Algebra, Calculus and Antidiagonal Types

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Dedekind "High School Identities"

$$\begin{aligned} x + y &= y + x \\ x + (y + z) &= (x + y) + z \\ x1 &= x \\ xy &= yx \\ x(yz) &= (xy)z \\ x(y + z) &= xy + xz \end{aligned}$$

Dedekind "High School Identities"

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x + y &= y + x \\
x + (y + z) &= (x + y) + z \\
x1 &= x \\
xy &= yx \\
x(yz) &= (xy)z \\
x(y + z) &= xy + xz \\
1^x &= 1 \\
x^1 &= x \\
x^{y+z} &= x^y x^z \\
(xy)^z &= x^z y^z \\
(x^y)^z &= x^{yz}
\end{aligned}
$$

Commutativity

$$a + b = b + a$$

```
commute::Either a b -> Either b a

commute (Left a) = Right a
commute (Right a) = Left a
```

Distributivity

$$a(b+c) = ab + ac$$

```
distribute::(a,Either b c) -> Either (a,b) (a,c)

distribute (a,Left b) -> Left (a,b)
distribute (a,Right b) -> Right (a,b)
```

Associativity of Exponentiation

$$(x^y)^z = x^{yz}$$

```
uncurry . flip::(z->y->z) -> (y,z) -> x
```

Missing Identities

$$\begin{aligned} 0 + x &= x \\ 0x &= 0 \\ x^0 &= 1 \end{aligned}$$

The Natural Type

As a fixed point

$$n = \mu n.1 + x$$

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As Haskell code

```
data N = Zero | Succ N
```

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As Haskell code
data N = Zero | Succ N

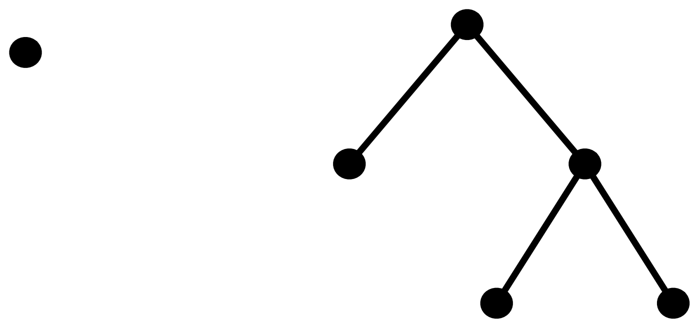
As the solution to an algebraic equation

$$n = 1 + n$$

Binary Trees

$$t = 1 + t^2$$

For example



Can we deduce...?

$$\begin{aligned} t &= \frac{1 \pm i\sqrt{3}}{2} \\ t^7 &= t \end{aligned}$$

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$$\begin{aligned} t &= \frac{1 \pm i\sqrt{3}}{2} \\ t^7 &= t \end{aligned}$$

Or even...

$$\begin{aligned} t^7 &= t^6 - t^5 \\ &= -t^4 \\ &= -t^3 + t^2 \\ &= t \end{aligned}$$

Justification

- Andreas Blass, *Seven trees in one*, J. Pure Appl. Alg. 103 (1995) 1-21
- Fiore, Leinster, *Objects of Categories as Complex Numbers*, Adv. in Math., Volume 190, Issue 2, 30 January 2005, Pages 264-277

The Theorems

Theorem 5.1 Let p, q_1, $q_2 \in N[x]$ be polynomials such that p has non-zero constant term and degree at least two and q_1 and q_2 have degree at least one. If

$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically.}$$

then the same is true rig-theoretically.

