Chinn and Ito (2006). The KAOPEN is a de jure indicator based on the Annual Report on Exchange Arrangements and Exchange Restrictions (AREAER henceforth) – a report the IMF has published since 1950. The AREAER provides information in the form of binary (exists/ does not exist) responses to a variety of possible restrictions and regulations in the domain of exchange rates and international finance for every member country. KAOPEN takes advantage of these binary responses provided in the IMF reports. KAOPEN is an “extensive” indicator of financial openness i.e., it measures the extent of these restrictions. The index is obtained through a principal component analysis on three categorical indicators of financial current account restrictions: (i) current account restrictions (ii) the requirement to surrender export proceeds surrender and (iii) the presence of multiple exchange rates plus the variable SHARE, which takes the rolling average of binary responses from AREAER’s categorical table over a ve-year window: $t - 4$ through $t$. KAOPEN is the first standardized principle component of these four AREAER table variables. Therefore, this index does not just capture capital account openness but also other factors that affect a country’s financial openness. The index is constructed for 182 countries for the time period of 1970-2013. The range of this index is $[-1.89, 2.39]$ with higher scores indicating greater openness. I include the KAOPEN in my analyses because regulations and restrictions which cause international financial frictions it could potentially affect the availability and value of dollars in a country’s exchange markets.

### 2.1 Panel VAR Approach

I consider a trivariate panel VAR of order $p$ with panel-specific fixed effects represented by the following system of linear equations:

$$Y_{it} = Y_{it-1}A_1 + Y_{it-2}A_2 + \cdots + Y_{it-p+1}A_{p-1} + Y_{it-p}A_p + u_i + e_{it} \quad (1)$$

where $i \in \{1, 2, \ldots, 20\}$ and $t \in \{1981, 1982, \ldots, 1997\}$. $Y_{it} = [KAOPEN, \ln(\pi_{LCU}/\pi_{USD}), \ln(e_{black}/e_{official})]$. KAOPEN is the

I base my model selection on the three model selection criteria by Andrews and Lu (2001) and select a first-order panel VAR since as this has the lowest mAIC, mBIC, and mQIC (see Table 3). A first order-panel VAR model is then estimated using GMM estimation
implemented by the Stata ‘pvar’ program developed by Abrigo and Love (2015). After estimating the model presented in Eq.(1) we generate orthogonalized impulse responses based on Cholesky decomposition and the impulse response of \( \ln(e_{\text{black}}/e_{\text{official}}) \) to one standard deviation shock to \( \ln(\pi_{LCU}/\pi_{USD}) \) are shown for 20 years in Figure 1

![Figure 1: One std. deviation shock to \( \ln(\pi_{LCU}/\pi_{USD}) \) for full sample](image)

From Figure 1, it is evident that for the full sample, a shock that raises the domestic inflation-US inflation ratio is associated with a small increase in the ratio of black market rate to official market exchange rate. This means an increase in domestic inflation relative to inflation in US dollar is associated with a slight increase the black market premium. This corresponds to the more common belief that an increase in inflation in the local currency is associated an increase in black market premium. However, these results change when I do a panel VAR analysis for the subset of six countries.

I specify the same panel VAR model as in Eq.(1) and estimate it for a smaller subset of countries which include Iran, Venezuela and the four South Asian countries of India, Nepal, Pakistan and Sri Lanka. For this sample of countries, I again select a first-order panel VAR since it has the lowest mAIC, mBIC, and mQIC (see Table 4) and after a GMM estimation of this first-order model I obtain the Cholesky decomposition-based orthogonalized impulse responses shown in Figure 2. For this set of countries a shock that raises the domestic inflation-US inflation ratio diminishes the ratio of black market rate to official market exchange rate from its current level. This effect persists in the for a while and the impulse response suggests a decline in black market premium with an increase in domestic inflation rate relative to US inflation. Furthermore, this panel VAR is stable as shown through eigenvalue stability conditions in Table 5. Therefore, it is invertible and has an infinite-
order vector moving-average representation, providing known interpretation to estimated impulse-response functions.

The results from this subset of countries is in contrast to the result from the full sample. We may predispose ourselves to misunderstand important aspects of the black market premia if we go by the results of the full sample and infer that rise in domestic inflation in relation to US inflation will always raise the black market rates vis-a-vis the official rate.

As a further confirmatory test in the next subsection I do a fixed effect panel regression with this subset of countries.

2.2 Fixed Effect Panel Regression Approach

In order to capture the effect of relative inflation rates on the ratio of black market rates to official market rates, for the subset of six countries discussed in the last subsection, I estimate the following fixed effect panel regression model

\[
\ln \left( \frac{e_{\text{black}it}}{e_{\text{official}it}} \right) = \beta_0 + \beta_\pi \ln \left( \frac{\pi_{\text{LCU}it}}{\pi_{\text{USD}it}} \right) + \beta_f KAOPENi_t + \beta_d D_{it} + \theta_i + \gamma_t + \epsilon_{it} \tag{2}
\]

where \(\ln(\frac{e_{\text{black}}}{e_{\text{official}}})_{it}\), \(\ln(\frac{\pi_{\text{LCU}}}{\pi_{\text{USD}}} )_{it}\) and \(KAOPENi_t\) have the same meaning as before. The subscript \(it\) denotes that the value is for country \(i\) in time period \(t\), where \(i\) denotes a country from the subset of six countries and \(t \in \{1981, 1982, \ldots, 1997\}\). Like before, due to the lack of data on CPI or inflation forecast at monthly frequency, I use annual averaged figures for \(e_{\text{black}}/e_{\text{official}}\). The country fixed effects are denoted by \(\theta_i\), while the time fixed effects
are denoted by $\gamma_t$ and $\epsilon_{it}$ is an i.i.d. error term. The inclusion of country fixed effects is important to remove the bias due to the omission of country-specific time-invariant variables. Finally, $D_{it}$ is a dummy variable that assumes the value 1 if inflation in country $i$ in year $t$ was high, i.e. $\pi_{LCU} > 10\%$.

The coefficient of interest is $\beta_\pi$. As seen in the panel VAR impulse responses, if indeed $\ln \left( \frac{e_{black}}{e_{official}} \right)_{it}$ (or the black market premium rate) goes down with an increase in domestic inflation in relation to the US inflation, this coefficient must be negative, i.e. $\beta_\pi < 0$.

Table 1: Fixed effect regression model: main results

<table>
<thead>
<tr>
<th>Dep. var.</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
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<td>LSDV</td>
<td>LSDV</td>
<td>LSDV</td>
<td>LSDV</td>
</tr>
<tr>
<td>$\ln(\pi_{LCU}/\pi_{USD})_{it}$</td>
<td>$-0.123^*$</td>
<td>$-0.150^*$</td>
<td>$-0.138^*$</td>
<td>$-0.206^{**}$</td>
<td>$-0.168^{**}$</td>
<td>$-0.211^{**}$</td>
</tr>
<tr>
<td></td>
<td>(0.065)</td>
<td>(0.082)</td>
<td>(0.070)</td>
<td>(0.094)</td>
<td>(0.083)</td>
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</tr>
<tr>
<td>$KAOPEN_{it}$</td>
<td>-0.071</td>
<td>-0.063</td>
<td>(0.060)</td>
<td>(0.063)</td>
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<td></td>
</tr>
<tr>
<td>$D_{it}$</td>
<td>0.049</td>
<td>0.104</td>
<td>0.084</td>
<td>(0.062)</td>
<td>(0.088)</td>
<td>(0.092)</td>
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<td>Country dummies</td>
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</tr>
<tr>
<td>Time dummies</td>
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<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
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<td>102</td>
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<tr>
<td>Countries</td>
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</tr>
<tr>
<td>Adj. $R^2$</td>
<td>0.741</td>
<td>0.752</td>
<td>0.739</td>
<td>0.751</td>
<td>0.753</td>
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</tr>
</tbody>
</table>

Note: Heteroskedasticity-robust (Huber-White) standard errors are in brackets. Asterisks denote significance levels (** significant at 5%; * significant at 10%).

To study the relationship between relative domestic inflation rate and black market premium rate, it is worthwhile to start investigating the relationship between relative domestic inflation rate as captured through $\ln(\pi_{LCU}/\pi_{USD})_{it}$ and black market premium rate as captured through $\ln(e_{black}/e_{official})_{it}$ assuming no differences between countries in terms of financial openness, no time fixed effects and no dummy for high inflation years. In fact, there is little variability in financial openness for these six countries. Column (1) report results obtained by Least Squares Dummy Variable (LSDV) of Eq/(2), without the index of financial openness, dummy variable for high inflation years and time fixed effects. The estimated coefficient for $\ln(\pi_{LCU}/\pi_{USD})_{it}$ is negative and significant at 10% level ($p$-value = 0.061). In column (2), I add the time fixed effects to the model specified in column
(1). The estimated coefficient of relative inflation, $\ln(\pi_{LCU}/\pi_{USD})_{it}$ is again negative and significant at 10% ($p$-value = 0.071).

Since, these countries have different political and economic regimes and in some cases different geographies, if we allow for the possibility of no single event affecting all these countries in a particular point of time, then the time fixed effects are not relevant. However, the dummy variable denoting high inflation (10%) years could be still be relevant. Therefore, in column (3) I estimate a model without index of financial openness and time fixed effects, but I include $D_{it}$. The estimated $\beta_\pi$ is negative and significant at 10% level ($p$-value = 0.052). In column (4), I include all variables and fixed effects except for the index of financial openness. The estimated coefficient of $\ln(\pi_{LCU}/\pi_{USD})_{it}$ is negative and significant at 5% level.

In column (5), all variables except the dummy variable indicating high inflation year in country $i$ are included and the estimated coefficient of $\ln(\pi_{LCU}/\pi_{USD})_{it}$ is negative and significant at 5% level suggesting a decline in $\ln(e_{black}/e_{official})_{it}$ with a rise in domestic inflation relative to the prevailing US inflation rate. This means black market premium rate is decreasing as domestic inflation rises with regards to US inflation. Finally in column (6), I present the estimated results of the Eq.(2) which includes all variables and both country and time fixed effects. In this last column, the estimated $\beta_\pi$ is again negative and significant.

These results are supportive of the hypothesis that an increase in domestic inflation with respect to the US inflation is associated with a decline in black market premium rate, at least for the countries - India, Iran, Nepal, Pakistan, Sri Lanka and Venezuela. In the next subsection, I analyze the case of one country, India for which monthly data on inflation rates as well as black market premium rate is available. I check whether this negative association is noticed in the case of India when using monthly data.

### 2.3 The Case of India

Monthly data on inflation rates is available both for India and the USA. Therefore, it would be prudent to check if the negative association between black market premium rate and the relative domestic inflation also holds for this country when using data at monthly frequency. For India, I specify the following time-series regression model

$$\ln\left(\frac{e_{black}}{e_{official}}\right)_t = \beta_0 + \beta_\pi \Delta \ln\left(\frac{1 + \pi_{INR}}{1 + \pi_{USD}}\right)_t + \beta_i \Delta i^{IN}_t + \beta_i \mathbb{1}\{t \geq 03/1993\} + \epsilon_t$$  \hspace{1cm} (3)
In India’s case the monthly inflation rates\textsuperscript{4} for some months are zero and sometimes negative. Therefore, instead of using \(\ln(\pi_{INR}/\pi_{USD})_t\), here I use \(\ln(1 + \pi_{INR}/1 + \pi_{USD})_t\). The variable \(\pi_{INR}\) denotes the inflation in Indian Rupee while \(i_{t}^{IN}\) is a measure of the prevailing interest rate in period \(t\). The interest rate I use here is the bank rate offered by the Reserve Bank of India. Information about the bank rate is available on a daily basis. For those months having multiple bank rates in different portions of the month, I use a weighted average representative interest rate for the month. The rationale behind including interest rates is that if interest rates for assets denominated in rupee are sufficiently high, then the real value of the assets might be preserved despite widening difference between rupee inflation and dollar inflation. This could in turn affect relative demand for the rupee and the dollar and might have some spillover effect to the black market. \(\mathbb{I}\{t \geq 03/1993\}\) is a dummy for all the months after March 1993. March 1993 is an important month in India’s exchange rate and exchange arrangement history because during this month India officially switched from a basket peg to managed float exchange rate regime. 

