A Risk-centric Model of Demand Recessions and Macroprudential Policy

Ricardo J. Caballero and Alp Simsek*

This draft: July 14, 2017

Abstract

A productive capacity generates output and risks, both of which need to be absorbed by economic agents. If they are unable to do so, output and risk gaps emerge. Risk gaps close quickly: A decline in the interest rate increases the Sharpe ratio of the risky assets and equilibrates the risk markets. If the interest rate is constrained from below (or the policy response is slow), the risk markets are instead equilibrated via a decline in asset prices. However, the drop in asset prices also drags down aggregate demand, which further drags prices down, and so on. If economic agents are optimistic about the speed of recovery, a decline in asset prices leads to a large increase in the Sharpe ratio that stabilizes the drop. If they are pessimistic, the economy becomes highly susceptible to downward spirals due to the feedback between asset prices and aggregate demand. When beliefs are heterogenous, optimists take too much risk from a social point of view since they do not internalize their positive effect on asset prices and aggregate demand during recessions. Macroprudential policy can improve outcomes, and is procyclical as the negative aggregate demand effect of prudential tightening is more easily offset by interest rate policy during booms than during recessions. Forward guidance policies are also effective, but their robustness weakens as agents become more pessimistic. Our model also illustrates that interest rate rigidities and speculation generate endogenous price volatility that exacerbates demand recessions.

JEL Codes: E00, E12, E21, E22, E30, E40, G00, G01, G11

Keywords: Risk-gap, output-gap, aggregate demand, aggregate supply, liquidity trap, interest rates, "rstar," portfolio decisions, Sharpe ratio, monetary and macroprudential policy, forward guidance, heterogeneous beliefs, speculation, tail risk, endogenous volatility.

---

*MIT and NBER. Contact information: caball@mit.edu and asimsek@mit.edu. We thank Emmanuel Farhi, Zhiguo He, Zilu Pan, Matthew Rognlie, Olivier Wang, Nathan Zorzi, and participants at the CCBS conference hosted by the Bank of England and MacCalm, Paris School of Economics, and BIS, for their comments. Simsek acknowledges support from the National Science Foundation (NSF) under Grant Number SES-1455319. First draft: May 11, 2017

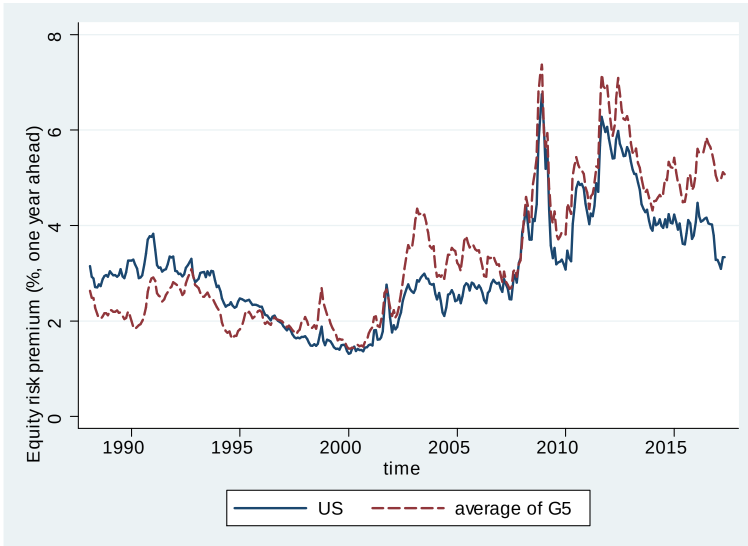

Figure 1: Solid line plots the (forward looking) equity risk premium for the US. Dashed line plots the unweighted average premium for the G5 countries (the US, Japan, the UK, Germany, France). Source: Constructed by Datastream as the median of nine different methods to calculate the ERP. Mean-based methods tend to give higher levels but similar shapes for the path of the ERP.

1. Introduction

A productive capacity generates output and risks, both of which need to be absorbed by economic agents. If they are unwilling or unable to do so, output- and risk-gaps emerge that require appropriate policy responses to prevent severe downward spirals. Macroeconomic modeling has focused primarily on the output-gap component, however risk considerations are central for private and policy decisions, and have become even more prominent since the subprime crisis. Figure 1 shows an estimate of the path of the expected equity risk premium (ERP) for the U.S. and the average of the G5 countries. Several risk-intolerance patterns are apparent in this figure: (i) the ERP spiked during the subprime and European crises; (ii) the ERP remained elevated through much of the U.S. recovery; and (iii) at the global level there is little evidence that the ERP will go to pre-crisis levels any time soon. Our main goal in this paper is to provide a dynamic macroeconomic model that highlights the role of risk markets equilibrium in business cycles (hence the "risk-centric" in the title).

We develop a continuous time macrofinance model with aggregate demand channels and speculative motives due to heterogeneous beliefs.[1] In this model, shocks interact with interest rate policy and its constraints in determining the output gap and the natural interest rate ("rstar"). Also, while the degree of optimism of economic agents is key in containing the fall during recessions, optimists' risk taking is potentially destabilizing, which generates a role for macroprudential policy.

The supply side of the (model-)economy is a stochastic AK model with capital-adjustment costs and sticky prices. The demand side has standard risk-averse consumer-investors that demand the goods and risky assets. In equilibrium, the volatility of their consumption is equal to "the Sharpe ratio" of capital (a measure of the risk-adjusted expected return in excess of the risk-free rate). Our analysis rests on the mechanism by which this risk condition is achieved. Economic agents only differ (when they do) in their beliefs with respect to the likelihood of a near-term recession or recovery. There are no financial frictions in the main environment (we introduce them in the last section). Instead, we focus on "interest-rate frictions": factors that might constrain or delay the adjustment of the risk-free interest rate to shocks. For concreteness, we work with a zero lower bound on the interest rate.

The model has productivity and volatility shocks; we view the latter as capturing a variety of factors that affect the risk premium. In the absence of interest-rate frictions, it is "rstar" that absorbs these types of shocks. The natural interest rate ensures that output is determined by the supply side of the economy. By Walras law, this also implies that there is no risk gap, as the desired volatility of consumption exactly matches the assets' fundamental volatility generated by the productive capacity of the economy. That is, when viewed from the perspective of risk markets, "rstar" ensures that the perceived Sharpe ratio of the returns of the fully utilized stock of capital is consistent with investor's desired risk holdings. It follows that "rstar" is not only a function of goods-markets but also of risk-markets conditions.