Theorem 5.2 Let p, q_1, q_2 be polynomials as in the first sentence of Theorem 5.1. Suppose that the polynomial $p(x)x \in Z[x]$ is primitive and has no repeated complex roots, and that each complex root t satisfies $q_1(t) = q_2(t)$. Then

$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ rig-theoretically.}$$

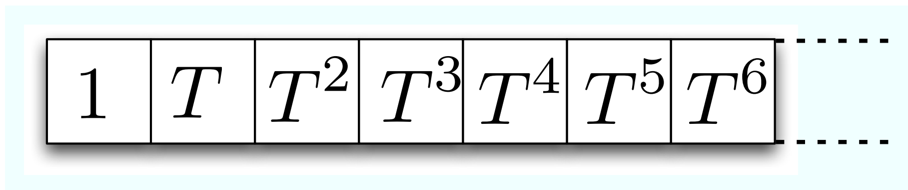
A Game

Each position is a monomial

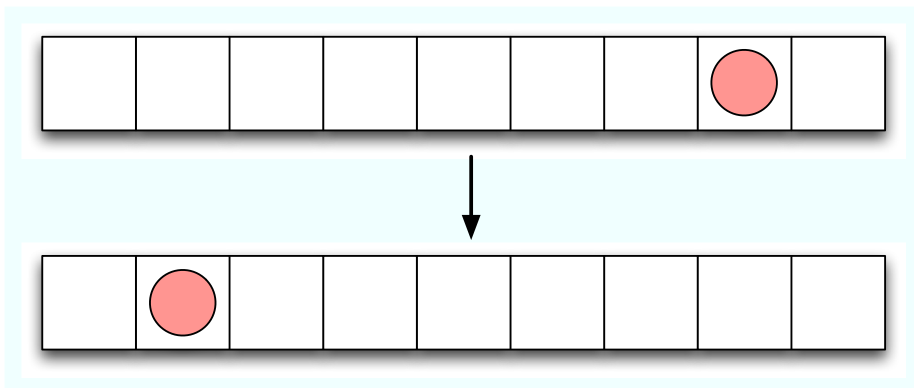
1	T	T^2	T^3	T^4	T^5	T^6

A Game

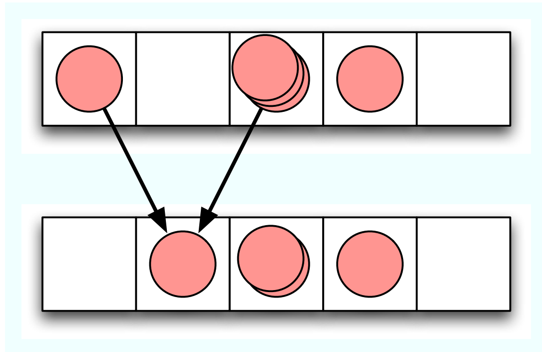
Each position is a monomial



The start and end points

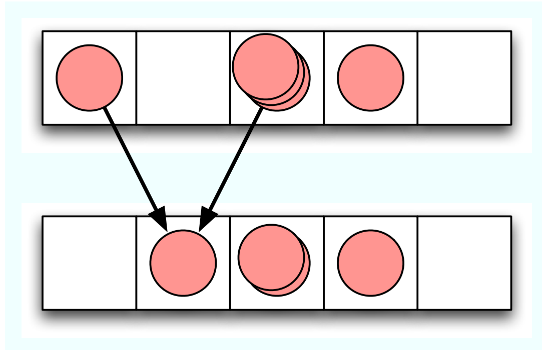


Rules



Fusion: $1 + T^2 \to T$

Rules



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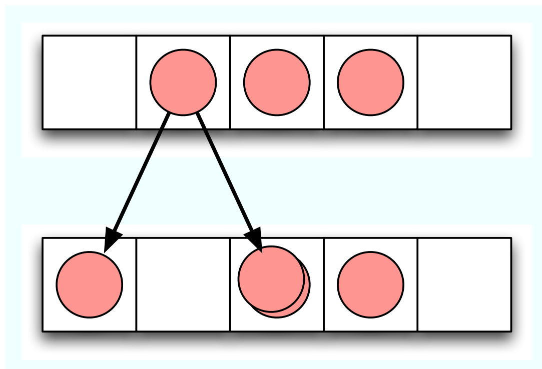
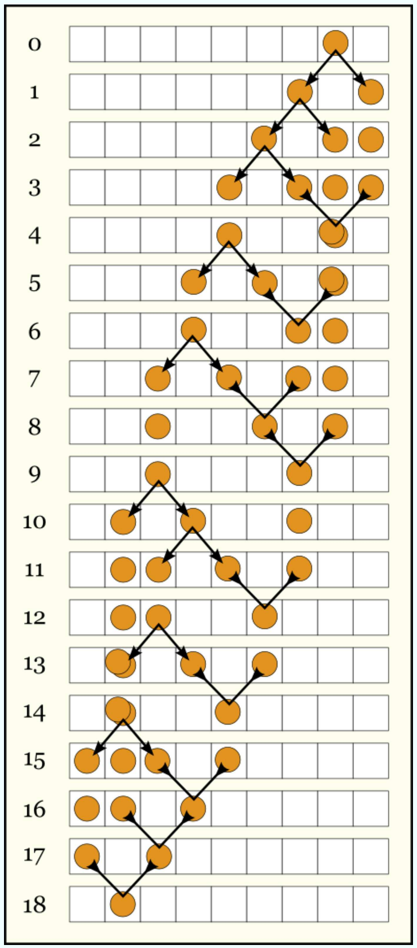


