Finite-state Rate Distortion

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Finite-state Compressibility

Given an individual sequence \( x \in B^\infty \), Lempel and Ziv (1978) defined

\[
\rho(x) \equiv \rho(x, B) = \lim_{s \to \infty} \limsup_{n \to \infty} \min_{\mu \in \mathbb{M}^i(s, B)} \frac{L(x^n_1, \mu)}{n},
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where

- $\mathcal{M}^i(s, B)$ is set of information lossless machines with at most $s$ states that map sequences in $B^\infty$ to $\{0, 1\}^\infty$.
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where

- \( M^i(s, B) \) is set of information lossless machines with at most \( s \) states that map sequences in \( B^\infty \) to \( \{0, 1\}^\infty \)
- \( L(x^n_1, \mu) \) is the number of bits output by the machine \( \mu \) upon seeing the first \( n \) bits of the input sequence.
Notation

- source alphabet: $B$
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- distortion measure $d : B \times \hat{B} \rightarrow [0, \infty)$

$$\max_{b \in B} \min_{\hat{b} \in \hat{B}} d(b, \hat{b}) = 0;$$
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Desired average per-letter distortion $D$;
Prior Work

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- Essence of the definition:

\[ R(D|x) = \inf \{ \rho(y, \hat{B}) : y \in \hat{B}^\infty, d^\infty(x, y) \leq D \} \]

where \( d^\infty(x, y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i) \)
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R(D|x) = \inf \left\{ \rho(y, \hat{B}) : y \in \hat{B}^\infty, d_\infty(x, y) \leq D \right\}
\]

where \( d_\infty(x, y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i) \)

- \( R(D|x) \) is an asymptotically attainable lower bound
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- \( R(D|x) \) is an asymptotically attainable lower bound

- If \( x \) is drawn from a stationary, ergodic process, then \( R(D|x) \) becomes the rate-distortion
A class of $D$-Lossy Machines

A deterministic machine $\hat{E} : B^\infty \to \hat{B}^\infty$ is $D$-lossy if

$$\sup_{x \in B^\infty} d_{\infty}(x, \hat{E}x) \leq D.$$
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Let $\hat{M}(s, D, B, \hat{B})$, $s \in \mathbb{N}$, denote the class of $D$-lossy machines with at most $s$ states.
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**Lemma 1:** Let $\hat{E}$ be in $\bigcup_{D \geq 0} \bigcup_{s = 0}^{\infty} \hat{M}(s, D, B, \hat{B})$, then

$$\rho(x) = \rho(\hat{E}x) + \rho(x|\hat{E}x),$$

where $\rho(x|\hat{E}x)$ is as in Merhav (2000).
For a fixed source sequence $x \in B^\infty$, define

$$\pi(x, D) = \lim_{s \to \infty} \min_{\hat{E} \in \hat{M}(s, D, B, \hat{B})} \rho(\hat{E}x, \hat{B}).$$
For a fixed source sequence $x \in B^\infty$, define

$$\pi(x, D) = \lim_{s \to \infty} \min_{\hat{E} \in \hat{M}(s, D, B, \hat{B})} \rho(\hat{E}x, \hat{B}).$$

**Lemma 2:** $\pi(x, D) = R(D|x)$. 
If $B = \hat{B} = \{0, 1\}$ and $d$ is Hamming distortion, then

$$\pi(x, D) \geq \rho(x, B) - h(D).$$
Main Theorem

If $\mathcal{B} = \hat{\mathcal{B}} = \{0, 1\}$ and $d$ is Hamming distortion, then

$$\pi(x, D) \geq \rho(x, B) - h(D).$$

Proof:

$$\pi(x, D) = \rho(x) - \lim_{s \to \infty} \max_{\hat{E} \in \hat{M}(s, D, B, \hat{B})} \rho(x | \hat{E}x)$$

$$\geq \rho(x) - \lim_{s \to \infty} \max_{\hat{E} \in \hat{M}(s, D, B, \hat{B})} \rho(x \oplus \hat{E}x | \hat{E}x)$$

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$$\geq \rho(x) - h(D).$$