To fix ideas, consider a shock that increases volatility. The immediate effect of this shock is to decrease the Sharpe ratio of capital. A risk gap develops, in the sense that the economy generates too much risk relative to what investors are willing to absorb. The natural response of the economy is a decrease in the interest rate, which increases the Sharpe ratio and restores equilibrium in risk markets (as well as goods markets).

If there is a lower bound on the interest rate, the economy loses its natural line of defense. Instead, the risk markets are equilibrated via a decline in asset prices, which increases the Sharpe ratio via expected capital gains. However, the wealth and investment effects of such price adjustment implies that the goods market becomes demand constrained and the economy experiences excess capacity, which further reduces asset prices, which again feeds into aggregate

[1] By a macrofinance model we mean, following (and quoting) Brunnermeier and Sannikov (2016b): "Instead of focusing only on levels, the first moments, the second moments, and movements in risk variables are all an integral part of the analysis, as they drive agents' consumption, (precautionary) savings and investment decisions."

Also, while in our model heterogenous beliefs have a specific formulation, we intend to capture common features, especially on the positive analysis, of many mechanisms that generate effective heterogeneity in asset valuation.

demand, and so on.

The severity of the recession following the drop in asset prices depends on the relative strength of the Sharpe and aggregate demand channels. If agents think of the decline in asset prices as largely temporary, then their perceived Sharpe ratio will rise quickly with a decline in current asset prices, so only a limited asset price drop is required to restore equilibrium, and hence the drop in aggregate demand will be mild. Conversely, if agents interpret the decline in asset prices as a lasting one, then it will take a large drop in asset prices to restore equilibrium in risk markets, and hence the feedbacks and drop in aggregate demand will be severe. Thus, the degree of optimism is a critical state variable in our economy, regardless of whether economic agents have homogeneous or heterogeneous beliefs.

With heterogeneous beliefs, which we analyze in the second part of the paper, the economy's degree of optimism depends on the share of wealth in the hands of optimistic and pessimistic investors. The value of rich optimists for the economy as a whole is high during recessions since they raise asset valuations, which in turn increases aggregate demand. However there is nothing in the economy that ensures this allocation of wealth. Differences in beliefs also lead to speculation which may introduce undesirable correlations between the state of the economy and the relative wealth of optimists and pessimists. For example, if the main source of discrepancy during a boom is in the likelihood of a near-term recession, optimists will sell put options which will impoverish them precisely in the state of the economy that needs them the most. Or, if during a recession the main source of discrepancy is about the speed of recovery, they will buy call options which will deplete their wealth if the recession lingers. That is, through relative wealth effects the economy becomes extrapolative: booms breed optimism and recessions breed pessimism. Moreover, for any given level of average optimism, as the dispersion of beliefs rises the anticipation of this extrapolative feature exacerbates the depth of the drop in asset prices and recession.

Our model generates scope for macroprudential policy, because optimists' (or more broadly, high-valuation investors') risk taking is associated with *aggregate demand externalities.* The depletion of optimists' wealth during a recession depresses asset prices and aggregate demand. Optimists do not internalize the effect of their portfolio risks on asset valuations (in subsequent periods), which leads to excessive risk taking from an aggregate point of view. We show that making optimistic agents behave as-if they were more pessimistic can lead to a *Pareto improvement* (that is, we evaluate investors' welfare according to their own beliefs). Moreover, the policy is naturally procyclical as the tightening of prudential regulation always depresses aggregate demand (in the current period), but this effect can be easily offset with interest rate policy during booms but not during severe recessions.

Forward guidance is also effective in our model, even when investors have homogeneous beliefs, since it affects the market's Sharpe ratio. A decline in future interest rates increases future asset prices, which increases the expected capital gains. These capital gains translate into a greater Sharpe ratio, and ultimately, greater asset valuations and aggregate demand. Perhaps surprisingly, forward guidance can be effective even if investors are very pessimistic about the likelihood of a near term recovery—a manifestation of the "forward guidance puzzle" in our framework. However, the effectiveness of forward guidance in this case is rather delicate as it relies on investors' understanding that the forward guidance would continue to stimulate the economy if the recession persists. In contrast, when investors are optimistic about a near-term recovery, forward guidance is a more robust policy in the sense that it continues to increase asset prices even if the policy is transient (or is perceived to be transient). In this sense, our results are related to a recent strand of the literature illustrating that forward guidance becomes weaker under informational or behavioral frictions that mitigate the effect of the policy in future periods (see, for instance, Gabaix (2017), Angeletos and Lian (2016), Farhi and Werning (2016a)).

While we emphasize the effect of exogenous risk shocks—such as changes in volatility or optimism—on macroeconomic outcomes, the model also generates endogenous price volatility that creates further amplification. Without interest rate rigidities, the interest rate policy optimally mitigates the impact of risk shocks on asset prices. When the interest rate is constrained, these shocks translate into price volatility. With heterogeneous beliefs, speculation exacerbates endogenous price volatility further by creating fluctuations in investors' wealth shares. These effects are already present in our main model without financial frictions, but they become particularly salient when we introduce financial frictions. For instance, with incomplete financial markets, optimists take leveraged positions on capital, and their (relative) net worth becomes exposed to plain-vanilla cyclical (productivity) shocks. The resulting changes in optimists' wealth share translate into endogenous fluctuations in asset prices as well as aggregate output. In recent work, Brunnermeier and Sannikov (2014) also obtain endogenous price volatility under a slightly different set of assumptions, but our model makes the additional prediction that volatility will be higher when the interest rate policy is constrained. This prediction lends support to the many unconventional tools aimed at reducing downward volatility, which the major central banks put in place once interest-rate policy was no longer available during the Great Recession.

Literature review. At a methodological level, our paper belongs in the new continuous time macrofinance literature started by the seminal work of Brunnermeier and Sannikov (2014, 2016a) and summarized in Brunnermeier and Sannikov (2016b) (see also Basak and Cuoco (1998); Adrian and Boyarchenko (2012); He and Krishnamurthy (2012, 2013); Di Tella (2012); Moreira and Savov (2017); Silva (2016)). This literature seeks to highlight the full macroeconomic dynamics induced by financial frictions, which force the reallocation of resources from high-productivity borrowers to low-productivity lenders after a sequence of negative shocks. While the structure of our economy shares many similarities with theirs, in our main model there are no financial frictions, and the macroeconomic dynamics stem not from the supply side (relative productivity) but from the aggregate demand side.