Fission: $T \rightarrow 1 + T^2$

Solution



Rendered by
Cale Gibbard

Functions of Types

$$t(x) = xxx = x^3$$

```
data Triple x = Triple x x x
```

$$l(x) = 1 + xl(x)$$

```
data List x = Nil | Cons x (List x)
```

Leibniz Rules

$$\begin{aligned}\frac{d1}{dx} &= 0\\ \frac{dx}{dx} &= 1\\ \frac{d}{dx}(f+g) &= \frac{df}{dx}+\frac{dg}{dx}\\ \frac{d}{dx}(fg) &= \frac{df}{dx}g+f\frac{dg}{dx}\end{aligned}$$

Leibniz Rules

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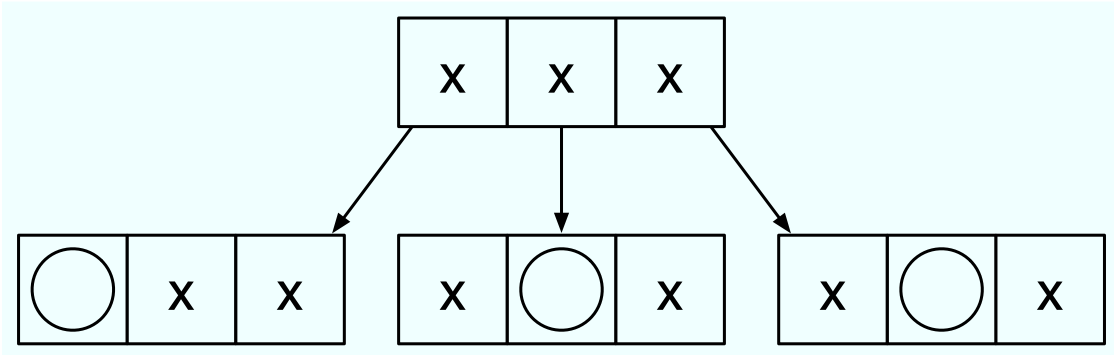
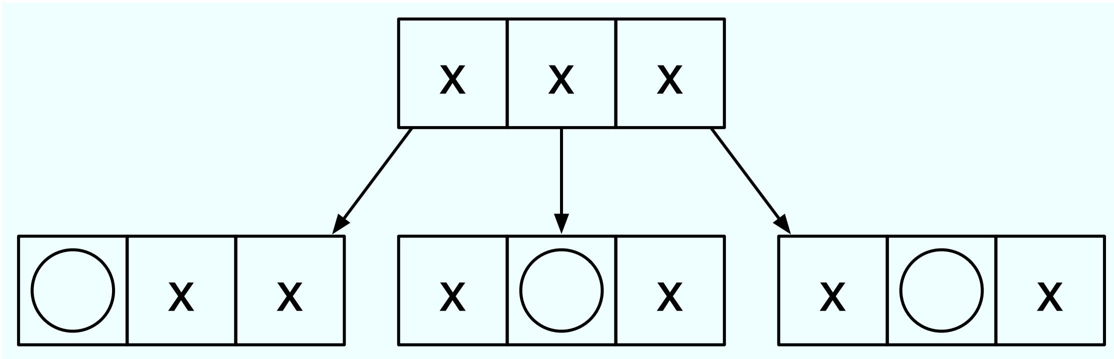
\frac{dx}{dx} &= 1 \\
\frac{d}{dx}(f+g) &= \frac{df}{dx} + \frac{dg}{dx} \\
\frac{d}{dx}(fg) &= \frac{df}{dx}g + f\frac{dg}{dx} \\
(f+g)'(x) &= f(x) + g(x) \\
(fg)'(x) &= f'(x)g(x) + f(x)g'(x) \\
(f \circ g)'(x) &= f'(g(x))g'(x)
\end{aligned}
$$