The variable \(\ln(e_{black}/e_{official})_t\) and the first difference variables \(\Delta \ln(1 + \pi_{INR}/1 + \pi_{USD})_t\), \(\Delta i_{t}^{IN}\) are all \(I(0)\). First, I estimate a time-series regression model without first difference values of interest rates and the dummy variable. These results are reported in column (1) or Table 2. The coefficient of \(\Delta \ln(1 + \pi_{INR}/1 + \pi_{USD})_t\) is negative and significant at 1% level suggesting that a relative increase in rupee inflation that widens the rupee inflation-

\begin{table}[h]
\centering
\begin{tabular}{lccc}
\hline
Dep. var. & (1) & (2) & (3) \\
Method & OLS & OLS & OLS \\
\(\Delta \ln\left(\frac{1 + \pi_{INR}}{1 + \pi_{USD}}\right)_t\) & -4.042*** & -3.883*** & -3.153*** \\
 & (1.371) & (1.387) & (0.974) \\
\(\Delta i_{t}^{IN}\) & 0.025* & -0.009 & \\
 & (0.013) & (0.010) & \\
\(\mathbb{I}\{t \geq 03/1993\}\) & & -0.118*** & \\
 & & (0.015) & \\
Observations & 210 & 210 & 210 \\
Adj. \(R^2\) & 0.039 & 0.039 & 0.528 \\
\hline
\end{tabular}
\caption{India: main results}
\end{table}

Note: Newey-West standard errors are in brackets. Asterisks denote significance levels (***/ significant at 1%; ** significant at 5%; * significant at 10%).

\textsuperscript{4}These rates are calculated as percentage change in CPI from the same month last year.
dollar inflation ratio will be associated with a decline in \( \ln(e_{\text{black}}/e_{\text{official}})_t \), i.e. a decline in the black market premium rate. Next, I add \( \Delta i_{IN}^t \) to the model. The estimated results are shown in column (2) of Table 2 and the coefficient of \( \Delta \ln(1 + \pi_{INR}/1 + \pi_{USD})_t \) is again negative and significant at 1% level. As a final analysis, I estimate Eq.(3) and the results are reported in column (3). The addition of the dummy for the months after exchange rate liberalization increases the explanatory power of the model but here also, the \( \beta_\pi \), the coefficient of \( \Delta \ln(1 + \pi_{INR}/1 + \pi_{USD})_t \) is negative and significant at 1%.

The analysis in this section lends support to the hypothesis that an increase in domestic inflation with respect to US (or foreign currency) inflation need not be associated with an increase in the black market premium rate. On the contrary, for some countries like Iran, Venezuela, the four South Asian countries in my sample we witness a negative association. This contradicts the general idea of rising black market premium rate with increase in domestic inflation. In the next section I propose a model in the New Monetarist framework that could explain this negative association between black market premium rate and the deviation between inflation in domestic currency inflation and that in foreign currency.

3 A Model of the Black Market

Time is discrete and continues forever. There are two countries, A and B, each populated with a continuum 2 of agents. Following ?, agents in each of the two countries are differentiated into two groups: a measure 1 of buyers and a measure 1 of sellers. Each period is divided into two stages where different activities take place. The first subperiod is for decentralized trades in local and foreign special goods and for currency exchange among country B buyers in a Walrasian black market. The labels ‘buyer’ and ‘seller’ refer to an agent’s role in the first subperiod and this role remains unchanged over periods. In the first subperiod, a seller can produce but does not want to consume, while a buyer wants to consume but cannot produce. Therefore, there is no double coincidence of wants. The black market is a perfectly competitive currency exchange market which country B buyers may choose to use in the first subperiod and this market is considered illegal by country B’s authorities. If they choose to go to the black market, country B buyers can successfully transact in the black market with an exogenous probability \( \alpha \in (0, 1) \) and with probability \( 1 - \alpha \) they lose their entire liquid wealth. This loss can be interpreted in several ways: (i) one may interpret this as confiscation by country B’s government as a penalty for participating in illegal currency exchange, or (ii) given the risky nature of the black market due to potential
involvement of dishonest elements the agent could lose his entire wealth as a result of being robbed or swindled (due to sale of fake currency). Therefore, the parameter $\alpha$ captures the frictions associated in accessing the illegal currency exchange market. The black market can be thought of as a reduced form of an over-the-counter market à la Duffie et al. (2005) where agents meet a dealer with probability $\alpha$ and a dealer’s bargaining power is zero. In this special case, dealers in Duffie et al. (2005) become redundant and it is as if agents access the perfectly competitive interdealer market with probability $\alpha$. Therefore, the black market in this paper is equivalent to the Walsrasian interdealer market in Duffie et al. (2005). To this special case of Duffie et al. (2005), I add the additional assumption of confiscation of wealth, by country $B$’s authorities with probability $1 - \alpha$. In the second subperiod, there is a frictionless centralized market where agents from both countries settle debts and exchange currencies. Therefore, the second subperiod acts as a global currency exchange market. This global currency exchange market is the legally recognized, official currency exchange market where agents from both countries participate. We label the first subperiod DM (decentralized markets) and the second subperiod as the CM (centralized market). All agents discount payoffs across periods with the same factor, $\beta \in (0, 1)$. Country $i$, $i \in \{A, B\}$ issues its own perfectly divisible and storable fiat currency which I will call $\text{money}_i$. We use $M^i_t$ to denote the stock of $\text{money}_i$ at time $t$. The initial stock of $\text{money}_i$ is given by $M^i_0 \in \mathbb{R}_+$ which grows at a constant rate $\pi_i$ over periods (therefore, $M^i_{t+1} = \pi_i M^i_t$). The growth rate of the stock of $\text{money}_i$, $\pi_i \geq \beta$ is chosen by the monetary authority in country-$i$.

In the CM of every period, all agents trade a consumption good produced in that stage, and the two monies, in a spot Walrasian market. The CM’s spot Walrasian market serves as the official channel for the exchange of two monies. During this stage, new $\text{money}_i$ ($i = A, B$) is injected ($\pi_i > 1$) or withdrawn ($\pi_i < 1$) from the economy via lump-sum transfers to buyers of country $i$. At the end of the second subperiod and at the start of next period’s first-stage, a distinct decentralized market opens up in each country for the trade of special goods. We denote the two decentralized markets for special goods as $SGM_i$, $i \in \{A, B\}$. The $SGM$s do not have any search frictions and the mass of bilateral matches in a $SGM$ is given by the minimum of buyers and sellers in a market. At the beginning of first subperiod, with probability $\delta \in [0, 1)$ a country $i$ buyer, obtains an opportunity to consume the foreign special good while with probability $(1 - \delta)$ he consumes the locally produced special good. However, I do not assume that all buyers get to consume the local special good. The buyers who get to consume the special good produced in $SGM_i$ are referred to as $SGM_i$ buyers ($i \in \{A, B\}$). A simple arithmetic shows the mass of $SGM_i$ buyers in $SGM_i$ is 1 of which a
fraction \((1 - \delta)\) are from country \(i\), while a fraction \(\delta\) are from country-\(j\) \((j \in \{A, B\}, j \neq i)\). Therefore, all buyers (all sellers) in \(SGM_i\) are matched with a seller (buyer). We assume that agents cannot make binding commitments, that there is no enforcement, and that histories of actions are private in a way that precludes any borrowing and lending. Therefore, all trade must be *quid pro quo*. We assume that sellers do not recognize a foreign currency\(^5\). Therefore, when a seller meets a foreign buyer, the buyer must pay the seller in the seller’s currency. This implies that a \(SGM_i\) country \(j\) buyer will not be able to participate in \(SGM_i\) unless he acquires \(money_i\). Finally, within any given match, buyers make a take-it-or-leave-it (TIOLI) offer to the seller.

An individual buyer’s preferences are given by

\[
E_0 \sum_{t=0}^{\infty} \beta^t (u(q_t) + c_t - h_t)
\]

where \(q_t\) is the quantity of the local or foreign special good that the buyer consumes at the end of the first subperiod of period \(t\), \(c_t\) is his consumption of the homogeneous good that is produced, traded and consumed in the second subperiod of period \(t\), and \(h_t\) is the utility cost from exerting \(h_t\) units of effort to produce this good. The function \(u(q_t)\) is the utility a buyer derives if he consumes \(q_t\) amount of local or foreign special good in the decentralized round of trade in period \(t\). The utility function, \(u(\cdot)\) is twice continuously differentiable with \(u(0) = 0, u'(\cdot) > 0, u''(\cdot) < 0\) and \(u'(q_t)q_t\) is decreasing. We also assume that \(u(\cdot)\) satisfies the Inada conditions: \(u'(0) = \infty\) and \(u'(\infty) = 0\) and that there exists a \(q^* \equiv \arg\max\{u(q) - q\}\). The expectation operator \(E_0\) is with respect to the random matching with local or foreign seller in the decentralized trades and the random success in transacting in the black market, if a buyer chooses to access it.

An individual country \(i\) \((i \in \{A, B\})\) seller’s preferences are given by

\[
E_0 \sum_{t=0}^{\infty} \beta^t (-q_t + c_t - h_t)
\]

where \(q_t\) is the quantity of the local or foreign special good that the seller produces at the end of the first subperiod of period \(t\), \(c_t\) is his consumption of the homogeneous good that is produced, traded and consumed in the second subperiod of period \(t\), \(h_t\) is the utility cost

\(^5\)This could be due to a sufficiently high cost of authentication incurred by a seller which leads to only local currencies being accepted in each \(DM\) as in Zhang (2014) or due to restriction imposed by the authorities that forbid sellers to accept foreign currency as in Curtis and Waller (2000)
from exerting \( h_t \) units of effort to produce this good. The expectation operator \( \mathbb{E}_0 \) is with respect to the random matching with local or foreign buyer.

## 4 Value Functions and Optimal Behavior

### 4.1 Value Functions

Let \( \phi_t^A \) be the real price of \( money_A \) and \( \phi_t^B \) be the real price of \( money_B \) both expressed in terms of the second-subperiod’s consumption good. Then, \( z_t^A = \phi_t^A m_t^A \) and \( z_t^B = \phi_t^B m_t^B \) are the real balances held by an agent in \( money_A \) and \( money_B \) respectively. We use \( V_t^B(z_{ti}) \) to denote the maximum expected discounted payoff of a country \( i \) buyer who enters the decentralized round of period \( t \) with portfolio \( z_{ti} \equiv (z_{ti}^A, z_{ti}^B) \). Let \( W_t^B(z_{ti}) \) denote the maximum expected discounted payoff of a country \( i \) buyer who is holding portfolio \( z_{ti} \) at the beginning of the second subperiod of period \( t \). Then,

\[
W_t^B(z_{ti}) = \max_{c_t, h_t, z_{t+1i}} \left[ c_t - h_t + \beta V_t^B(z_{t+1i}) \right]
\]

s.t.

\[
c_t + \phi_t m_{t+1i} = h_t + \phi_t m_{ti} + T_t^i
\]

\[
c_t, h_t \in \mathbb{R}_+; m_{t+1i}, z_{t+1i} \in \mathbb{R}_+^2
\]

\[
m_{t+1i} = (m_{t+1i}^A, m_{t+1i}^B), z_{t+1i} = (z_{t+1i}^A, z_{t+1i}^B)
\]

(4)

where \( \phi_t = (\phi_t^A, \phi_t^B) \), \( m_{ti} = (m_{ti}^A, m_{ti}^B) \) and \( \phi_t m_{ti} \) (or \( \phi_t m_{t+1i} \)) denotes the dot product of \( \phi_t \) and \( m_{ti} \) (or, \( m_{t+1i} \)) which is equivalent to \( z_{ti} \) (or, \( z_{t+1i} \)). \( T_t^i = \phi_t^i (\pi_i - 1) M_{i,t} \) is the real value of the time \( t \) lump-sum monetary transfer (or, tax, if \( \pi_i < 1 \)). One can further simplify Eq (4) and it can be rewritten entirely in terms of real balances as

\[
W_t^B(z_{ti}) = \max_{c_t, h_t, z_{t+1i}} \left[ c_t - h_t + \beta V_t^B(z_{t+1i}) \right]
\]

s.t.

\[
c_t + \beta(1 + \epsilon) \cdot z_{t+1i} = h_t + 1 \cdot z_{ti} + T_t^i
\]

\[
c_t, h_t \in \mathbb{R}_+, z_{t+1i} \in \mathbb{R}_+^2
\]

\[
z_{t+1i} = (z_{t+1i}^A, z_{t+1i}^B)
\]

(5)
where \( \iota_i = \frac{\delta_i}{\beta \delta_{i+1}} - 1 \) and it is the cost of holding money, \( i \in \{A, B\} \). In (5), eliminating \( h_t \) from the budget constraint yields

\[
W^G_i(z_{ti}) = 1 \cdot z_{ti} + W^G_i(0)
\]

\[
W^G_i(0) \equiv T^i_t + \max_{z_{t+1}} \beta[-(1 + \iota) \cdot z_{t+1} + V^G_i(z_{t+1})]
\]

\[
z_{t+1} = (z^A_{t+1}, z^B_{t+1}) \in \mathbb{R}_+^2
\]

(6)

As is standard in models that build on Lagos and Wright (2005), the buyer’s value function is linear in the real balances, implying that there are no wealth effects on the choice of \( z_{t+1} \).