Our paper is also related to an extensive New Keynesian literature that emphasizes the role of financial frictions and nominal rigidities in driving business cycle fluctuations (see, for instance, Bernanke et al. (1999); Curdia and Woodford (2010); Gertler and Karadi (2011); Gilchrist and Zakrajšek (2012); Christiano et al. (2014)). Like this literature, we focus on episodes with high risk premia, but we generate these episodes from changes in risk (or risk perceptions) as opposed to financial frictions. We also emphasize the role of beliefs (optimism/pessimism) as well as speculation in exacerbating risk-driven business cycles.

A strand of the literature emphasizes the role of "risk shocks" in exacerbating financial frictions (see, for instance, Christiano et al. (2014); Di Tella (2012)). We share with this literature the emphasis on uncertainty, but we focus on changes in aggregate risk—as opposed to idiosyncratic uncertainty—which increases risk premia even in absence of frictions. More broadly, there is an extensive recent empirical literature documenting the importance of uncertainty shocks in causing and worsening recessions (see, for instance, Bloom (2009)).

The interactions between risk shocks and interest rate lower bounds is also a central theme of the literature on safe asset shortages and safety traps (see, for instance, Caballero and Farhi (2017); Caballero et al. (2017b)). We extend this literature by analyzing recurrent business cycles with multiple shocks, speculation, as well as integrated interest-rate and macroprudential policies. In recent work, Del Negro et al. (2017) provide a comprehensive empirical evaluation of the different mechanisms that have put downward pressure on interest rate and argue convincingly that risk and liquidity considerations played a central role (see also Caballero et al. (2017a)). More broadly, the literature on liquidity traps is extensive and has been rekindled by the Great Recession (see, for instance, Tobin (1975); Krugman (1998); Eggertsson and Woodford (2006); Eggertsson and Krugman (2012); Guerrieri and Lorenzoni (2017); Werning (2012); Hall (2011); Christiano et al. (2015); Eggertsson et al. (2017); Rognlie et al. (2017); Midrigan et al. (2016); Bacchetta et al. (2016)). We extend this literature by focusing on the risk aspects (both shocks and mechanisms) behind the drop in the natural rate below its lower bound, as well as on the interaction between speculation and the severity of recessions.

Our results on macroprudential policy are related to a recent literature that analyzes the implications of aggregate demand externalities for the optimal regulation of financial markets. For instance, Korinek and Simsek (2016) show that, in the run-up to deleveraging episodes that coincide with a zero-lower-bound on the interest rate, welfare can be improved by policies targeted toward reducing household leverage. In Farhi and Werning (2017), the key constraint is instead a fixed exchange rate, and the aggregate demand externality calls for ex-ante regulation but also ex-post redistribution, in the form of a fiscal union. In these papers, heterogeneity in agents' marginal propensities to consume (MPC) is the key determinant of optimal macroprudential policy. The policy works by reallocating wealth across agents and states in a way that high-MPC agents hold relatively more wealth when the economy is more depressed due to deficient demand. The mechanism in our paper is different and works through *heterogeneous asset valuations.* In fact, we work with a log-utility setting in which all agents have the same marginal propensity to consume. The policy operates by transferring wealth to optimists during recessions, not because optimists spend more than other agents, but because they raise the asset valuations and induce *all investors to spend more* (while also increasing aggregate investment).[2]

Beyond aggregate demand externalities, the macroprudential literature is also extensive, and mostly motivated by the presence of pecuniary externalities that make the competitive equilibrium constrained inefficient (e.g., Caballero and Krishnamurthy (2003); Lorenzoni (2008); Bianchi and Mendoza (2013); Jeanne and Korinek (2010)). The friction in this case is not "nominal" rigidities, but market incompleteness or collateral constraints that depend on asset prices (see Davila and Korinek (2016) for a detailed exposition). Macroprudential policy typically improves outcomes by mitigating fire sales that exacerbate financial frictions. The policy in our model also operates through asset prices but through a different channel. We show that a decline in asset prices is damaging not only because of the fire-sale reasons emphasized in this literature, but also because it lowers aggregate demand through standard wealth and investment channels. Moreover, most of our analysis (except Section 7.2) does not feature the incomplete markets or collateral constraints that are central in this literature.

Our results with heterogeneous beliefs are related to a large literature that analyzes the effect of belief disagreements and speculation on financial markets (e.g., Lintner (1969); Miller (1977); Harrison and Kreps (1978); Varian (1989); Harris and Raviv (1993); Chen et al. (2002); Scheinkman and Xiong (2003); Fostel and Geanakoplos (2008); Geanakoplos (2010); Simsek (2013a,b); Iachan et al. (2015)). One strand of this literature emphasizes that disagreements can exacerbate asset price fluctuations by creating endogenous fluctuations in agents' wealth distribution (see, for instance, Basak (2000, 2005); Xiong and Yan (2010); Kubler and Schmedders (2012); Korinek and Nowak (2016)). Our paper features similar forces but explores them in an environment in which output is not necessarily at its supply-determined level due to interest rate rigidities. In fact, our framework is similar to the models analyzed by Detemple and Murthy (1994); Zapatero (1998), who show that financial speculation between optimists and pessimists (with log utility) can increase the volatility of the interest rate. In our model, these results apply when the interest rate is unconstrained but they are modified if the interest rate is constrained in downward adjustments. In the latter case, speculation translates into (inefficient) fluctuations in asset prices as well as aggregate demand. Among other things, we show that (controlling for the average belief) speculation driven by belief disagreement depresses aggregate demand and lowers output during recessions. We also show that belief disagreements create scope for macroprudential policy.

The rest of the paper is organized as follows. Section 2 presents the general environment and defines the equilibrium. Section 3 characterizes the equilibrium in a benchmark setting with homogeneous beliefs. This section illustrates how risk premium shocks can induce a demand recession, and how optimism helps to mitigate the recession. Section 4 characterizes the equilibrium with heterogeneous beliefs, and illustrates how speculation exacerbates the recession. Section 5 establishes our normative results in two steps. Section 5.1 characterizes the value

[2] Also, see Farhi and Werning (2016b) for a synthesis of some of the key mechanisms that justify macroprudential policies in models that exhibit aggregate demand externalities.

functions in the equilibrium with heterogeneous beliefs, and illustrates the aggregate demand externalities. Section 5.2 analyzes the effect of introducing risk limits on optimists, and presents our results on (procyclical) macroprudential policy. Section 6 establishes our results on forward guidance. This section builds upon the benchmark model with homogeneous beliefs, and it can be read independently of our analysis of heterogeneous beliefs. Section 7 establishes our results on endogenous volatility in two steps. Section 7.1 illustrates how the presence of interest rate frictions generates endogenous volatility, and Section 7.2 shows how speculation generates endogenous volatility. In this section we also extend our model to the case of incomplete markets, where the rise in endogenous volatility is particularly salient. Section 8 concludes and is followed by two appendices that contain the omitted derivations and proofs.