Holey Triple



Holey Triple



$$\frac{d}{dx}x^3 = 1xx + x1x + xx1$$

$$= 3x^2$$

A List with a Hole...

$$
\begin{aligned}
l(x) &= 1 + xl(x) \\
l'(x) &= l(x) + xl'(x)
\end{aligned}
$$

A List with a Hole...

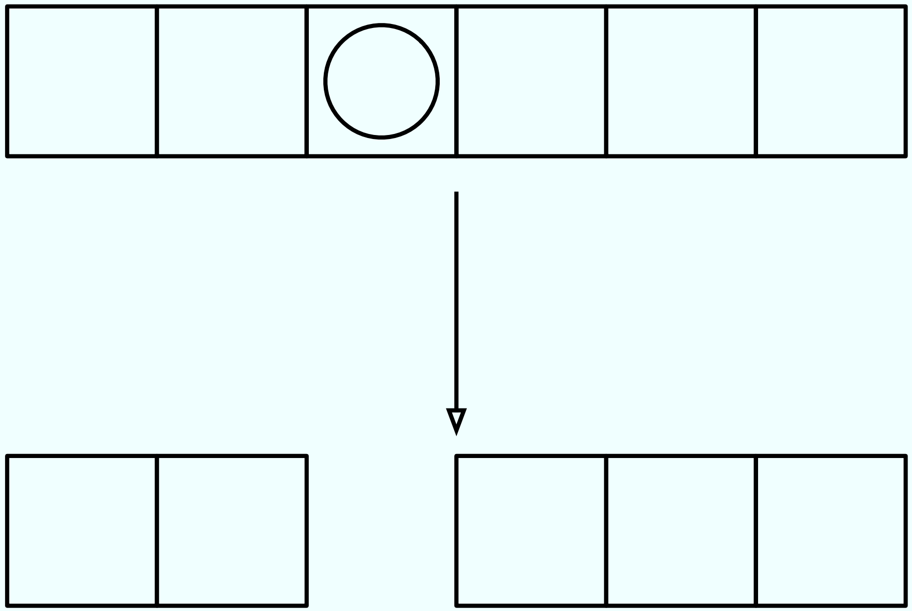
$$\begin{aligned} l(x) &= 1 + xl(x) \\ l'(x) &= l(x) + xl'(x) \end{aligned}$$