Next, consider country \( i \) seller’s value function for the second subperiod of period \( t \). Let \( W^S_i(z_{ti}) \) denote the maximum expected discounted payoff of a country \( i \) seller who is holding portfolio \( z_{ti} \) at the beginning of the second subperiod of period \( t \). This agent will never want to leave the CM with any money holdings, since he does not participate in the black market and does not want to consume in the decentralized round of trade (see Rocheteau and Wright (2005) for a rigorous proof). Then,

\[
W^S_i(z_{ti}) = \max_{c_t, h_t, z_{t+1}} [c_t - h_t + \beta V^S_i(0)]
\]

s.t. \( c_t + \beta(1 + \iota) \cdot z_{t+1} = h_t + 1 \cdot z_{ti} \)

\[
c_t, h_t \in \mathbb{R}_+, z_{t+1} \in \mathbb{R}_+^2
\]

\[
z_{t+1} = (z^A_{t+1}, z^B_{t+1})
\]

(7)

Again, eliminating \( h_t \) from the budget constraint in (7), I get

\[
W^S_i(z_{ti}) = 1 \cdot z_{ti} + W^S_i(0)
\]

\[
W^S_i(0) \equiv \max_{z_{t+1}} \beta[-(1 + \iota) \cdot z_{t+1} + V^S_i(0)]
\]

\[
z_{t+1} = (z^A_{t+1}, z^B_{t+1}) \in \mathbb{R}_+^2
\]

(8)

In the first subperiod with a probability \( \delta \) buyers of each country get the opportunity to consume a foreign special good, while with probability \( (1 - \delta) \) they get to consume a local special good. However, country \( B \) buyers can successfully transact in the black market for currency exchange with probability \( \alpha \in (0, 1] \) and with probability \( 1 - \alpha \) has their entire liquid wealth confiscated. Since sellers of country \( A \) do not accept money \( B \) and sellers of country \( B \) do not accept money \( A \), country \( B \) buyers would want to access the black market.
and readjust their portfolio of real balances. If a country $B$ buyer, who gets the opportunity to consume a foreign special good, chooses to access the black market, he would want to convert his entire real holdings of $money_B$ into real holdings of $money_A$ and vice-versa. The resulting post-trade real portfolios of country $B$ buyer who gets to consume a special good from country $A$ and that of a country $B$ buyer who gets to consume a special good from country $B$ are denoted

\[
\begin{align*}
[z^A_B(z_{tB}; \psi_t), z^B_B(z_{tB}; \psi_t)] \\
[z^A_B(z_{tB}; \psi_t), z^B_B(z_{tB}; \psi_t)]
\end{align*}
\]

respectively, where $\psi_t \equiv (\phi^A_t, \phi^B_t, \epsilon_t)$ and $z_{tB} = (z^A_{tB}, z^B_{tB})$ is the pre-trade real portfolio of country $B$. The asterisk (*) over $B$ in the second post-trade real portfolio indicates that the buyer is matched with a local seller (i.e. from country $B$). The black market is effectively the market where country $B$ buyers trade real balances held in the two currencies and readjust their portfolios after realizing the shock. Here real balance of $money_B$ trades at $\epsilon_t^{-1}$ against real balance of $money_A$. In nominal terms this means, in the black market, $\epsilon_t^{-1} \phi^A_t / \phi^B_t$ is the price of $money_A$ in terms of $money_B$. Note that $\phi^A_t / \phi^B_t$ is the nominal price of $money_A$ in terms of $money_B$ (or, nominal exchange rate) in the CM which also acts as the officially recognized foreign exchange market. Therefore, $\epsilon_t^{-1} - 1$ is the black market premium. If $\epsilon_t^{-1} > 1$ the premium is positive. If $\epsilon_t^{-1} < 1$, then the premium is negative and when $\epsilon_t^{-1} = 1$, the exchange rate of the official market and the black market coincide.

We can now write the value function of a country $B$ buyer who enters the decentralized round of period $t$ with portfolio $z_{tB}$ and chooses to access the black market,

\[
\begin{align*}
V^B_B(z_{tB}) &= \alpha \delta [u(\tilde{q}_t|_{bm}) + W^B_B(\tilde{z}^A_B - \tilde{d}^A_{bm}, \tilde{z}^B)] \\
&\quad + (1 - \alpha) \delta W^B_B(0) \\
&\quad + \alpha (1 - \delta) [u(q_t|_{bm}) + W^B_B(\tilde{z}^A_{B*} - \tilde{d}^B_{bm})] \\
&\quad + (1 - \alpha) (1 - \delta) W^B_B(0) 
\end{align*}
\]

where $\tilde{q}_t|_{bm}$ denotes the amount of special good bought by a country $B$ buyer in $SGM_A$ (foreign market) given that he has adjusted his portfolio of real balances in the black market. The variable $q_t|_{bm}$ denote the special good bought by a country $B$ buyer in $SGM_B$ (local market) conditional on him adjusting his portfolio of real balances in the black market. The real payments made with $money_A$ by a country $B$ buyer to country $A$ (foreign) seller after having readjusted his portfolio in the black market is $\tilde{d}^A_{bm}$. Similarly $d^B_{bm}$ denotes the real payment made with $money_B$ by a country $B$ buyer buyer to country $B$ (local) seller after
readjusting his portfolio in the black market. The value function of a country B buyer who enters the decentralized round of period \( t \) with portfolio \( z_{tB} \) and chooses not to access the black market is:

\[
V^B_B(z_{tB}) = \delta[u(\tilde{q}_B|nbm) + W^B_B(z_{tB} - \tilde{d}_{nbm}, z_{tB})] \\
+ (1 - \delta)[u(q_B|nbm) + W^B_B(z_{tB}, z_{tB} - d_{nbm})] \quad (10)
\]

The value function of a country A buyer who enters the decentralized round of period \( t \) with portfolio \( z_{tA} \),

\[
V^B_A(z_{tA}) = \delta[u(\tilde{q}_A) + W^B_A(z_{tA}, z_{tA} - \tilde{d})] \\
+ (1 - \delta)[u(q_A) + W^B_A(z_{tA}, d_{tA})] \quad (11)
\]

where \( \tilde{q}_A \) denotes the amount of special good bought by a country A buyer in SGM\(_B\) (foreign market). The variable \( q_A \) denotes the amount of special good bought by a country A buyer in SGM\(_A\) (local market). The real payments made with money\(_B\) in SGM\(_B\) and with money\(_A\) in SGM\(_A\) by country A buyer are denoted by \( \tilde{d} \) and \( d^A \) respectively.

### 4.2 Terms of Trade

In this section, I discuss the determination of the terms of trade in the two SGMs. Consider a meeting in SGM\(_i\), between a country \( i \) seller and a buyer (from any country) who carries a real portfolio \( z_t = (z^A_t, z^B_t) \) composed of money\(_A\) and money\(_B\). The two parties negotiate over a quantity of special good, \( q_t \), to be produced, and an amount of real payment in money\(_A\), \( d^A_t \) and real payment in money\(_B\), \( d^B_t \) to be delivered to the seller. Define \( d_t \equiv (d^A_t, d^B_t) \). Given that the buyer makes a TIOLI offer to the seller, the bargaining problem can be expressed as

\[
\max_{q_t, d_t} \{ u(q_t) + W^B_i(z_t - d_t) - W^B_i(z_t) \} \\
\text{s.t. } q_t = W^S_i(\tilde{z}_t + d_t) - W^S_i(\tilde{z}_t) \\
\text{and } q_t \in \mathbb{R}_+, d_t \in [0, z^A_t] \times [0, z^B_t] \quad (12)
\]

Now, I discuss the more specific cases. Consider a meeting in SGM\(_i\) between a seller from country \( i \) and a buyer from any country who carries a real portfolio \( z_t = (z^A_t, z^B_t) \). Since the seller won’t accept any payment in money\(_j\) (\( j \neq i \)) it must be that \( d^j_t = 0 \). With \( d^1_t = 0 \), the
bargaining problem described in (12) reduces to

$$\max_{q_t,d_t^i} \{u(q_t) - d_t^i\}$$

s.t. \( q_t = d_t^i \)  \hspace{1cm} (13)

The next lemma describes the solution to this bargaining problem.

**Lemma 1.** Define \( q^* = \{q : u'(q) = 1\} \). Then in a SGM\(_i\) meeting between a seller from country \( i \) and a buyer from any country, who carries a real portfolio \( z_t = (z_t^A, z_t^B) \), the bargaining solution is given by \( q_t = \min\{z_t^i, q^*\} \), \( d_t^i = \min\{z_t^i, q^*\} \) and \( d_t^j = 0 \) where \( i, j \in \{A, B\}, i \neq j \).

**Proof.** In appendix.  

The interpretation of Lemma 1 is standard. The terms of trade depend only on the buyer’s real holdings of money\(_i\). When \( z_t^i \) exceeds a certain level \( q^* \), then the buyer purchases the first-best quantity, \( q^* \), and gives up exactly \( q^* \) units of his real holdings of money\(_i\). On the other hand, if \( z_t^i \) is less than \( q^* \), then the buyer is liquidity constrained and he gives up his entire real holding of money\(_i\) to receive the amount of good that the seller is willing to produce for that money, i.e., \( q_t = z_t^i \).

We now proceed to the characterization of the terms of trade in the black market. Consider a country \( B \) buyer who gets the opportunity to consume foreign special goods, i.e. he is matched with a country \( A \) seller (foreign seller). This buyer would want to exchange some (or all) of his money\(_B\) for money\(_A\) so if he accesses the frictionless, competitive black market, he can acquire money\(_A\) from other country \( B \) buyers who buys locally (and needs to exchange money\(_A\) for money\(_B\)). The problem of the country \( B \) buyer buying foreign special goods is given by

$$\max_{\tilde{z}_B^A, \tilde{z}_B^B} \left[u(\tilde{q}_B|\tilde{z}_B) + W_B^B(\tilde{z}_B^A - \tilde{d}_B^A, \tilde{z}_B^B)\right]$$

s.t. \( \tilde{z}_B^A + \varepsilon_{t_B} \tilde{z}_B^B = \tilde{z}_{t_B}^A + \varepsilon_t \tilde{z}_{t_B}^B \)

\( \tilde{z}_B^A, \tilde{z}_B^B \geq 0 \)  \hspace{1cm} (14)

The country \( B \) buyer matched with a foreign seller trades money\(_B\) for money\(_A\) to readjust his portfolio so as to maximize the sum of his utility from consumption of the foreign special good and the continuation value. However, if he successfully transacts in the black market, he cannot leave with any more than what he entered with. So, the budget constraint must
be satisfied. Furthermore, since the objective function is monotonic, the budget constraint
must hold with equality. The following lemma solves the above problem.

**Lemma 2.** Consider a country B buyer with portfolio \((z^A_{tB}, z^B_{tB})\) who gets the opportunity to
consume foreign good in period \(t\). If he successfully transacts in the black market, he leaves
with a post-trade portfolio \((\bar{z}^A_{tB}, \bar{z}^B_{tB})\) such that

\[
\begin{align*}
(a) & \quad \text{If } \varepsilon_t > 1, \quad \begin{cases} 
\bar{z}^A_{tB} = z^A_{tB} + \varepsilon_t z^B_{tB} \\
\bar{z}^B_{tB} = 0 
\end{cases} \\
(b) & \quad \text{If } \varepsilon_t = 1 \text{ and } z^A_{tB} + \varepsilon_t z^B_{tB} \geq q^*, \quad \begin{cases} 
\bar{z}^A_{tB} = z^A_{tB} + z^B_{tB} \\
\bar{z}^B_{tB} = z^B_{tB} + z^A_{tB} - \bar{z}^A_{tB} 
\end{cases} \\
(c) & \quad \text{If } \varepsilon_t = 1 \text{ and } z^A_{tB} + \varepsilon_t z^B_{tB} < q^*, \quad \begin{cases} 
\bar{z}^A_{tB} = z^A_{tB} + z^B_{tB} \\
\bar{z}^B_{tB} = 0 
\end{cases} \\
(d) & \quad \text{If } \varepsilon_t < 1 \text{ and } z^A_{tB} + \varepsilon_t z^B_{tB} \geq \bar{\chi}, \quad \begin{cases} 
\bar{z}^A_{tB} = \bar{\chi} \\
\bar{z}^B_{tB} = z^B_{tB} + \varepsilon_t^{-1}(z^A_{tB} - \bar{\chi}) 
\end{cases} \\
(e) & \quad \text{If } \varepsilon_t < 1 \text{ and } z^A_{tB} + \varepsilon_t z^B_{tB} < \bar{\chi}, \quad \begin{cases} 
\bar{z}^A_{tB} = z^A_{tB} + \varepsilon_t z^B_{tB} \\
\bar{z}^B_{tB} = 0 
\end{cases}
\end{align*}
\]

where \(\varepsilon_t\) is the black market price of real balance of money\(A\) in terms real balance of money\(B\)
and \(\bar{\chi}\) such that \(u'(\bar{\chi}) = \varepsilon_t^{-1}\).