2. General environment and equilibrium

In this section, we introduce our general environment and define the equilibrium. In subsequent sections, we will characterize this equilibrium in various special cases of interests. We start by describing the production and investment technology, as well as the risk-premium shocks that play the central role in our analysis. We then describe the firms' investment decisions, followed by the investors' consumption and portfolio choice decisions. Then, we introduce the nominal and the interest rate rigidities that ensure output is determined by aggregate demand. We finally introduce the goods and asset market clearing conditions and define the equilibrium.

Potential output and risk-premium shocks. The economy is set in infinite continuous time, $t \in [0, \infty)$, with a single consumption good and a single factor of production: capital. Let $k_{t,s}$ denote the capital stock at time t and the aggregate state $s \in S$. Suppose that, when fully utilized, $k_{t,s}$ units of capital produces $Ak_{t,s}$ units of the consumption good. Hence, $Ak_{t,s}$ denotes the potential output in this economy. As we will see, actual output might be lower than this level due to interest rate rigidities and aggregate demand shortages.

Capital follows the process,

$$\frac{dk_{t,s}}{k_{t,s}} = (\varphi(\iota_{t,s}) - \delta)\, dt + \sigma_s dZ_t. \tag{1}$$

Here, $\iota_{t,s}$ denotes the investment rate, $\varphi(\iota_{t,s})$ denotes the production function for capital (that will be specified below), and δ denotes the depreciation rate. We also let $g_{t,s} \equiv \varphi(\iota_{t,s}) - \delta$ denote the expected growth rate of capital and potential output. The last term, dZ_t, denotes the standard Brownian motion, which captures "aggregate productivity shocks."[3]

The states, $s \in S$, differ only in terms of the volatility of aggregate productivity, σ_s. At every instant, the economy in state s transitions to state s' according to Poisson transition probabilities

[3]Note that fluctuations in $k_{t,s}$ generate fluctuations in potential output, $Ak_{t,s}$. We introduce Brownian shocks to capital, $k_{t,s}$, as opposed to the total factor productivity, A, since this leads to a slightly more tractable analysis. See Footnote 2 in Brunnermeier and Sannikov (2014) for an equivalent formulation in terms of shocks to A.

that will be specified below. We will define the equilibrium for an arbitrary number of states. However for most of our analysis we will focus on a special case with two states—a low volatility state and a high volatility state.

Remark 1 (Interpreting the Volatility Shocks)**.** *We work with volatility shocks mainly because they lead to a tractable analysis. The key feature of these shocks is that they increase the risk premium on capital, and might push the economy into a liquidity trap in which the risk-free interest rate is at its lower bound. Many other shocks that increase the risk premium would lead to a similar analysis. In fact, we view the volatility parameters,* $\{\sigma_s\}_s$*, as capturing in reduced form various unmodeled objective and subjective factors that might affect the risk premium (such as long-run risks, Knightian uncertainty, or financial panics).*

Aggregate wealth and investment. We let $Q_{t,s}$ denote the price of capital. Absent volatility regime transitions, this price follows an (endogenous) diffusion process,

$$\frac{dQ_{t,s}}{Q_{t,s}} = \mu_{t,s}^{Q} dt + \sigma_{t,s}^{Q} dZ_t. \tag{2}$$

If there is a transition, the price $Q_{t,s}$ makes a discrete adjustment to $Q_{t,s'}$.

Combining Eqs. (1) and (2) the aggregate wealth (absent a transition) evolves according to

$$\frac{d\left(Q_{t,s}k_{t,s}\right)}{Q_{t,s}k_{t,s}} = \left(\varphi\left(\iota_{t,s}\right) - \delta + \mu_{t,s}^{Q} + \sigma_s \sigma_{t,s}^{Q}\right) dt + \left(\sigma_s + \sigma_{t,s}^{Q}\right) dZ_t. \tag{3}$$

This implies that the instantaneous expected return (conditional on no transition) and the volatility of capital are, respectively:

$$r_{t,s}^{k} = \frac{R_{t,s} - \iota_{t,s}}{Q_{t,s}} + \varphi\left(\iota_{t,s}\right) - \delta + \mu_{t,s}^{Q} + \sigma_s \sigma_{t,s}^{Q}, \tag{4}$$

$$\sigma_{t,s}^{k} = \sigma_s + \sigma_{t,s}^{Q}. \tag{5}$$

Note that $r_{t,s}^{k}$ has two components. The first term can be thought of as the "dividend yield," which captures the instantaneous rental rate of capital, $R_{t,s}$, as well as the reinvestment costs. The second component is the capital gains conditional on no transition, which captures the expected changes in the value of capital due to investment, depreciation, or price changes.

There is a continuum of identical firms that manage capital. These firms rent capital to production firms (that will be described below) to earn the instantaneous rate, $R_{t,s}$. They also make investment decisions to maximize the return to capital in (4). Their investment problem can be rewritten as,

$$\max_{\iota_{t,s}} Q_{t,s}\varphi\left(\iota_{t,s}\right) - \iota_{t,s}. \tag{6}$$

Under standard regularity conditions for $\varphi\left(\iota\right)$, investment is determined by the optimality condition, $\varphi'\left(\iota_{t,s}\right) = 1/Q_{t,s}$ We will work with the special and convenient case proposed by Brunnermeier and Sannikov (2016b): $\varphi(\iota) = \psi \log\left(\frac{\iota}{\psi} + 1\right)$. In this case, we obtain the closed form solution,

$$\iota(Q_{t,s}) = \psi(Q_{t,s} - 1). \tag{7}$$

The parameter, ψ, captures the sensitivity of investment to asset prices. Note also that the amount of capital produced is given by,

$$\varphi(\iota(Q_{t,s})) = \psi q_{t,s}, \text{ where } q_{t,s} \equiv \log(Q_{t,s}). \tag{8}$$

The log price level, $q_{t,s}$, will simplify some of the expressions below.