Multiply first equation by $l(x)$ to get

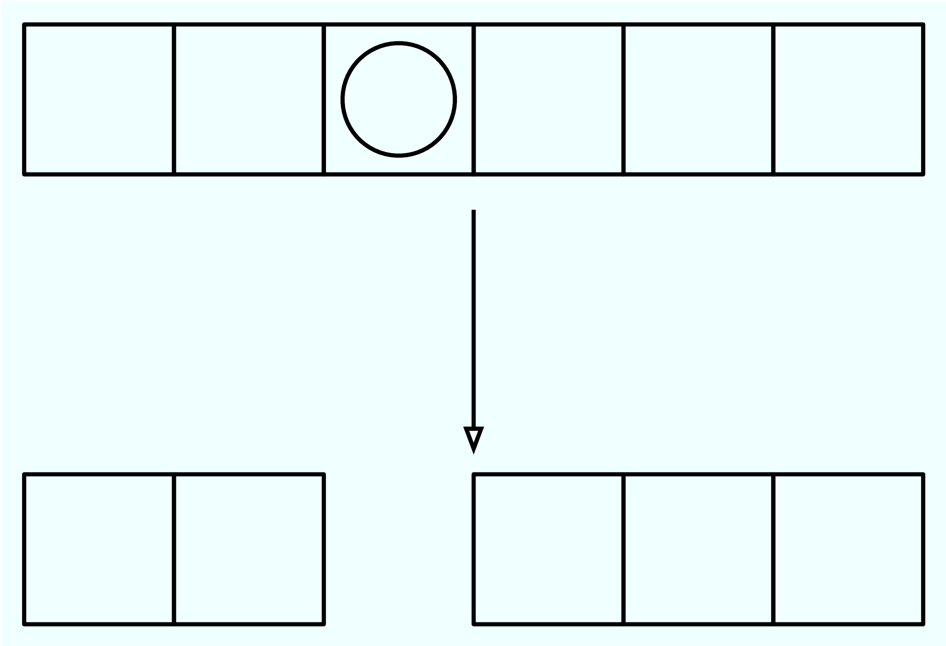
$$l(x)^2 = l(x) + xl(x)^2$$

So $l'(x)$ and $l(x)^2$ satisfy the same equation.

...is Two Lists



...is Two Lists



- Gérard Huet, *The Zipper*, J. Functional Programming 7 (5): 549-554, September 1997.

The Illicit Way to do That

$l(x) = 1 + xl(x)$. So $(1-x)l(x) = 1$ and $l(x) = \frac{1}{1-x}$.
Therefore

$$l(x) = 1 + x + x^2 + x^3 + \ldots$$

Makes sense, a list of x's is an empty list, or one x, or two x's and so on. Elementary calculus now gives

$$l'(x) = \frac{1}{(1-x)^2} = l(x)^2$$

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Pretend you didn't see that!

More Illicit Stuff

$$f(x+\delta) = f(x) + \delta f'(x) + O(\delta^2)$$

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$$f(x+\delta) = f(x) + \delta f'(x) + O(\delta^2)$$

Choosing δ s.t. $\delta^2 = 0$ makes the above exact.

$$f(x+\delta) = f(x) + \delta f'(x)$$

Differentiating Exponentials

$$
\begin{aligned}
\frac{dx^n}{dx} &= nx^{n-1} \\
\frac{d^2x^n}{dx^2} &= n(n-1)x^{n-1}
\end{aligned}
$$

Maybe Identical Pairs

First attempt: $2x^2$

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Second attempt: $x + x^2$

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Define $x^{\underline{n}} = x(x-1)(x-2)\ldots(x-n+1)$ so we want $x^{\underline{2}}$.

Differentiating Exponentials

$$\begin{aligned} \frac{dx^n}{dx} &= nx^{n-1} \\ \frac{d^2x^n}{dx^2} &= n(n-1)x^{n-1} \end{aligned}$$

Exponentials

$$
\begin{aligned}
x^0 &= 1 \\
x^1 &= x \\
x^{y+z} &= x^y x^z \\
x^{yz} &= (x^y)^z
\end{aligned}
$$