**Proof.** In appendix. ■

To interpret the result in Lemma 2 first observe that the objective function in (14) can
be written as \(u(\tilde{q}_{B|bm}) + \bar{z}^A_{tB} - \tilde{q}_{B|bm} + \bar{z}^B_{tB}\). When \(\varepsilon_t > 1\), in the black market using a unit real
baalnce of money\(B\), real balance of money\(A\) can be increased by more than one unit. Since
the objective function is strictly increasing in \(\bar{z}^A_{tB}\) the best thing to do is to convert entire
money\(B\) into money\(A\). When \(\varepsilon_t = 1\), real balance of money\(A\) and that of money\(B\) trades
one to one. Due to the strict concavity of the utility function, an increase in real balance of
money\(A\) strictly increases the objective function as long as \(\bar{z}^A_{tB} < q^*\) after that the objective
function increases in \(\bar{z}^A_{tB} + \bar{z}^B_{tB}\). Since real balance of money\(A\) and that of money\(B\) trade one to
one, the buyer is indifferent between increasing or not increasing his real holding of money\(A\).
When \(\varepsilon_t < 1\), in the black market it gets more expensive to acquire real balance of money\(A\)
as a unit real balance of money\(A\) can be only bought with more than one unit of money\(B\),
while there is increase in utility from increasing real balance of money\(A\), there is a cost to
it as well. The $\bar{\chi}$ represent that amount of real money holding at which marginal utility equals marginal cost. If a buyer’s real wealth is more than $\bar{\chi}$, he will increase his real balance of money no more than $\bar{\chi}$, otherwise he will increase it as much as his wealth would permit.

Now, consider a country B buyer who is matched with a seller from his own country and gets to consume local goods. This buyer would want to exchange some (or all) of his money for money B. The problem of the country B buyer buying local special goods is given by

$$\max_{\bar{z}_B^*, \bar{z}_B^*} \left[ u(q_{B|bm}) + W_B(z_B^*, z_B^* - q_{bm}) \right]$$

s.t. $\bar{z}_B^* + \varepsilon_t\bar{z}_B^* = z_{tB}^* + \varepsilon_tz_{tB}^*$

$$\bar{z}_B^*, \bar{z}_B^* \geq 0$$ (15)

The following lemma provides the solution to the above problem.

**Lemma 3.** Consider a country B buyer with portfolio $(z_{tB}^*, z_{tB}^*)$ who gets the opportunity to consume local good in period $t$. If he successfully transacts in the black market, he leaves with a post-trade portfolio $(\bar{z}_B^*, \bar{z}_B^*)$ such that

(a) If $\varepsilon_t > 1$ and $z_{tB}^* + \varepsilon_tz_{tB}^* \geq \varepsilon_t\bar{\psi}$, then

$$\begin{cases} 
\bar{z}_B^* = z_{tB}^* + \varepsilon_t(z_{tB}^* - \bar{\psi}) \\
\bar{z}_B^* = \bar{\psi}
\end{cases}$$

(b) If $\varepsilon_t > 1$ and $z_{tB}^* + \varepsilon_tz_{tB}^* < \varepsilon_t\bar{\psi}$, then

$$\begin{cases} 
\bar{z}_B^* = 0 \\
\bar{z}_B^* = \varepsilon_t^{-1}z_{tB}^* + z_{tB}^*
\end{cases}$$

(c) If $\varepsilon_t = 1$ and $z_{tB}^* + z_{tB}^* \geq q^*$, then

$$\begin{cases} 
\bar{z}_B^* = z_{tB}^* + z_{tB}^* - \bar{z}_B^* \\
\bar{z}_B^* \in [q^*, z_{tB}^* + z_{tB}^*]
\end{cases}$$

(d) If $\varepsilon_t = 1$ and $z_{tB}^* + z_{tB}^* < q^*$, then

$$\begin{cases} 
\bar{z}_B^* = 0 \\
\bar{z}_B^* = z_{tB}^* + z_{tB}^*
\end{cases}$$

(e) If $\varepsilon_t < 1$, then

$$\begin{cases} 
\bar{z}_B^* = 0 \\
\bar{z}_B^* = \varepsilon_t^{-1}z_{tB}^* + z_{tB}^*
\end{cases}$$

where $\varepsilon_t$ is the black market price of real balance of moneyB in terms real balance of moneyB and $\bar{\psi}$ such that $u'(\bar{\psi}) = \varepsilon_t$.

**Proof.** In appendix. ■ To interpretation of in Lemma 3 is similar to that of Lemma 2. When $\varepsilon_t > 1$, in the black market a unit real balance of moneyB costs more than one unit of moneyA, while utility increases with real balance of moneyB, there is a cost to it as well. The
If the buyer successfully readjusts here portfolio to $\psi$, otherwise he will increase it as much as his wealth would permit. When $\varepsilon_t = 1$, real balance of $\text{money}_A$ and that of $\text{money}_B$ trades one to one. Due to the strict concavity of the utility function, an increase in real balance of $\text{money}_B$ strictly increases the objective function as long as $z^B_\ast < q^\ast$ after that the objective function increases in $z^A_\ast + z^B_\ast$. Since real balance of $\text{money}_A$ and that of $\text{money}_B$ trade one to one, the buyer is indifferent between increasing or not increasing his real holding of $\text{money}_B$. When $\varepsilon_t < 1$, it becomes cheaper to increase real balance of $\text{money}_B$ and more than one unit of real balance of $\text{money}_B$ can be obtained with a unit real balance of $\text{money}_A$. Therefore, it is optimal to convert entire $\text{money}_A$ into $\text{money}_B$.

Now, that I have specified the exact portfolios resulting from trades in the black market, I can find the exact terms of trade in the special goods market when a country $B$ buyer successfully transacts in the black market. First consider a country $B$ buyer visiting $SGM_A$. The exact terms of trade will depend on the readjusted portfolio of the buyer. Given that payments are always made in $\text{money}_A$, from Lemma 1, whenever his real balance of $\text{money}_A$ is $q^\ast$ or more, he would buy $q^\ast$ amounts of the special good. Otherwise he will buy whatever amount he can afford by spending his entire holdings of $\text{money}_A$. Combining Lemmas 1 and 2 I can write down a country $B$ buyer’s terms of trade in $SGM_A$ after he successfully readjusts his portfolio in the black market. This can be summarized by the following Lemma

**Lemma 4.** Consider a country $B$ buyer who is matched with a country $A$ seller in $SGM_A$. If the buyer successfully readjusts here portfolio to $(\tilde{z}^A_B, \tilde{z}^B_B)$ in the black market then, the terms of trade in $SGM_A$ meeting between country $B$ buyer and country $A$ seller is given by

(a) If $\varepsilon_t > 1$ and $z^A_{tB} + \varepsilon_t z^B_{tB} \geq q^\ast$, then $\{\tilde{q}_{B|bm} = q^\ast, \tilde{d}^A_{bm} = q^\ast, \tilde{d}^B_{bm} = 0\}$

(b) If $\varepsilon_t > 1$ and $z^A_{tB} + \varepsilon_t z^B_{tB} < q^\ast$, then $\{\tilde{q}_{B|bm} = z^A_B, \tilde{d}^A_{bm} = \tilde{z}^A_B, \tilde{d}^B_{bm} = 0\}$

(c) If $\varepsilon_t = 1$ and $z^A_{tB} + z^B_{tB} \geq q^\ast$, then $\{\tilde{q}_{B|bm} = q^\ast, \tilde{d}^A_{bm} = q^\ast, \tilde{d}^B_{bm} = 0\}$

(d) If $\varepsilon_t = 1$ and $z^A_{tB} + z^B_{tB} < q^\ast$, then $\{\tilde{q}_{B|bm} = z^A_B, \tilde{d}^A_{bm} = \tilde{z}^A_B, \tilde{d}^B_{bm} = 0\}$

(e) If $\varepsilon_t < 1$ and $z^A_{tB} + \varepsilon_t z^B_{tB} \geq \bar{\chi}$, then $\{\tilde{q}_{B|bm} = \bar{\chi}, \tilde{d}^A_{bm} = \bar{\chi}, \tilde{d}^B_{bm} = 0\}$

(f) If $\varepsilon_t < 1$ and $z^A_{tB} + \varepsilon_t z^B_{tB} < \bar{\chi}$, then $\{\tilde{q}_{B|bm} = \tilde{z}^A_B, \tilde{d}^A_{bm} = \tilde{z}^A_B, \tilde{d}^B_{bm} = 0\}$

**Proof.** This proof is trivial and therefore omitted. \[\blacksquare\]
On similar lines, I can specify the exact terms of trade in $SGM_B$ for a country $B$ buyer who has successfully traded in the black market and readjusted his portfolio. Given that payments are always made in $money_B$ in this market, from Lemma 1, whenever his real balance of $money_B$ is $q^*$ or more, he would buy $q^*$ amounts of the special good. Else, he will buy whatever amount he can afford by spending his entire holdings of $money_B$. Combining Lemmas 1 and 3 I can write down a country $B$ buyer’s terms of trade in $SGM_A$ after he successfully readjusts his portfolio in the black market. This can be summarized by the following Lemma

**Lemma 5.** Consider a country $B$ buyer who is matched with a country $B$ seller in $SGM_B$. If the buyer successfully readjusts here portfolio to $(z_{t+1}^A, z_{t+1}^B)$ in the black market then, the terms of trade in $SGM_B$ meeting between country $B$ buyer and country $B$ seller is given by

(a) If $\varepsilon_t > 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B \geq \varepsilon_t \bar{\psi}$, then $\{q_B | bm = \bar{\psi}, d_{bm}^A = 0, d_{bm}^B = \bar{\psi}\}$

(b) If $\varepsilon_t > 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B < \varepsilon_t \bar{\psi}$, then $\{q_B | bm = \bar{\psi}, d_{bm}^A = 0, d_{bm}^B = \bar{\psi}\}$

(c) If $\varepsilon_t = 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B \geq q^*$, then $\{q_B | bm = q^*, d_{bm}^A = 0, d_{bm}^B = q^*\}$

(d) If $\varepsilon_t = 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B < q^*$, then $\{q_B | bm = q^*, d_{bm}^A = 0, d_{bm}^B = q^*\}$

(e) If $\varepsilon_t < 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B \geq \varepsilon_t q^*$, then $\{q_B | bm = q^*, d_{bm}^A = 0, d_{bm}^B = q^*\}$

(f) If $\varepsilon_t < 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B < \varepsilon_t q^*$, then $\{q_B | bm = q^*, d_{bm}^A = 0, d_{bm}^B = q^*\}$

### 4.3 CM Portfolio Problem

In this section, I describe the optimal portfolio choices of buyers from the countries $A$ and $B$. As a first step I characterize the objective functions for these agents. First, consider a country $B$ buyer who goes to the black market. The max operator problem in Eq. (6) represents the portfolio choice problem (i.e. choosing $z_{t+1}^A, z_{t+1}^B$) of a country $B$ buyer. To derive the portfolio choice problem of a country $B$ buyer who chooses to go to the black market I lead Eq (9) by one period and substitute it in the max operator of Eq. (6). After replacing the $W_{B}(.)$s with their linear expression and dropping the constant terms, I get

$$
\max_{z_{t+1}^A, z_{t+1}^B} \left[ \alpha \delta \{u(q_B | bm) + \bar{z}_B^A - \bar{d}_{bm}^A + \bar{z}_B^B \}ight. \\
+ \alpha (1 - \delta) \{u(q_B | bm) + \bar{z}_B^A + \bar{z}_B^B - d_{bm}^B \} \\
- \{(1 + t_A)z_{t+1}^A + (1 + t_B)z_{t+1}^B \} \right]
$$

(16)
where \((\bar{z}^A_B, \bar{z}^B_B)\) and \((\tilde{z}^A_B, \tilde{z}^B_B)\) are the country B buyer’s post-black market trade readjusted portfolios when he is matched with a foreign (country A) seller and when he is matched with a local seller (county B) respectively. These readjusted portfolios are given by Lemmas 2 and 3. The pairs \((\bar{q}_{B|bn}, \bar{a}^A_B)\) and \((\tilde{q}_{B|bn}, \tilde{a}^B_B)\) represent the quantity bought in \(SGM_A\) \((SGM_B)\) and real payment in \(money_A\) \((money_B)\) after having readjusted his portfolio in the black market. These are given by Lemmas 4 and 5.