Consumption and portfolio choice. Suppose there is a continuum of mass one of investors denoted by $i \in I$. Investors are identical in all respect except possibly their assessment of the likelihood of state transition events. Specifically, investor i believes that at every instant the economy transitions from state s to state $s' \neq s$ with Poisson probability $\lambda^i_{s,s'}$. Investors' beliefs are dogmatic: that is, they know each others' beliefs and they agree to disagree. Common beliefs, which we will analyze in the next section, is a special case in which $\lambda^i_{s,s'} = \lambda_{s,s'}$ for each i.

Investors continuously make consumption and portfolio allocation decisions. Each investor has access to three types of assets. First, the investor can invest in capital (more precisely, in shares of firms that manage the capital). The instantaneous return and volatility of capital (conditional on no transition), $r^k_{t,s}$ and $\sigma^k_{t,s}$, are described respectively in Eqs. (4) and (5). Second, she can invest in a risk-free asset with return, $r^f_{t,s}$. The risk-free asset is in zero net supply. Third, for each $s' \neq s$, she can also invest in a contingent Arrow-Debreu security that trades at the (endogenous) instantaneous price $p^{s'}_{t,s}$, and that pays 1 dollar if the economy transitions to state s'. These securities are also in zero net supply, and they ensure that the financial markets are complete.

Let $a^i_{t,s}$ denote the wealth level for investor i, at time t, in state s. For analytical tractability, we assume the investor has log utility. Let $c^i_{t,s}$ be the investor's consumption rate, $\omega^{k,i}_{t,s}$ denotes the fraction of her wealth she allocates to capital, and $\omega^{s',i}_{t,s}$ denotes the fraction of her wealth she allocates to the Arrow-Debreu security $s' \neq s$. The residual fraction of the investor's wealth, $1 - \omega^{k,i}_{t,s} - \sum_{s' \neq s} \omega^{s',i}_{t,s}$, is invested in the risk-free asset. The investor's optimization problem (at some time t and state s) can be written as,

$$V_{t,s}^{i}\left(a_{t,s}^{i}\right) = \max_{\left[\tilde{c}_{\tilde{t},\tilde{s}},\tilde{\omega}_{\tilde{t},\tilde{s}}^{k},\left\{\tilde{\omega}_{\tilde{t},\tilde{s}}^{\tilde{s}'}\right\}_{\tilde{s}'\neq\tilde{s}}\right]_{\tilde{t}\geq t,\tilde{s}}} E_{t,s}^{i}\left[\int_{t}^{\infty} e^{-\rho\tilde{t}}\log\tilde{c}_{\tilde{t},\tilde{s}}^{i}d\tilde{t}\right] \tag{9}$$

$$\text{s.t.}\quad \begin{cases} da_{t,s}^{i} = \left(a_{t,s}^{i}\left(r_{t,s}^{f}+\tilde{\omega}_{t,s}^{k}\left(r_{t,s}^{k}-r_{t,s}^{f}\right)-\sum_{s'\neq s}\tilde{\omega}^{s'}\right)-\tilde{c}_{t,s}\right)dt+\tilde{\omega}_{t,s}^{k}a_{t,s}^{i}\sigma_{t,s}^{k}dZ_{t} & \text{absent transition,} \\ a_{t,s'}^{i} = a_{t,s}^{i}\left(1+\tilde{\omega}_{t,s}^{k}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}+\tilde{\omega}_{t,s}^{s'}\frac{1}{p_{t,s}^{s'}}\right) & \text{if there is a transition to state } s'\neq s. \end{cases} \tag{10}$$

Here, $E_{t,s}^{i}[\cdot]$ denotes the expectations operator that corresponds to the investor i's beliefs for state transition probabilities.

In Appendix A.1.1, we characterize the solution to the investor's optimization problem using recursive optimization techniques (in particular the value function solves the HJB equation (A.1)). Log utility implies that the value function has the form,

$$V_{t,s}^{i}\left(a_{t,s}^{i}\right) = \frac{\log\left(a_{t,s}^{i}/Q_{t,s}\right)}{\rho} + v_{t,s}^{i}. \tag{11}$$

The first term in the value function captures the effect of holding a greater capital stock (or greater wealth), which scales the investors consumption proportionally at all times and states. The second term, $v_{t,s}^{i}$, is the normalized value function when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^{i} = Q_{t,s}$). Combining the functional form in (11) with the HJB equation, the optimal consumption is given by,

$$c_{t,s}^{i} = \rho a_{t,s}^{i}. \tag{12}$$

Likewise, the optimal portfolio allocation to capital is determined by,

$$\omega_{t,s}^{k,i}\sigma_{t,s}^{k} = \frac{1}{\sigma_{t,s}^{k}}\left(r_{t,s}^{k}-r_{t,s}^{f}+\sum_{s'\neq s}\lambda_{s,s'}^{i}\frac{a_{t,s}^{i}}{a_{t,s'}^{i}}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right). \tag{13}$$

Intuitively, the investor invests in capital up to the point at which the risk of her portfolio (left side) is equal to "the Sharpe ratio" of capital (right side). The Sharpe ratio provides a measure of the risk-adjusted expected return on capital. Our notion of the Sharpe ratio accounts for potential revaluation gains or losses from state transitions (the term, $\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}$) as well as the adjustment of marginal utility in case there is a transition (the term, $\frac{a_{t,s}^{i}}{a_{t,s'}^{i}}$).[4]

[4] The presence of state transitions makes the Sharpe ratio in our model slightly different than the common definition of the Sharpe ratio, which corresponds to the expected return in excess of the risk-free rate normalized by volatility.

Finally, the optimal portfolio allocation to the contingent securities implies,

$$\frac{p_{t,s}^{s'}}{\lambda_{s,s'}^{i}} = \frac{a_{t,s}^{i}}{a_{t,s'}^{i}} \text{ for each } s'. \tag{14}$$

The portfolio weight, $\omega_{t,s}^{s',i}$, is implicitly determined as the level that ensures that this equality holds. The investor invests in the contingent securities up to the point at which the price-to-probability ratio of a state (or the state price) is equated to the investor's relative marginal utility in that state. Note that replacing (14) into (13) shows that investors allocate identical portfolio weights to capital , $\omega_{t,s}^{k}$, and express their differences in beliefs through their holdings of contingent securities.