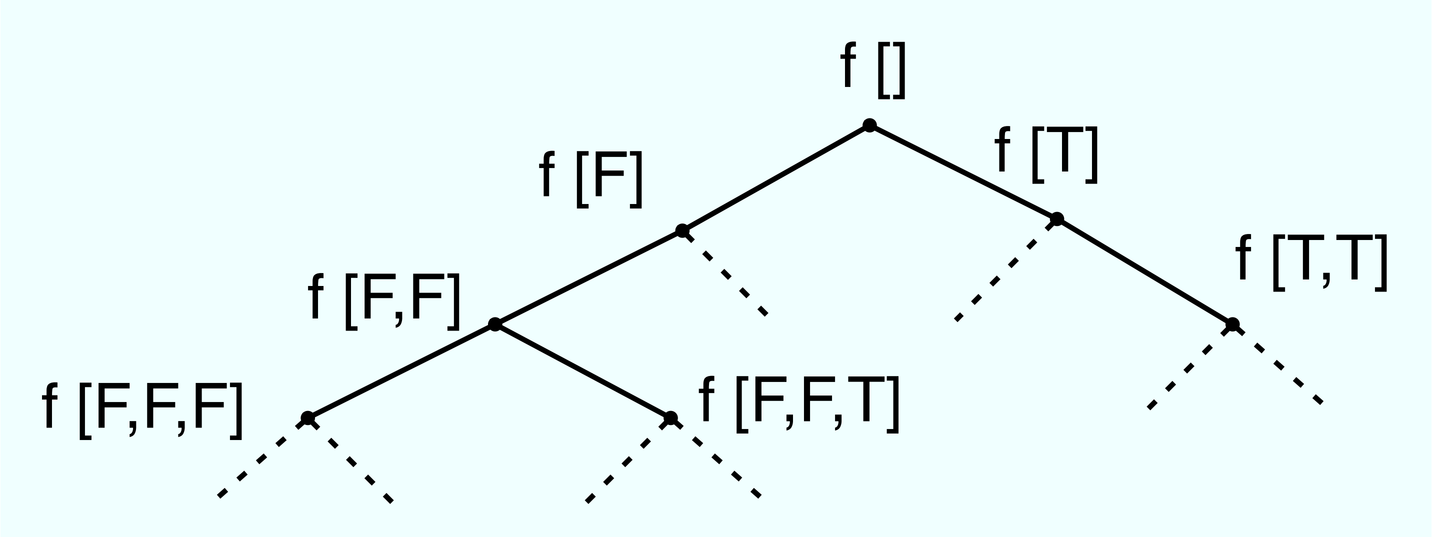
Binary Lists

$$\begin{aligned} x &= 1 + 2x \\ y^x &= y^{1+2x} \\ &= yy^{2x} \\ &= y(y^x)^2 \end{aligned}$$

So if we define $t(y) = y^x$ then $t(y) = yt(y)^2$.

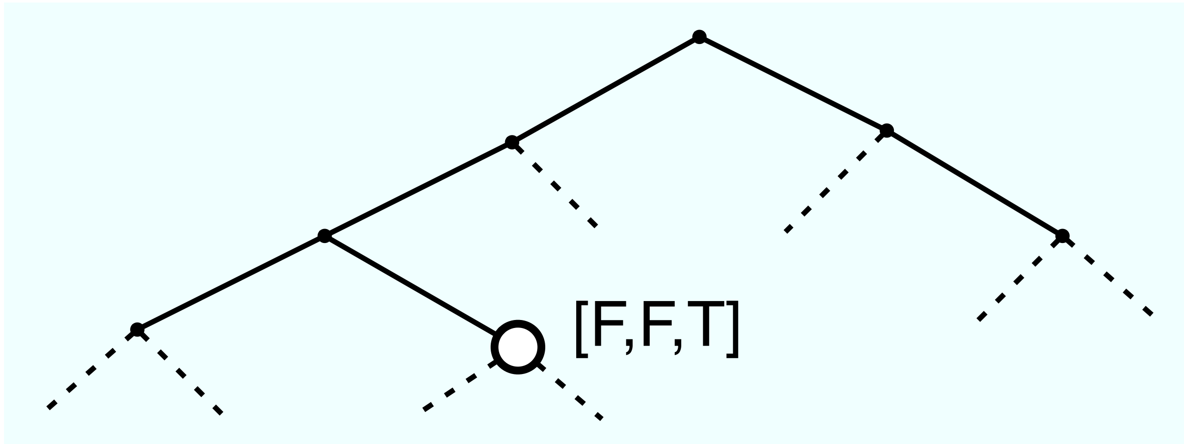
- Ralf Hinze, *Memo functions, polytypically!*, Proceedings of the Second Workshop on Generic Programming, WGP 2000