Next, I consider the portfolio choice problem of a country B buyer who does not access the black market. Using Eq. (10), the max operator of Eq. (6) and after replacing the \(W_{B|B}(.).
\)s with their linear expression and dropping the constant terms, I get

\[
\max_{z^A_{t+1B}, z^B_{t+1B}} \left[ \delta \{u(\bar{q}_B) - \bar{q}_B\} + (1 - \delta) \{u(q_B) - q_B\} - \iota_A z^A_{t+1B} - \iota_B z^B_{t+1B} \right] (17)
\]

Finally, I consider a country A buyer’s portfolio choice problem. To do this I lead Eq. (11) by one period and substitute it in the max operator of Eq. (6). After replacing the \(W^B_B(.)\)s with their linear expression and dropping the constant terms, I get

\[
\max_{z^A_{t+1A}, z^B_{t+1A}} \left[ \delta \{u(\bar{q}_A) - \bar{q}_A\} + (1 - \delta) \{u(q_A) - q_A\} - \iota_A z^A_{t+1A} - \iota_B z^B_{t+1A} \right] (18)
\]

**Lemma 6.** In any equilibrium, \(\iota_i > 0, i \in \{A, B\}\).

**Proof.** This is a standard result in monetary theory. If \(\iota_i < 0\) (i.e. if \(\phi_i < \beta \phi_{i+1}\)) for any \(i \in \{A, B\}\), then country A buyers will have an infinite demand for \(money_i\). Therefore, the equilibrium is not well defined. Since \(\iota_i\) is never negative for the extreme case of \(\iota_i = 0\) following Lagos and Wright (2005) I assume that \(\iota_i\) approaches 0 from above. This rules out the indeterminacy of optimal portfolio. ■

### 4.4 Entry to the Black Market

In this section I discuss a country B buyer’s decision to enter the black market. After entering the first subperiod, once buyers realize the idiosyncratic shock that allows them the opportunity to consume a foreign special good or a local special good, country B buyers are separated into two groups. There is a mass \(\delta \in (0, 1)\) of country B buyers who buy from country A sellers. These agents would like to sell their real holdings of \(money_B\) and increase their real holding of \(money_A\). This will allow them to buy a greater quantity of foreign goods. There is a mutually exclusive group of country B buyers of mass \(1 - \delta\) that buy locally and pay in \(money_B\) – these buyers would want to get rid of their real balance
of money\textsubscript{A} and increase their real balance of money\textsubscript{B} so that they can buy more of the local goods. Therefore, these two sets of buyers create the two sides of the black market for currency exchange. For a black market to exist, both sets must want to use the black market. This would be the case when for both sets of buyers the payoff from using the black market is strictly greater than the payoff from not using it. The following lemma captures this idea

**Lemma 7.** The black market exists if and only if the following conditions hold

\[
\alpha [u(\tilde{q}_{B|bm}) + \tilde{z}_B^A + q_B|bm] - u(q_{B|nbm}) + z_B^A + \tilde{z}_B B > u(\tilde{q}_{B|bm}) - q_B|bm - \tilde{z}_B B
\]

and

\[
\alpha [u(q_{B|bm}) + \tilde{z}_B^A + \tilde{z}_B B] > u(q_{B|nbm}) + z_B^A + \tilde{z}_B B - \tilde{d}_B
\]

**Proof.** In appendix. ■

The first condition of this lemma suggests that the payoff from entering the black market for a country B buyer matched with a foreign (country A) seller is greater than the payoff from not entering it. The second condition suggests the same for a country B buyer matched with a local (country B) seller. Assuming that \(\alpha\) and other macroeconomic fundamentals are such that these conditions are satisfied, in the next section I characterize the equilibrium in this two-country model with a black market in country B.

## 5 Equilibrium in the Two-Country Model with Black Market

This section describes the equilibrium of the two-country, two-monies model. The focus is on a stationary equilibrium where aggregate real balances in each country are constant over time. Therefore, the rate of return of money\textsubscript{i} in each country is constant and will equal \(\pi_i = \frac{\phi_{i,t+1}}{\phi_i t}\). Since, the focus is on stationary monetary equilibrium, I drop the time subscripts from the variables.

**Definition 1.** Given \(\alpha\), a stationary monetary equilibrium for the two-country economy with a black market for currencies is a list of quantities traded in SGM\(_i\), \(i \in \{A, B\}\): \(\{(q_A, d_A^A, d_B^A), (\tilde{q}_B^A, \tilde{d}_B^A, \tilde{d}_B^B)\}\) and \(\{(q_{B|bm}, q_{bm}^A, q_{bm}^B), (\tilde{q}_{B|bm}, \tilde{d}_{bm}^A, \tilde{d}_{bm}^B)\}\), end of period real balances \(\{z_A \equiv (z_A^A, z_A^B), z_B \equiv (z_B^A, z_B^B)\}\) of country A buyers and B respectively, post-black market trade portfolios for country B buyers, \(\{(\tilde{z}_B^A, \tilde{z}_B^B), (\tilde{z}_B^A, \tilde{z}_B^B)\}\) and the black market terms of trade between money\textsubscript{A} and money\textsubscript{B}, \(\varepsilon\), such that
1. \((q_A, d^A, d^B)\) and \((\overline{q}^A, \overline{d}^A, \overline{d}^B)\) solves country A buyer’s bargaining problem when he is matched with a country A seller in SGM\(_A\) and with a country B seller in SGM\(_B\) respectively.

2. \((q_{B|m}, d_{B|m}, d_{B|m}^B)\) and \((\overline{q}_{B|m}, \overline{d}_{B|m}, \overline{d}_{B|m}^B)\) solves country B buyer’s bargaining problem when he is matched with a country B seller in SGM\(_B\) and with a country A seller in SGM\(_A\) respectively.

3. \((z^A_A, z^B_B)\) and \((z^A_B, z^B_B)\) solves the portfolio problem in the second subperiod for a country A buyer and country B respectively.

4. Taking \(\varepsilon\) as given \(\{(\overline{z}^A_B, \overline{z}^B_B), (\overline{z}^A_B, \overline{z}^B_B)\}\) solves the country B buyer’s black market portfolio readjustment problem.

5. \(\varepsilon\) clears the black market: \(\delta \overline{z}^A_B + (1 - \delta) \overline{z}^B_B = \overline{z}^A_B\) and \(\delta \overline{z}^B_B + (1 - \delta) \overline{z}^B_B = \overline{z}^B_B\)

6. CM money market clears: \(z^A_A + z^A_B = \phi^M_t A\) and \(z^A_A + z^B_B = \phi^M_t B\)

Now, I discuss some aspects of the equilibrium pertaining to the black market terms of trade and portfolio choice by agents from both countries.

**Proposition 1.** The portfolio choice of a country A buyer is unaffected by the presence of black market in country B and \((z^A_A, z^B_B)\), its optimal end of period portfolio in the stationary monetary equilibrium solves

\[
\begin{align*}
u'(z^A_A) &= 1 + \frac{\iota_A}{1 - \delta} \quad \text{and} \quad u'(z^B_B) = 1 + \frac{\iota_B}{\delta}
\end{align*}
\]

This is expected since agents (i.e. buyers or sellers) from country A or sellers from country B do not participate in the black market. Also, the terms of trade in SGM\(_i\) between country A buyer and country i seller \((i \in \{A, B\})\) depends only on the amount of money \(i\) a buyer from country A carries. It is important to note the implication of the above proposition. It implies \(z^A_A, z^B_B < q^*\) at the stationary monetary equilibrium. As a result a buyer from country A would always consume less than the optimal level of consumption, \(q^*\). When \(z^A_A \geq q^*\) (or \(z^B_B \geq q^*\)), \(u(q^*) - q^*\) is flat in \(z^A_A\) (or \(z^B_B\)) and first order conditions of (18) would imply \(\iota_A = 0\) or \(\iota_B = 0\). Therefore, it is not possible to have \(z^A_A \geq q^*\) or \(z^B_B \geq q^*\).

**Proposition 2.** In the stationary monetary equilibrium, \(\varepsilon = \frac{1 + \iota_B}{1 + \iota_A}\)
Proof. In appendix. ■

This result is intuitive. If \( \iota_B > \iota_A \), it is costlier to hold \( \text{money}_B \) and ideally agents would want to hold less of \( \text{money}_B \) real balance. However, due to the presence of the black market in future there will be an additional demand of \( \text{money}_B \) real balance by country \( B \) buyers matched with local sellers. Therefore, country \( B \) buyers who have been matched with local sellers, in order to acquire the extra \( \text{money}_B \) real balance will have to compensate other country \( B \) buyers not matched will local sellers for holding the low return money. As a result, in the black market, a unit real balance of \( \text{money}_B \) trades for more than one unit real balance of \( \text{money}_A \). In terms, of black market premium, \( \varepsilon^{-1} - 1 = (\iota_A - \iota_B)/(1 + \iota_B) \), a higher \( \iota_B \) implies negative black market premium, i.e. as inflation rates in a country goes up, in the stationary monetary equilibrium it will witness a decline in black market premium. If \( \iota_B > \iota_A \), then in the stationary monetary equilibrium the premium would be negative. The exact opposite happens when \( \iota_B < \iota_A \): agents would want to hold less of \( \text{money}_A \) real balance. However, country \( B \) buyers who have been matched with foreign sellers, and who want to acquire the extra \( \text{money}_A \) real balance will have to compensate other country \( B \) buyers not matched will foreign sellers for holding the low return money. As a result a unit real balance of \( \text{money}_A \) trades for more than one unit real balance of \( \text{money}_B \). In this case the black market is flushed with \( \text{money}_B \), while \( \text{money}_A \) is scarce. Therefore, in the stationary monetary equilibrium, \( \text{money}_A \) despite being a low return money, will fetch a positive premium. Finally, in the case of \( \iota_A = \iota_B \) agents value both currency equally and in the black market real balance of \( \text{money}_A \) will trade one-to-one for real balance of \( \text{money}_B \). In this case the premium would be zero.

Connection to Covered Interest Parity Condition: The result presented in Proposition 2 bears a striking resemblance to the covered interest parity condition in international finance. This is reasonable because both are arbitrage condition. In international finance the CIP condition states that one cannot buy one country’s asset that pays higher rate of interest and make a profit because exchange rates will adjust and all such profit making opportunities would be eroded away. Here instead of asset markets in two countries we have two currency markets in the same country where exchange rates could be different. One could, for all practical purpose, buy a currency in one of the foreign exchange markets (e.g. official) in Country \( B \) and sell it another foreign exchange market (black) and make a profit. However, at steady-state, the exchange rates in these two markets would adjust in such a way and align with the interest rates of the two currencies such that these arbitrage
opportunities would be removed.

Before, I discuss the implications of optimal portfolio choice of a country \( B \) buyer who accesses the black market, let us define the following objects

\[
G(\varepsilon_t) = \alpha(1 - \delta)\varepsilon_t^{-1}u'(\varepsilon_t q^*) + \alpha \delta - 1
\]

\[
H(\varepsilon_t) = \alpha \delta u'(\varepsilon_t q^*) + \alpha(1 - \delta)\varepsilon_t^{-1} - 1
\]

\[
\bar{\alpha} = \frac{\varepsilon_t}{(1 - \delta)u'(\varepsilon_t q^*) + \delta \varepsilon_t}
\]

\[
\hat{\alpha} = \frac{1}{\delta \varepsilon_t u'(\varepsilon_t q^*) + (1 - \delta)}
\]

Details of end of period optimal portfolio choice for a country \( B \) buyer accessing the black market is provided in the Appendix. Here I discuss the implications of this portfolio choice regarding their consumption levels in foreign and domestic special goods market. The next proposition summarizes these implications.

**Proposition 3.** At the stationary monetary equilibrium \( 1 + \iota_B = \varepsilon(1 + \iota_A) \)

1. If \( \iota_B > \iota_A \) and
   
   a. \( \iota_A \leq G(\varepsilon) \) with \( \alpha > \bar{\alpha} \), then \( \tilde{q}_{B|bm} = q^* \), \( q_{B|bm} < q^* \).
   
   b. \( \iota_A > G(\varepsilon) \), then \( \tilde{q}_{B|bm} < q^* \), \( q_{B|bm} < q^* \).

2. If \( \iota_B = \iota_A \), then \( \tilde{q}_{B|bm} < q^* \), \( q_{B|bm} < q^* \).

3. If \( \iota_B < \iota_A \) and
   
   a. \( \iota_A \leq H(\varepsilon) \) with \( \alpha > \hat{\alpha} \), then \( \tilde{q}_{B|bm} < q^* \), \( q_{B|bm} = q^* \).
   
   b. \( \iota_A > H(\varepsilon) \), then \( \tilde{q}_{B|bm} < q^* \), \( q_{B|bm} < q^* \).

Case 1(a) of the Proposition 3 implies while \( money_B \) may be costlier to hold, if country \( B \) buyers have sufficient access to the black market (\( \alpha > \bar{\alpha} \)) and if the cost of holding \( money_A \) is not too high, then since a unit real balance of \( money_B \) real balance trades for more than one unit of \( money_A \) real balance, it is possible for the buyer to trade his holdings of \( money_B \) to sufficiently increase his real balance of \( money_A \) in the black market and consume the optimal level \( q^* \) when matched with a foreign seller. On the other hand, if \( \iota_A \) is higher than a certain level (\( > G(\varepsilon) \)), then the rate at which a unit \( money_A \) real balance trades for real balance of \( money_A \) goes down and despite access to black market a buyer cannot increase
his consumption to \( q^* \) in either special goods markets. In Case 2, if both currencies have equal cost, real balances of monies trade one to one in the black market and it presents no advantage like before and buyers consume below \( q^* \). Case 3(a) implies that with sufficient access to the black market (\( \alpha > \hat{\alpha} \)), \( \iota_A \) bounded below \( H(\varepsilon) \) and a unit real balance of money \( A \) trading for more than a unit real balance of money \( B \) in the black market buyers can convert their entire money \( A \) into money \( B \) and increase their consumption to \( q^* \) when matched with a local seller. These results are in sharp contrast to the case when buyers do not access the black market. When country \( B \) buyers do not access the black market, their portfolio choice problem is given by (17). The optimal portfolios in that case always satisfy \( u'(z_{tB}^A) = 1 + \frac{\iota_A}{\delta} \), \( u'(z_{tB}^B) = 1 + \frac{\iota_B}{1-\delta} \) suggesting that \( q_{B|bm} > q^* \), \( q_{B|bm} < q^* \) always.