Nominal rigidities and demand-determined output. The supply side of our model features nominal rigidities similar to the standard New Keynesian model. We relegate the details to Appendix A.1.2 and describe the main implications relevant for our analysis. There is a continuum of monopolistically competitive production firms that rent capital from investment firms and produce intermediate goods (which are then converted into the final good). For simplicity, these production firms have preset prices that they never change. The firms meet the available demand (as long as they find it optimal to do so). In equilibrium, these features imply that output is determined by aggregate demand,

$$y_{t,s} = \eta_{t,s} A k_{t,s} = \int_I c_{t,s}^i di + k_{t,s}\iota_{t,s}, \text{ where } \eta_{t,s} \in [0,1]. \tag{15}$$

Here, $\eta_{t,s}$ denotes the instantaneous factor utilization rate for capital. We assume firms can increase factor utilization for free until $\eta_{t,s} = 1$ and they cannot increase it beyond this level (we relax the latter assumption in Section 6). Aggregate demand corresponds to the sum of aggregate consumption and aggregate investment.

There are also lump sum taxes on the production firms' profits combined with linear subsidies to capital. In equilibrium, these features imply that the rental rate of capital is given by,

$$R_{t,s} = A\eta_{t,s}. \tag{16}$$

This also implies, $y_{t,s} = R_{t,s} k_{t,s}$, that is: all output accrues to the agents in the form of return to capital, which simplifies our analysis.[5]

Interest rate rigidity. Our assumption that production firms do not change their prices implies that the aggregate price level is fixed. The real risk-free interest rate is then equal to the nominal risk-free interest rate, which is determined by the interest rate policy of the monetary

[5] Without this type of taxes and subsidies, firms would also make pure profits that are not necessarily linked to the capital they use in production. The analysis of the portfolio problem would then require introducing a second risky asset (claims on pure profits).

authority. We assume there is a lower bound on the nominal interest rate, which we take to be zero for convenience,[6]

$$r_{t,s}^{f} \geq 0. \tag{17}$$

In practice, this type of constraint emerges naturally from a variety of factors. The zero lower bound in particular can be motivated by the presence of cash in circulation (which we leave unmodeled for simplicity). Since cash offers zero interest rate, the monetary authority cannot lower the interest rate (much) below zero—a constraint that appeared to be binding for major central banks in the aftermath of the Great Recession.

We assume that the interest rate policy focuses on replicating the level of output that would obtain absent nominal rigidities subject to the constraint in (17). Appendix A.1.2 illustrates that, without nominal rigidities, capital is fully utilized, $\eta_{t,s} = 1$. Thus, we assume the interest rate policy follows the rule,

$$r_{t,s}^{f} = \max\left(0, r_{t,s}^{f,*}\right) \text{ for each } t \geq 0 \text{ and } s \in S. \tag{18}$$

Here, $r_{t,s}^{f,*}$ is recursively defined as the (instantaneous) natural interest rate that obtains when the (instantaneous) utilization is given by $\eta_{t,s} = 1$, and the monetary policy follows the rule in (18) at all future times and states.

Equilibrium in the goods market. Eq. (18) implies the following complementary slackness condition for the goods market equilibrium,

$$\eta_{t,s} \leq 1, r_{t,s}^{f} \geq 0, \text{ with at least one condition satisfied as equality.} \tag{19}$$

Combining this with Eq. (15) implies that the equilibrium at any time and state takes one of two forms. If the natural interest rate is nonnegative, then the interest rate policy ensures that capital is fully utilized, $\eta_{t,s} = 1$, and output is equal to its potential, $y_{t,s} = Ak_{t,s}$. Otherwise, the interest rate policy is constrained, $r_{t,s}^{f} = 0$, capital utilization satisfies, $\eta_{t,s} \leq 1$, and output is determined by aggregate demand at the zero interest rate according to Eq. (15).

[6]In practice, the lower bound on the real interest rate seems to be slightly below zero due to steady-state inflation. We could also assume that firms set their prices at every period mechanically according to a predetermined inflation target. This formulation yields a very similar bound as in (17) and results in the same economic trade-offs. We normalize inflation to zero so as to economize on notation.

Our assumption that the aggregate price (or inflation) level is fixed is admittedly extreme. It captures in reduced form a situation in which inflation is sticky in the upward direction during a demand recession. In practice, this type of stickiness could be driven by nominal rigidities at the micro level, or due to constraints on monetary policy against creating inflation. Note also that making the prices more flexible at the micro level does not necessarily circumvent the bound in (17). In fact, if monetary policy follows an inflation targeting policy regime, then limited price flexibility exacerbates the bound in (17) (see Korinek and Simsek (2016); Caballero and Farhi (2017) for further discussion).

Equilibrium in asset markets. Asset markets clearing requires that the total wealth held by investors is equal to the value of aggregate capital before and after the portfolio allocation decisions,

$$\int_I a^i_{t,s} di = Q_{t,s} k_{t,s} \text{ and } \int_I \omega^{k,i}_{t,s} a^i_{t,s} di = Q_{t,s} k_{t,s}. \tag{20}$$

Contingent securities are in zero net supply, which implies,

$$\int_I a^i_{t,s} \omega^{s',i}_{t,s} di = 0. \tag{21}$$

The market clearing condition for the risk-free asset (which is also in zero net supply) holds when conditions (20) and (21) are satisfied.

We can now define the equilibrium as follows.

Definition 1. *The equilibrium is a collection of processes for allocations, prices, and returns such that capital and prices evolve according to respectively Eqs.* (1) *and* (2)*, the return and the volatility of capital is given by respectively Eqs.* (4) *and* (5)*, investment firms maximize (cf. Eq.* (7)*), the investors maximize (cf. Eqs.* (12)*,* (13)*, and* (14)*), output is determined by aggregate demand (cf. Eq.* (15)*), the rental rate of capital is given by Eq.* (16)*, the interest rate policy follows the rule in* (18)*, the goods market clears (cf. Eq.* (19)*), and the asset markets clear (cf. Eqs.* (20) *and* (21)*).*

Next we provide a characterization of the equilibrium in the goods market, which applies in all of our subsequent analyses. Note that Eqs. (12) and (20) imply that aggregate consumption is a constant fraction of aggregate wealth, $\int_I c^i_{t,s} di = \rho Q_{t,s} k_{t,s}$. Plugging this into Eq. (15), and using the investment equation (7), we obtain,

$$A\eta_{t,s} = \rho Q_{t,s} + \psi \left(Q_{t,s} - 1\right) = Q_{t,s} \left(\rho + \psi\right) - \psi.$$

Rewriting this expression, we obtain,

$$q_{t,s} = q\left(\eta_{t,s}\right) = \log\left(\frac{A\eta_{t,s} + \psi}{\rho + \psi}\right). \tag{22}$$

Hence, there is an increasing relationship between factor utilization and asset prices. Full factor utilization, $\eta_{t,s} = 1$, obtains only if the log price is at a particular level $q^* = q(1)$. This is the level of the price that ensures that the implied consumption and investment clears the goods market.