A Binary List Trie



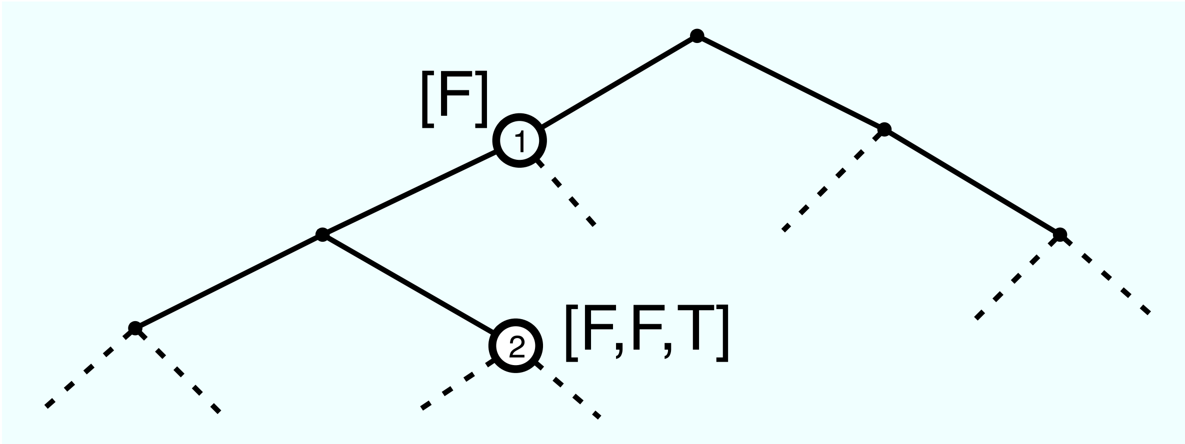
$$t(y) = 2t(y)^2$$

A Trie with a Hole



$$t'(y) = t(y)^2 + \underline{2yt(y)t'(y)}$$

A Trie with Two Holes



$$t''(y) = 4t(y)t'(y) + 2t'(y)^2 + \underline{2t(y)t''(y)}$$

The Details

$$t''(1) = 4x + 2x^2 + 2t''(1)$$

So

$$p = t''(1) = 4x + 2p + 2x + 2p$$

where we have used the antidiagonal property $l^2 = p + x$.

p	$=$	$2x$	data ListPair	=	LeftEmpty Bool [Bool]
	$+$	$2x$		|	RightEmpty Bool [Bool]
	$+$	$2p$		|	TailSame Bool [Bool]
	$+$	$2x$		|	BothDiffer Bool ListPair
	$+$	$2p$		|	HeadSame Bool ListPair

More General Types

$$\frac{d^2 x^{m+n}}{dx^2} = \frac{d^2}{dx^2}(x^m x^n) = x^m \frac{d^2 x^m}{dx^2} + 2\frac{dx^m}{dx}\frac{dx^n}{dx} + \frac{d^2 x^m}{dx^2} x^n$$

Evaluating at $x = 1$ gives

$$(m+n)^{\underline{2}} = m^{\underline{2}} + 2mn + n^{\underline{2}}$$

Similarly, using the chain rule gives

$$\frac{d^2}{dx^2}(x^m)^n = (mn)^{\underline{2}} = n^{\underline{2}} m^2 + n m^{\underline{2}}$$

More General Antidiagonals

$$(a+b)^{\underline{n}} = \sum_{r=0}^{n} \binom{n}{r} a^{\underline{n}} b^{\underline{n-r}}$$

$$(ab)^{\underline{n}} = \sum_{r=0}^{n} a^{\underline{n}} B_{n,r}(b, b^{\underline{2}}, b^{\underline{3}}, \ldots)$$

where the $B_{n,r}$ are known as *Bell* polynomials. They have all positive integer coefficients so they make sense for types.