5.1 Welfare in the Presence of Black Market

This section concludes with a discussion of the model’s welfare properties. Welfare in country \( i \in \{A, B\} \) is defined as the steady-state sum of buyers’ and sellers’ utilities in country \( i \), weighted by their respective measures in the first subperiod:

\[
W_i = V^{B}_i + V^{S}_i
\]

Now, because sellers do not bring any real balances to the first subperiod and because their real payment is exactly equal to their amount of production through TIOLI offers from buyer, sellers’ utility in the first subperiod, \( V^{S}_i = 0 \). Therefore, welfare in country \( i \), \( W_i = V^{B}_i \). Therefore, welfare of country \( B \) when buyers use the black market

\[
W_{B|bm} = \alpha \delta [u(\bar{q}_{B|bm}) - d_{bm}^A + z^A + z^B] + \alpha (1 - \delta) [u(q_{B|bm}) - d_{bm}^B + z^A + z^B] + W^B_B(0)
\]

Welfare of country \( B \) when buyers do not use the black market

\[
W_{B|nbm} = \delta [u(\bar{q}_{B|nbm}) - d_{nbm}^A + z_{tB}^A + z_{tB}^B] + (1 - \delta) [u(q_{B|nbm}) - d_{nbm}^B + z_{tB}^A + z_{tB}^B] + W^B_B(0)
\]

The existence of black market requires participation from both sides of the market: country \( B \) buyers matched with country \( A \) sellers as well as country \( B \) buyers matched with country \( B \) sellers. Therefore, existence of black market implies Lemma 7 is satisfied. If Lemma 7 is satisfied then the two conditions imply \( W_{B|bm} > W_{B|nbm} \). Therefore, in this model, if the black market exists, it strictly raises welfare for country \( B \)'s residents. Since, the presence of black market doesn’t affect country \( A \) buyers’ portfolio choice and consumption patterns,
\( \mathcal{W}_A \) is unchanged by the presence of black market in country \( B \).

6 Asymmetric Penalty

So far, we have considered a model that only explains why black market premium goes down with rise in domestic inflation in relation to foreign inflation. While this maybe true for a subset of countries, the proposed model do not explain what is observed more often, i.e. premium rises with rise in inflation rate. In this section, I present an alternative confiscation rule where residents are penalized for participating in the black market by confiscating only their foreign currency holding (\( \text{money}_A \)). This rule seems more realistic and is often in place in many countries. Thus, this is an asymmetric penalty. Proposition 4 summarizes the outcome in this case

**Proposition 4.** The black market premium on foreign currency is rising in domestic inflation if the buyer’s DM utility function is sufficiently elastic, else it is decreasing in domestic inflation.

*Proof.* In appendix. ■

High elasticity of the buyer’s DM utility function implies greater risk aversion. When there is an increase in domestic inflation, to avoid inflation tax agents would find it beneficial to hold more of the foreign currency. As a result, the black market would be flush with foreign currency since in every agents’ portfolio the proportion of foreign currency increases. As a result, the black market premium on foreign currency decreases. However, this increase in foreign currency holding comes with a risk. Now they are susceptible to a greater amount of consumption loss if confiscation happens which wouldn’t be the case had they held more of domestic currency. Therefore, agents need to be compensated for the risk they are taking in holding more foreign currency. This has an increasing effect on the premium on foreign currency. The net effect would depend on agents’ risk aversion. Higher the risk aversion, higher is the premium on black market. If the utility function is sufficiently elastic, i.e. if agents are sufficiently risk averse then the increasing effect on premium due to risk aversion outweighs the decreasing effect on premium due to increased supply of foreign currency.
7 Conclusion

In this paper I show that it is not necessarily true that there is a positive correlation between black market premium rate and the domestic inflation (relative to a foreign currency). As shown through the empirical exercise, in fact, this correlation could be negative for some countries. While a sudden unexpected rise in inflation rate could spur residents of a country to quickly sell off their domestic currency at a discount in the black market thereby raising black market premium rate, the effect would be quite different when a country has gone through high periods of domestic inflation and inflation has stabilized at a high value. Agents of such a country will then build this information of high but stable domestic inflation in their portfolio decision. Since real balances in the portfolio of currencies are chosen according to their rates of return, agents would hold less of their wealth in domestic currency and more of it in the low inflation/ high return currency. Thus allowing for currency substitution (commonly known as dollarization), we could get a negative correlation between black market premium on foreign currency and domestic inflation if agents are not too risk averse. Unlike other papers where the black market is a free market that is a consequence of foreign exchange controls or those models where the black market is used to channel earnings from illegal production, the black market we present here is rather benign. It is an informal, unregulated market of currency exchange that operates in the absence of a formal market of currency exchange and increases welfare. One could extend this model by adding foreign exchange controls in the \( CM \) by adding a constraint that sets an upper bound to the real balance of \( money_A \) a buyer from country \( B \) can hold. However, the result would be the same. If the optimal choice for real \( money_A \) balance is less than the upper bound, the constraint is non-binding and it would be just like the model presented here. If it is binding, then buyers will hold \( money_A \) balances up to the limit, but now the optimal choice of real balance of \( money_B \) will be reduced considerably as well. Since premium rates are determined by relative excess supply of real balances of the two monies, it will have a decreasing (increasing) effect on the premium as inflation (rate of return) for \( money_B \) goes up (down) in the stationary monetary equilibrium. The net effect on the premium would also be determined with agents’ attitude towards risk as discussed in the penultimate section of this paper. This model is not without limitations though. Firstly, it assumes away many other uses of the black market - most importantly the features such as laundering money earned through illegal production. Secondly, this model studies behavior in the stationary monetary equilibrium. It would be also interesting to extend this model in order to study behavior of the premium to an unanticipated inflation shock in the short run. Adding some shocks to the model would be
References


Appendix

Table 3: Selection criteria for full sample

<table>
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<tr>
<th>Lag</th>
<th>mBIC</th>
<th>mAIC</th>
<th>mQIC</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-17.865</td>
<td>-55.731</td>
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<td>2</td>
<td>-75.445</td>
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<td>3</td>
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Table 4: Selection criteria for Iran, India, Nepal, Pakistan, Sri Lanka and Venezuela

<table>
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<th>mAIC</th>
<th>mQIC</th>
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</thead>
<tbody>
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<td>-51.294</td>
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<td>2</td>
<td>-59.401</td>
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<td>3</td>
<td>-31.183</td>
<td>-10.693</td>
<td>-18.850</td>
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Table 5: Eigenvalue stability condition in panel VAR with Iran, India, Nepal, Pakistan, Sri Lanka and Venezuela

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<th>Real</th>
<th>Imaginary</th>
<th>Modulus</th>
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<td>-0.069</td>
<td>0.804</td>
</tr>
<tr>
<td>0.801</td>
<td>0.069</td>
<td>0.804</td>
</tr>
<tr>
<td>0.548</td>
<td>0.000</td>
<td>0.548</td>
</tr>
</tbody>
</table>
Lemma 1.

Proof.

\[
\max_{q_t,d_t} \{ u(q_t) - d_t \} \\
\text{s.t. } q_t = d_t
\]

The above problem can be rewritten as: \(\max_{q_t} \{ u(q_t) - q_t \} \) which yields \(q_t = q^* \) and therefore \(d_t = q^* \). Note that for \(q_t < q^* \), \(u(q_t) - q_t \) is strictly increasing in \(q_t \). Therefore, if \(z_t^i < q^* \), the optimal solution would entail \(q_t = z_t^i \) and \(d_t = z_t^i \). ■

Lemma 2.

Proof. Using linearity of \(W_B^B(.)\), the objective function can be rewritten as \(u(\tilde{q}_B|bm) + \tilde{z}_B^A + \tilde{z}_B^B - \tilde{d} + W_B^B(0)\). Dropping the term \(W_B^B(0)\) and using \(\tilde{d} = \tilde{q}_B|bm \) (from Lemma 1), the problem simplifies to maximization of \(u(\tilde{q}_B|bm) - \tilde{q}_B|bm + \tilde{z}_B^A + \tilde{z}_B^B \), with respect to \(\tilde{z}_B^A \) and \(\tilde{z}_B^B \) subject to the budget constraint. From the budget constraint: \(\tilde{z}_B^B = \tilde{z}_B^* - \tilde{z}_B^A = \tilde{z}_B^* - \tilde{d}_B^* - \tilde{d}_B^A \).

We plug this back in the objective function and a country \(B\) buyer’s portfolio readjustment, when he is matched with a foreign seller, becomes:

\[
\max_{\tilde{z}_B^A} \left( u(\tilde{q}_B|bm) - \tilde{q}_B|bm + \tilde{z}_B^A + \tilde{z}_B^B + (1 - \varepsilon_t^{-1})\tilde{z}_B^A \right)
\]

When \(1 - \varepsilon_t^{-1} > 0\), i.e. \(\varepsilon_t > 1\) : there would be two cases

Case 1: If \(z_t^A + \varepsilon_t z_t^B < q^*\), then it must be \(\tilde{z}_B^A < q^*\). When \(\tilde{z}_B^A < q^*\), then the objective function is strictly increasing in \(\tilde{z}_B^A\). Therefore, the solution is \(\tilde{z}_B^* = z_t^A + \varepsilon_t z_t^B \) and \(\tilde{z}_B^B = 0\).

Case 2: If \(z_t^A + \varepsilon_t z_t^B \geq q^*\). From Lemma 1, \(\tilde{q}_B|bm\) is increasing in \(\tilde{z}_B^A\) as long as \(\tilde{z}_B^A < q^*\). And, for \(\tilde{z}_B^A \geq q^*\), \(\tilde{q}_B|bm = q^*\). So \(u(\tilde{q}_B|bm) - \tilde{q}_B|bm\) is weakly increasing in \(\tilde{z}_B^A\), while \((1 - \varepsilon_t^{-1})\tilde{z}_B^A\) is strictly increasing in \(\tilde{z}_B^A\). Therefore, it is optimal to convert entire holdings of \(money_B\) to \(money_A\). Therefore, the solution is \(\tilde{z}_B^* = z_t^A + \varepsilon_t z_t^B \) and \(\tilde{z}_B^B = 0\).

When \(1 - \varepsilon_t^{-1} = 0\), i.e. \(\varepsilon_t = 1\) : the objective function becomes \(u(\tilde{q}_B|bm) - \tilde{q}_B|bm + z_t^A + z_t^B \).

There would be two cases:

Case 1: If \(z_t^A + z_t^B < q^*\), it must be always \(\tilde{z}_B^A < q^*\). When \(\tilde{z}_B^A < q^*\), then using Lemma 1, \(u(\tilde{q}_B|bm) - \tilde{q}_B|bm\) is strictly increasing in \(\tilde{z}_B^A\). Therefore, the optimal solution will entail \(\tilde{z}_B^* = z_t^A + z_t^B \) and \(\tilde{z}_B^B = 0\).

Case 2: If \(z_t^A + z_t^B \geq q^*\), then the objective function, \(u(\tilde{q}_B|bm) - \tilde{q}_B|bm + z_t^A + z_t^B\) is increasing

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in \( z^A_B \) only until \( z^A_B \rightarrow q^* \). For \( z^A_B \geq q^* \) it becomes flat – so the buyer is indifferent between increasing or not increasing his real balance of \( \text{money}_A \) beyond \( q^* \). Therefore, the solution is \( z^A_B \in [q^*, z^*_t + z^B_t] \) and \( z^B_t = z^*_t + z^A_t - z^A_B \).

**When** \( 1 - \varepsilon_t^{-1} < 0 \), i.e. \( \varepsilon_t < 1 \) : The \((1 - \varepsilon_t^{-1})z^A_B\) part of the objective function is always decreasing in \( z^A_B \) while \( u(q_B|bm) - q_B|bm \) is increasing in \( z^A_B \) as long as \( z^A_B < q^* \). In the case of \( \varepsilon_t < 1 \) we can never have \( z^A_B = q^* \) since that would mean \( q_B|bm = q^* \) and derivative of the objective function with respect to \( z^A_B \) would be \( 1 - \varepsilon_t^{-1} < 0 \). So, the optimal solution entails \( z^A_B < q^* \). In the set \( 0 \leq z^A_B < q^* \), after plugging in values given by Lemma 4 the derivative of the objective function w.r.t. \( z^A_B \) is

\[
u'(z^A_B) - \varepsilon_t^{-1}\]

Now, at \( z^A_B = 0 \), we have \( u'(z^A_B) \rightarrow \infty \). Therefore, we will have an interior solution for \( z^A_B \).