Combining Eq. (22) with the equilibrium condition in (19), the goods market side of the economy can be summarized with,

$$q_{t,s} \leq q^*, r^f_{t,s} \geq 0, \text{ with at least one condition satisfied as equality.} \tag{23}$$

Either the interest rate policy is unconstrained and asset prices are at the level consistent with full factor utilization; or the policy is constrained, asset prices are at a lower level, and factor utilization and output are below their efficient levels. As we will see, an imbalance in risk markets can push the economy into the latter equilibrium.

It is also useful to characterize the expected return to capital (conditional on no transition) in equilibrium. Note that Eqs. (15) and (12) imply $A\eta_{t,s} - \iota_{t,s} = \frac{1}{k_{t,s}} \int_I c^i_{t,s} di = \rho Q_{t,s}$. Combining this with Eqs. (4) and (16), and using Eq. (8), the return to capital can be written as,

$$r^k_{t,s} = \rho + \psi q_{t,s} - \delta + \mu^Q_{t,s} + \sigma_s \sigma^Q_{t,s}. \quad (24)$$

Hence, controlling for the drift and the volatility of the price level, and conditional on no transition, lower asset prices lead to *lower* return. This result is somewhat surprising, and it reflects two potentially destabilizing forces. First, lower asset prices reduce aggregate demand, which ensures that the dividend yield remains constant and equal to the consumption rate despite the fact that capital is cheaper. Second, lower asset prices also reduce investment, which leads to a lower return to capital (note that $g_{t,s} = \psi q_{t,s} - \delta$ is the expected growth rate of capital). The net effect of lower prices is negative, and depends on the sensitivity of investment to asset prices, captured by ψ. Note, however, that $r^k_{t,s}$ describes only part of the return to capital. The total return also depends on the expected capital gains from transition events, which is greater when the current asset prices are lower (see Eq. (13)). We will make assumptions to ensure that the latter effect dominates and lower asset prices increase the Sharpe ratio, consistent with conventional wisdom. Nonetheless, the instability highlighted here will be latent, and will be the source of deep recessions when optimism is depressed.

For future reference, we also note that the first-best equilibrium obtains when price is at its efficient level at all times and states, $q_{t,s} = q^*$. This also implies that the growth rate of and the return to capital are constant and given by, respectively, $g = \psi q^* - \delta$ and $r^k = \rho + \psi q^* - \delta$ (see Eq. (24)). We next turn to the characterization of equilibrium with interest rate rigidities.

3. Common beliefs benchmark

In this section, we characterize the equilibrium for a benchmark case in which investors are identical (and therefore, also share common beliefs). We denote the variables related to the representative investor by dropping the superscript i. We first derive a general characterization in terms of a system of equations. We then solve the system for a special case with two states, $S = \{1, 2\}$, with $\sigma_1 < \sigma_2$. In this special case, $s = 1$ corresponds to a low-volatility state, whereas state $s = 2$ corresponds to a high-volatility state. When we are in the context of two states, we will also simplify the notation by letting $\lambda_s = \lambda_{s,s'}$ denote the transition probability in state s (into the other state s'). We will be particularly interested in the comparative statics with respect to the transition from the high-volatility state into the low-volatility state, λ_2, which provides a measure of optimism at times of distress.

With a representative investor, the market clearing conditions (20) and (21) imply $\omega^k_{t,s} = 1$ and $\omega^{s'}_{t,s} = 0$ for each s'. Combining these observations with Eqs. (13) and (14), and using $a_{t,s} = Q_{t,s} k_{t,s}$, we obtain the following risk balance condition for each state s,

$$\sigma^k_{t,s} = \frac{1}{\sigma^k_{t,s}} \left(r^k_{t,s} + \sum_{s'} \lambda_{s,s'} \left(\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \right) - r^f_{t,s} \right), \tag{25}$$
$$\text{where } \sigma^k_{t,s} = \sigma_s + \sigma^Q_{t,s} \text{ and } r^k_{t,s} = \rho + \psi q_{t,s} - \delta + \mu^Q_{t,s} + \sigma_s \sigma^Q_{t,s}.$$

The equation says that in equilibrium the total risk in the economy (the left side) is equal to the Sharpe ratio perceived by the representative investor (the right side). Note that the Sharpe ratio accounts for the fact that the aggregate wealth (as well as the marginal utility) will change in case there is a state transition.[7]

We next conjecture an equilibrium in which there is no price drift and volatility, $\mu^Q_{t,s} = \sigma^Q_{t,s} = 0$. In particular, we conjecture that the price and the interest rate are constant within states, $Q_{t,s} = Q_s$ and $r^f_{t,s} = r_s$. Under this conjecture, Eqs. (25) and the goods market equilibrium conditions (23) represent a system of $2|S|$ equations in $2|S|$ unknowns, $\{Q_s, r_s\}_S$.

Two-states special case. We characterize the equilibrium further for the special case with two states, $S = \{1, 2\}$ with $\sigma_1 < \sigma_2$. In this case, Eq. (25) can be written as,

$$\sigma_s = \frac{\rho - \delta + \psi q_s + \lambda_s \left(1 - \frac{Q_s}{Q_{s'}}\right) - r^f_s}{\sigma_s}. \tag{26}$$

After rearranging terms, we obtain,

$$\begin{aligned} &R(q_s, q_{s'}, \lambda_s) - \sigma_s^2 = r^f_s, \text{ where} \\ &R(q_s, q_{s'}, \lambda_s) = \rho + \psi q_s - \delta + \lambda_s (1 - \exp(q_s - q_{s'})). \end{aligned} \tag{27}$$

Here, the function $R(q_s, q_{s'}, \lambda_s)$ captures the expected total return to capital when the current price is q_s, the price after the transition is $q_{s'}$, and the transition probability is λ_s. Condition (27) says that the risk-adjusted expected return to capital must be equal to the risk-free rate that determines the cost of capital. Note that $R(q_s, q_{s'}, \lambda_s)$ satisfies some intuitive comparative statics. It is increasing in the transition probability, λ_s, if and only if the future price level is greater than the current level, $q_{s'} > q_s$. It is always increasing in the future price level, $q_{s'}$. However, it is not necessarily decreasing in the current price level, q_s, due to the potentially destabilizing aggregate demand and growth effects that we described earlier (see Eq. (24) and

[7] To see this, observe that the term, $\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}$, in the equation is actually equal to, $\frac{Q_{t,s}}{Q_{t,s'}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}$. Here, $\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}$ denotes the capital gains and $\frac{Q_{t,s}}{Q_{t,s'}}$ denotes the marginal utility adjustment when there is a representative investor (see (13)).

the subsequent discussion). Assumption 2 below will ensure that, in the relevant range, these effects are dominated by the capital gains effect and the return is decreasing in the price level.