First order conditions for interior solution imply \( u'(z^A_B) - \varepsilon_t^{-1} = 0 \) or \( u'(z^A_B) = \varepsilon_t^{-1} \). Define \( \bar{\chi} \) such that \( u'(\bar{\chi}) = \varepsilon_t^{-1} \). Then, there are two possibilities:

**Case 1:** If \( z^A_B + \varepsilon_t z^B_t < \bar{\chi} \), then \( u'(z^A_B) - \varepsilon_t^{-1} \) the derivative of the objective function w.r.t. \( z^A_B \) is strictly positive. So, \( z^A_B = z^*_t + \varepsilon_t z^B_t \) and \( z^B_t = 0 \).

**Case 2:** If \( z^A_B + \varepsilon_t z^B_t \geq \bar{\chi} \), then as discussed above \( z^A_B = \bar{\chi} \) and \( z^B_t = z^*_t + \varepsilon_t^{-1}(z^A_B - \bar{\chi}) \).

**Lemma 3.**

**Proof.** Using linearity of \( W_B^B(\cdot) \), the objective function can be rewritten as \( u(q_B|bm) + z^A_B + z^B_B - d_B + W_B^B(0) \). Dropping the term \( W_B^B(0) \) and using \( d_B = q_B|bm \) (from Lemma 1), the problem simplifies to maximization of \( u(q_B|bm) - q_B|bm + z^A_B + z^B_B \), with respect to \( z^A_B \) and \( z^B_B \) subject to the budget constraint. From the budget constraint: \( z^A_B = z^*_t + \varepsilon_t z^B_t - \varepsilon_t z^B_B \).

We plug this back in the objective function and a country B buyer’s portfolio readjustment, when he is matched with a local seller, becomes:

\[
\max_{z^B_B} u(q_B|bm) - q_B|bm + z^A_B + \varepsilon_t z^B_B + (1 - \varepsilon_t)z^B_B
\]

**When** \( 1 - \varepsilon_t < 0 \), i.e. \( \varepsilon_t > 1 \) We claim that \( z^B_B \) cannot be \( \geq q^* \) because for \( z^B_B \geq q^* \), using Lemma 1, the derivative of the objective function with respect to \( z^B_B \) is strictly negative. So, optimal \( z^B_B < q^* \). In this range, the derivative of the objective function is:

\[
u'(z^B_B) - \varepsilon_t\]

39
At \( z_{B*}^B = 0 \) we have \( u'(z_{B*}^B) \to \infty \). Therefore, there exists an interior solution. First order condition for interior solution gives us \( u'(\bar{z}^B_{B*}) = \varepsilon_t \). Define \( \bar{\psi} \) such that \( u'(\bar{\psi}) = \varepsilon_t \). Then,

**Case 1:** If \( z^A_{iB} + \varepsilon_t \bar{z}^B_{iB} \geq \varepsilon_t \bar{\psi} \), then \( \bar{z}^A_{B*} = z^A_{iB} + \varepsilon_t(z^B_{B*} - \bar{\psi}) \) and \( \bar{z}^B_{B*} = \bar{\psi} \).

**Case 2:** If \( z^A_{iB} + \varepsilon_t \bar{z}^B_{iB} < \varepsilon_t \bar{\psi} \), then it must be \( \bar{z}^B_{B*} < \bar{\psi} \). In this range the objective function is strictly increasing in \( \bar{z}^B_{B*} \). Therefore, optimal portfolio: \( \bar{z}^A_{B*} = 0 \) and \( \bar{z}^B_{B*} = \varepsilon_t^{-1} z^A_{iB} + z^B_{iB} \).

**When** \( 1 - \varepsilon_t = 0 \), **i.e.** \( \varepsilon_t = 1 \) : the objective function becomes \( u(q_{B|bm}) - q_{B|bm} + z^A_{iB} + z^B_{iB} \).

There would be two cases:

**Case 1:** If \( z^A_{iB} + z^B_{iB} < q^* \), it must be always \( \bar{z}^B_{B*} < q^* \). When \( \bar{z}^B_{B*} < q^* \), then using Lemma 1, \( u(q_{B|bm}) - q_{B|bm} \) is strictly increasing in \( \bar{z}^B_{B*} \). Therefore, the optimal solution will entail \( \bar{z}^A_{B*} = 0 \) and \( \bar{z}^B_{B*} = z^A_{iB} + z^B_{iB} \).

**Case 2:** If \( z^A_{iB} + z^B_{iB} \geq q^* \), then the objective function, \( u(q_{B|bm}) - q_{B|bm} + z^A_{iB} + z^B_{iB} \) is increasing in \( z^B_{B*} \), as long as \( \bar{z}^B_{B*} < q^* \). For \( \bar{z}^B_{B*} \geq q^* \) it becomes flat - so the buyer is indifferent between increasing or not increasing his real balance of \( money_B \) beyond \( q^* \). Therefore, the solution is \( \bar{z}^A_{B*} = z^B_{iB} + z^A_{iB} - \bar{z}^B_{B*} \) and \( \bar{z}^B_{B*} \in \{q^*, z^A_{iB} + z^B_{iB}\} \).

**When** \( 1 - \varepsilon_t > 0 \), **i.e.** \( \varepsilon_t < 1 \) there would be two cases

**Case 1:** If \( z^A_{iB} + \varepsilon_t z^B_{iB} < q^* \), then it must be \( \bar{z}^B_{B*} < q^* \). When \( \bar{z}^B_{B*} < q^* \), then using Lemma 1, the objective function is \( u(\bar{z}^B_{B*}) - \bar{z}^B_{B*} + z^A_{iB} + \varepsilon_t \bar{z}^B_{iB} + (1 - \varepsilon_t) \bar{z}^B_{B*} \), which strictly increasing in \( \bar{z}^B_{B*} \). Therefore, the solution is \( \bar{z}^A_{B*} = 0 \) and \( \bar{z}^B_{B*} = \varepsilon_t^{-1} z^A_{iB} + z^B_{iB} \).

**Case 2:** If \( z^A_{iB} + \varepsilon_t z^B_{iB} \geq \varepsilon_t q^* \). From Lemma 1, \( q_{B|bm} \) is increasing in \( \bar{z}^B_{B*} \) as long as \( \bar{z}^B_{B*} < q^* \). And, for \( \bar{z}^B_{B*} \geq q^* \), \( q_{B|bm} = q^* \). So \( u(q_{B|bm}) - q_{B|bm} \) is weakly increasing in \( z^B_{B*} \), while \( (1 - \varepsilon_t) z^B_{B*} \) is strictly increasing in \( \bar{z}^B_{B*} \). Therefore, it is optimal to convert entire holdings of \( money_A \) to \( money_B \). Therefore, the solution is \( z^A_{B*} = 0 \) and \( \bar{z}^B_{B*} = \varepsilon_t^{-1} z^A_{iB} + z^B_{iB} \). 

**Lemma 7.**

**Proof.** The payoff from going to the black market for a country \( B \) buyer matched with foreign (country \( A \) seller) = \( \alpha[u(\bar{q}_{B|bm}) + W_B^{B}(\bar{z}^A_{B} - \bar{d}^A_{bm}, \bar{z}^B_{B})] + (1 - \alpha)W_B^{B}(0) = \alpha[u(\bar{q}_{B|bm}) - \bar{d}^A_{bm} + \bar{z}^A_{B} + \bar{z}^B_{B}] + W_B^{B}(0) \)

The payoff from not going to the black market for a country \( B \) buyer matched with foreign (country \( A \) seller) = \( [u(\bar{q}_{B|nbm}) + W_B^{B}(\bar{z}^A_{iB} - \bar{d}^A_{nbm}, \bar{z}^B_{iB})] = [u(\bar{q}_{B|nbm}) - \bar{d}^A_{nbm} + z^A_{iB} + z^B_{iB}] + W_B^{B}(0) \)

Therefore, a country \( B \) buyer matched with a foreign seller will access the black market if and only if: \( \alpha[u(\bar{q}_{B|bm}) - \bar{d}^A_{bm} + \bar{z}^A_{B} + \bar{z}^B_{B}] > [u(\bar{q}_{B|nbm}) - \bar{d}^A_{nbm} + z^A_{iB} + z^B_{iB}] \)
The payoff from going to the black market for a country B buyer matched with local (country B seller) = \( \alpha[u(q_{B|bm}) + W_B^B(\bar{z}_B - d_{nm} + z_B^B)] + (1 - \alpha)W_B^B(0) = \alpha[u(q_{B|bm}) - d_{nm} + \bar{z}_B^B] + W_B^B(0) \)

The payoff from going to the black market for a country B buyer matched with local (country B seller) = \( u(q_{B|nbm}) + W_B^B(z_A - d_{nm} + z_{tB}) = [u(q_{B|nbm}) - d_{nm} + z_A^B + z_{tB}] + W_B^B(0) \)

Therefore, a country B buyer matched with a foreign seller will access the black market if and only if: \( \alpha[u(q_{B|bm}) - d_{nm} + \bar{z}_B^A + \bar{z}_B^B] > [u(q_{B|nbm}) - d_{nm} + z_A^B + z_{tB}] \) ■

**Proposition 2 and 3.**

**Proof.** Using Lemma 1, country B buyer’s end of period (CM) portfolio choice problem (described in 16) can be rewritten as:

\[
\max_{z_A^B, z_{tB}} \left[ \alpha \delta \{ u(\bar{q}_{B|bm}) + \bar{z}_B^A - \bar{q}_{B|bm} + \bar{z}_B^B \} + \alpha(1 - \delta) \{ u(q_{B|bm}) + z_B^A - q_{B|bm} \} - \{(1 + t_A)z_{tB}^A + (1 + t_B)z_{tB}^B \} \right]
\]

The first order conditions with respect to \( z_{tB}^A \):

\[
\alpha \delta \left\{ u'(\tilde{q}_{B|bm}) \frac{\partial q_{B|bm}}{\partial z_{tB}^A} \frac{\partial z_{tB}^B}{\partial z_{tB}^A} + \frac{\partial z_{tB}^A}{\partial z_{tB}^A} - \frac{\partial q_{B|bm}}{\partial z_{tB}^B} \frac{\partial z_{tB}^B}{\partial z_{tB}^A} + \frac{\partial z_{tB}^B}{\partial z_{tB}^B} \right\} + \alpha(1 - \delta) \left\{ u'(q_{B|bm}) \frac{\partial q_{B|bm}}{\partial z_{tB}^B} \frac{\partial z_{tB}^A}{\partial z_{tB}^B} + \frac{\partial z_{tB}^A}{\partial z_{tB}^A} - \frac{\partial q_{B|bm}}{\partial z_{tB}^B} \frac{\partial z_{tB}^B}{\partial z_{tB}^B} \right\} - (1 + t_A) = 0 \quad (19)
\]

The first order conditions with respect to \( z_{tB}^B \):

\[
\alpha \delta \left\{ u'(\tilde{q}_{B|bm}) \frac{\partial q_{B|bm}}{\partial z_{tB}^B} \frac{\partial z_{tB}^A}{\partial z_{tB}^B} + \frac{\partial z_{tB}^A}{\partial z_{tB}^A} - \frac{\partial q_{B|bm}}{\partial z_{tB}^B} \frac{\partial z_{tB}^B}{\partial z_{tB}^B} + \frac{\partial z_{tB}^B}{\partial z_{tB}^B} \right\} + \alpha(1 - \delta) \left\{ u'(q_{B|bm}) \frac{\partial q_{B|bm}}{\partial z_{tB}^B} \frac{\partial z_{tB}^B}{\partial z_{tB}^B} + \frac{\partial z_{tB}^B}{\partial z_{tB}^B} - \frac{\partial q_{B|bm}}{\partial z_{tB}^B} \frac{\partial z_{tB}^B}{\partial z_{tB}^B} \right\} - (1 + t_B) = 0 \quad (20)
\]

When \( u'(q)q \) is strictly decreasing, given \( \varepsilon_t \), there are eight possible regions in the \((z_{tB}^A, z_{tB}^B)\) space that supports CM portfolio choice by country B buyers who access the black market.
for currencies:

1. $\varepsilon_t > 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B \geq \varepsilon_t \bar{\psi} > q^*$
2. $\varepsilon_t > 1$ and $q^* \leq z_{tB}^A + \varepsilon_t z_{tB}^B < \varepsilon_t \bar{\psi}$
3. $\varepsilon_t > 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B < q^* < \varepsilon_t \bar{\psi}$
4. $\varepsilon_t = 1$ and $z_{tB}^A + z_{tB}^B \geq q^*$
5. $\varepsilon_t = 1$ and $z_{tB}^A + z_{tB}^B < q^*$
6. $\varepsilon_t < 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B \geq \bar{x} > \varepsilon_t q^*$
7. $\varepsilon_t < 1$ and $\varepsilon_t q^* \leq z_{tB}^A + \varepsilon_t z_{tB}^B < \bar{x}$
8. $\varepsilon_t < 1$ and $z_{tB}^A + \varepsilon_t z_{tB}^B < \varepsilon_t q^* < \bar{x}$

We show that cases 1 and will violate Lemma 6 and then characterize the conditions under which an optimum will exist in the rest.

**Case 1:** The first order conditions become:

$$\alpha \delta + \alpha (1-\delta) - (1 + \iota_A) = 0$$
$$\alpha \delta \varepsilon_t + \alpha (1-\delta) \varepsilon_t - (1 + \iota_B) = 0$$

These yield: $\iota_B = \alpha \varepsilon_t - 1$ and $\iota_A = \alpha - 1 < 0$ which violates Lemma 6. Therefore, there cannot be an optimal portfolio in this region when $\varepsilon_t > 1$.

**Case 2:** The first order conditions are:

$$\alpha \delta \varepsilon_t + \alpha (1-\delta) u'(\varepsilon_t^{-1} z_{tB}^A + z_{tB}^B) - (1 + \iota_A) \varepsilon_t = 0$$
$$\alpha \delta \varepsilon_t + \alpha (1-\delta) u'(\varepsilon_t^{-1} z_{tB}^A + z_{tB}^B) - (1 + \iota_B) = 0$$

These conditions yield: $(1 + \iota_A) \varepsilon_t = (1 + \iota_B)$ or, $\varepsilon_t = (1 + \iota_B)/(1 + \iota_A)$. Also, $u'(\varepsilon_t^{-1} z_{tB}^A + z_{tB}^B) = (1 + \iota_A) \varepsilon_t^{-1} z_{tB}^A + z_{tB}^B$. Since, $q^* \leq z_{tB}^A + \varepsilon_t z_{tB}^B$, it must be $u'(\varepsilon_t^{-1} z_{tB}^A + z_{tB}^B) = (1 + \iota_A) \varepsilon_t^{-1} z_{tB}^A + z_{tB}^B \leq u'(\varepsilon_t q^*) \implies \iota_A \leq \alpha (1-\delta) \varepsilon_t^{-1} u'(\varepsilon_t^{-1} q^*) + \alpha \delta - 1$. The expression $\alpha (1-\delta) \varepsilon_t^{-1} u'(\varepsilon_t^{-1} q^*) + \alpha \delta - 1$ is increasing in $\alpha$, so to make sure that $\iota_A$ is not less than zero, it must be $\alpha \geq [(1-\delta) \varepsilon_t^{-1} u'(\varepsilon_t^{-1} q^*) + \delta]^{-1}$. On the other hand, since, $z_{tB}^A + \varepsilon_t z_{tB}^B < \varepsilon_t \bar{\psi}$, so $(1 + \iota_A) \varepsilon_t^{-1} \psi - \varepsilon_t^{-1} \varepsilon_t \bar{\psi} > \varepsilon_t \implies \iota_A > \alpha - 1$ which will always be satisfied since at the equilibrium $\iota_A > 0$. 

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Similarly, \((1+\mu_B)\frac{\alpha \delta \varepsilon_t}{\alpha(1-\delta)} \leq u'(\varepsilon_t q^*) \implies \mu_B \leq \alpha(1-\delta)u'(\varepsilon_t^{-1} q^*) + \alpha \delta \varepsilon_t - 1\). The expression \(\alpha(1-\delta)u'(\varepsilon_t^{-1} q^*) + \alpha \delta \varepsilon_t - 1\) is increasing in \(\alpha\), so to make sure that \(\mu_B\) is not less than zero, it must be \(\alpha \geq [(1-\delta)u'(\varepsilon_t^{-1} q^*) + \delta \varepsilon_t]^{-1}\). On the other hand, since \(z_{tB}^A + \varepsilon_t z_{tB}^B < \varepsilon_t \bar{\psi}\), the first order conditions are:

\[
\frac{(1+\mu_B)\delta \varepsilon_t - \alpha \delta \varepsilon_t}{\alpha(1-\delta)} > \varepsilon_t \implies \mu_B > \alpha \varepsilon_t - 1,
\]

which after substituting for \(\varepsilon_t\), boils down to \(\mu_B > 0\).

If \(\varepsilon_t > 1\), then \([(1-\delta)\varepsilon_t^{-1}u'(\varepsilon_t^{-1} q^*) + \delta]^{-1} > [(1-\delta)u'(\varepsilon_t^{-1} q^*) + \delta \varepsilon_t]^{-1}\). SO a necessary condition for optimal portfolio to exist in this region is \(\alpha \geq [(1-\delta)u'(\varepsilon_t^{-1} q^*) + \delta \varepsilon_t]^{-1}\). Also, \(0 < \mu_A \leq \alpha(1-\delta)\varepsilon_t^{-1}u'(\varepsilon_t^{-1} q^*) + \alpha \delta - 1\). Since \((1 + \mu_A)\varepsilon_t = (1 + \mu_B)\), the upper bound for \(\mu_B\): \(\mu_B \leq \alpha(1-\delta)u'(\varepsilon_t^{-1} q^*) + \alpha \delta \varepsilon_t - 1\) will be satisfied if the upper bound for \(\mu_A\) is satisfied.

For this optimal end of period portfolio \(\bar{z}_{tB}^A\), \(\bar{z}_{tB}^B\), we get \(u'(\frac{z_{tB}^A}{\delta \varepsilon_t}) = \frac{1 + \mu_B - \alpha \delta \varepsilon_t}{\alpha(1-\delta)} = \frac{(1+\mu_A)\varepsilon_t - \alpha \delta \varepsilon_t}{\alpha(1-\delta)}\).

Case 3: The first order conditions are:

\[
\alpha \delta u'(\varepsilon_t z_{tB}^A + \varepsilon_t z_{tB}^B) + \alpha(1-\delta)\varepsilon_t^{-1}u'(\varepsilon_t^{-1} z_{tB}^A + z_{tB}^B) - (1 + \mu_A) = 0
\]
\[
\alpha \delta \varepsilon_t u'(\varepsilon_t z_{tB}^A + \varepsilon_t z_{tB}^B) + (1-\alpha)\delta u'(\varepsilon_t^{-1} z_{tB}^A + z_{tB}^B) - (1 + \mu_B) = 0
\]

These conditions yield: \((1 + \mu_A)\varepsilon_t = (1 + \mu_B)/(1 + \mu_A)\). Also, \(u'(\varepsilon_t^{-1} z_{tB}^A + z_{tB}^B) > u'(\varepsilon_t^{-1} q^*)\) and \(u'(\varepsilon_t^{-1} z_{tB}^A + z_{tB}^B) > 1\). Therefore, \((1 + \mu_A) = \alpha \delta u'(\varepsilon_t z_{tB}^A + \varepsilon_t z_{tB}^B) + \alpha(1-\delta)\varepsilon_t^{-1}u'(\varepsilon_t^{-1} q^*) + \alpha \delta \implies \mu_A > (1-\delta)\varepsilon_t^{-1}u'(\varepsilon_t^{-1} q^*) + \alpha \delta - 1\).

Therefore, \(\mu_B > (1-\delta)u'(\varepsilon_t^{-1} q^*) + \alpha \delta \varepsilon_t - 1\).

For this optimal end of period portfolio \(\bar{z}_{tB}^A\), \(\bar{z}_{tB}^B\), we get \(u'(\frac{z_{tB}^A}{\delta}) = \frac{1 + \mu_B - \alpha \delta}{\alpha(1-\delta)}\cdot \frac{z_{tB}^A}{\delta \varepsilon_t} - (1 + \mu_A)\).

Case 4: The first order conditions are:

\[
\alpha \delta + \alpha(1-\delta) - (1 + \mu_A) = 0
\]
\[
\alpha \delta + \alpha(1-\delta) - (1 + \mu_B) = 0
\]

These conditions yield: \(\mu_A = \mu_B = \alpha - 1 < 0\) which is not possible. Therefore, there cannot be an optimal portfolio in this region.
Case 5: The first order conditions

\[ \alpha u'(z_{tB}^A + z_{tB}^B) - (1 + \iota_A) = 0 \]
\[ \alpha u'(z_{tB}^A + z_{tB}^B) - (1 + \iota_B) = 0 \]

These yield: \( \iota_A = \iota_B \). Note here also \( \varepsilon_t = \frac{1 + \iota_B}{1 + \iota_A} \) is satisfied (trivially). Also, it must be \( u'(z_{tB}^A + z_{tB}^B) = \frac{1 + \iota_A}{\alpha} = \frac{1 + \iota_B}{\alpha} > 1 \) which will be satisfied as long as \( \iota_A, \iota_B \geq 0 \). For this optimal end of period portfolio \( \bar{z}_{tB}^A = 0, \bar{z}_{tB}^B = z_{tB}^A + z_{tB}^B, \bar{z}_{tB}^A = z_{tB}^A + z_{tB}^B, \bar{z}_{tB}^B = 0 \). From the market clearing conditions of black market it can be show that \( z_{tB}^B = \frac{1 - \delta}{\delta} \bar{z}_{tB}^A \). Therefore, the optimal \( z_{tB}^A \) solves \( u'\left(\frac{z_{tB}^A}{\delta}\right) = \frac{1 + \iota_A}{\alpha} = \frac{1 + \iota_B}{\alpha} \).

Case 6: first order conditions are:

\[ \alpha \delta u'(z_{tB}^A + \varepsilon_t z_{tB}^B) + \alpha(1 - \delta)(1 + \iota_B) = 0 \]
\[ \alpha \delta u'(z_{tB}^A + \varepsilon_t z_{tB}^B) + \alpha(1 - \delta) - (1 + \iota_B) = 0 \]

These yield: \( \iota_A = \varepsilon_t^{-1} - 1 \) and \( \iota_B = \alpha - 1 < 0 \). Now, \( \iota_B \) cannot be negative. Therefore, an optimal end of period (CM) portfolio in this region is inadmissible.

Case 7: first order conditions are:

\[ \alpha \delta u'(z_{tB}^A + \varepsilon_t z_{tB}^B) + \alpha(1 - \delta)(1 + \iota_A) = 0 \]
\[ \alpha \delta u'(z_{tB}^A + \varepsilon_t z_{tB}^B) + \alpha(1 - \delta) - (1 + \iota_B) = 0 \]

Again, these yield: \( \varepsilon_t = \frac{1 + \iota_B}{1 + \iota_A} \) and \( u'(z_{tB}^A + \varepsilon_t z_{tB}^B) = \frac{1 + \iota_A - \alpha(1 - \delta)\varepsilon_t^{-1}}{\alpha \delta} = \frac{1 + \iota_B - \alpha(1 - \delta)\varepsilon_t^{-1}}{\alpha \delta \varepsilon_t} \). Now \( u'(z_{tB}^A + \varepsilon_t z_{tB}^B) \leq u'(\varepsilon_t q^*) \implies \iota_A \leq \alpha u'(\varepsilon_t q^*) + \alpha(1 - \delta)\varepsilon_t^{-1} - 1 \). Now, \( \alpha \delta u'(\varepsilon_t q^*) + \alpha(1 - \delta)\varepsilon_t^{-1} - 1 \) is increasing in \( \alpha \) and we cannot have \( \iota_A < 0 \). Therefore, one necessary condition is \( \alpha \geq \lceil (1 - \delta)\varepsilon_t^{-1} \rceil - 1 \). Also, \( u'(z_{tB}^A + \varepsilon_t z_{tB}^B) > \varepsilon_t^{-1} \implies \iota_A > \alpha \varepsilon_t^{-1} - 1 \implies (1 + \iota_B - \alpha)(1 + \iota_A) > 0 \) which is always true because \( \iota_A, \iota_B > 0 \).

Proof of Proposition 4:

Proof. As in the proof for Proposition 2 and 3, on similar lines it can be shown that when
only money_A is confiscated with probability \((1 - \alpha)\), the following condition holds true:

\[
\frac{e_{formal}}{e_{black}} = \varepsilon_t = \frac{1 + \nabla_B - (1 - \alpha)[(1 - \delta)u'(z_{tB}^B) + \delta]}{1 + \nabla_B}
\]

Since they are price takers in the competitive black market, buyers take \(\varepsilon\) as given and choose the optimum \(z_{tB}^B\). The optimal real balance of money_B for Country B buyers, \(z_{tB}^B\) is necessarily decreasing in \(\nabla_B\) and \(\varepsilon\) adjusts accordingly so as to satisfy one of the eight cases mentioned in the proof of Proposition 2 and 3.

So, an increase in domestic inflation would increase \(\nabla_B\) leading to an increase in \(\varepsilon\) or fall in the premium on money_A. But at the same time \(u'(z_{tB}^B)\) would increase since \(z_{tB}^B\) decreases. This leads to a fall in \(\varepsilon\) or a rise in the premium on money_A. The reverse happens when domestic inflation decreases or \(\nabla_B\) falls. Now which effect will be stronger depends on the elasticity of \(u(.).\) If \(u(.).\) is more elastic, then the drop in \(\nabla_B\) will be stronger leading to a larger increase in \(u'(z_{tB}^B)\). If \(u(.).\) is sufficiently elastic, it would lead to a fall in \(\varepsilon\) with an increase in \(\nabla_B\) and a rise in premium on money_A, else it would lead to a drop in premium on money_A with an increase in \(\nabla_b\). ■