To solve for the equilibrium, let $R(q^*, q^*, \lambda_s) \equiv \rho - \delta + \psi q^*$ denote the return to capital (in either state) when the price is at its efficient level in both states. If the parameters are such that this return exceeds σ_2^2 (and thus, σ_1^2), then it is easy to check that the first-best equilibrium obtains. We focus on the more interesting case with the parameters that satisfy the following.

Assumption 1. $\sigma_2^2 > R(q^*, q^*, \lambda_s) = \rho + \psi q^* - \delta > \sigma_1^2$.

That is, the parameters are such that the risk-free rate in the first-best equilibrium would be strictly positive in state 1 but strictly negative in state 2. In this case, we conjecture that (under further parametric conditions) the low-volatility state 1 features positive interest rates, efficient prices, and full factor utilization, $r_1^f > 0, q_1 = q^*$ and $\eta_1 = 1$, whereas the high-volatility state 2 features zero interest rates, lower prices, and imperfect factor utilization, $r_2^f = 0, q_2 < q^*$ and $\eta_2 < 1$.

First consider the equilibrium in the high-volatility state. Combining this conjecture with Eq. (27), we obtain,

$$R(q_2, q^*, \lambda_2) - \sigma_2^2 = r_2^f = 0. \tag{28}$$

Recall also that $R(q^*, q^*, \lambda_2) - \sigma_2^2 < 0$ by assumption. Hence, the price level needs to decline below its efficient level, $q_2 < q^*$, to ensure that the return to capital is sufficiently high and the risk-adjusted return is equal to the risk-free interest rate, $r_2^f = 0$. Intuitively, since the interest rate is constrained, the risk balance condition (26) cannot be equilibrated with a decline in the interest rate. Instead, it is equilibrated via a decline in asset prices, q_2, which increases the expected asset return and ultimately the Sharpe ratio. This adjustment also leads to an inefficient recession. The following assumption ensures the existence of a stable equilibrium.

Assumption 2. $\lambda_2 \geq \lambda_2^{\min}$, where $\lambda_2^{\min}$ is the unique solution to $R\left(q^*, q^*, \lambda_2^{\min}\right) + \lambda_2^{\min} - \psi + \psi \log\left(\psi/\lambda_2^{\min}\right) = \sigma_2^2$ over the range $\lambda_2 \geq \psi$.

This condition ensures that there is a unique solution to Eq. (28). When the condition holds as strict inequality, the unique equilibrium price also satisfies, $\frac{\partial R(q_2, q^*, \lambda_2)}{\partial q_2} < 0$, that is, the decline in prices increases the expected return to capital. Intuitively, we need optimism to be sufficiently large that the capital gains effect (from a transition into the low-volatility state) dominates the destabilizing aggregate demand and growth effects that we described earlier (see Eq. (27)). When the condition is violated, a lower price level would lower the return further, which would trigger a downward spiral that would lead to an equilibrium with zero asset prices and output. When the condition holds as equality, the stabilizing force barely balances the destabilizing forces so that the equilibrium price satisfies, $\frac{\partial R(q_2, q^*, \lambda_2)}{\partial q_2} = 0$. As we will see below, this case features positive but very low asset prices and output due to relatively strong destabilizing forces.

Next consider the equilibrium in the low-volatility state 1. Combining our conjecture with

Eq. (27), we have,

$$R\left(q^{*},q_{2},\lambda_{1}\right)-\sigma_{1}^{2}=r_{1}^{f}. \tag{29}$$

Given q_2, this equation determines the interest rate, r_1^f. Intuitively, given the expected return on capital (that depends on—among other things—q_2), the interest rate adjusts to ensure that the risk-balance condition is satisfied with the efficient price level, $q_1 = q^*$. For our conjectured equilibrium, we also require that the implied interest rate to be nonnegative, $r_1^f \geq 0$. The following parametric condition ensures that this is the case.

Assumption 3. $R\left(q^*, q_2, \lambda_1\right) \geq \sigma_1^2$ where q_2 is the unique solution to (28).

Proposition 1. *Consider the model with two states, $s \in \{1,2\}$, with common beliefs and Assumptions 1-3. The low-volatility state 1 features a nonnegative interest rate, efficient asset prices and full factor utilization, $r_1^f \geq 0, q_1 = q^*$ and $\eta_1 = 1$, whereas the high-volatility state 2 features zero interest rate, lower asset prices, and a demand-driven recession, $r_2^f = 0, q_2 < q^*$, and $\eta_2 < 1$. The price level in state 2 is characterized as the unique solution to Eq.* (28)*, and the risk-free rate in state 1 is characterized by Eq.* (29)*.*

Comparative statics of equilibrium. We next establish comparative statics of the equilibrium, focusing on the endogenous price level in the high-volatility state, q_2 (the effects on r_1^f are straightforward conditional on q_2). First consider the effect of a change in optimism, λ_2. Implicitly differentiating Eq. (28), we obtain,

$$\frac{dq_2}{d\lambda_2}=\frac{\partial R\left(q_2,q^*,\lambda_2\right)/\partial\lambda_2}{-\partial R\left(q_2,q^*,\lambda_2\right)/\partial q_2}>0. \tag{30}$$

Here, the inequality follows since the denominator is positive in view of Assumption 2. Hence, the effect of optimism on the price is determined by its direct effect on the expected return to capital, which is positive. Intuitively, greater optimism increases the expected return, which leads to greater asset prices in equilibrium.

Next consider this expression for the special case in which optimism is at its lowest allowed level, $\lambda_2 = \lambda_2^{\min}$ (so that Assumption 2 holds as equality). In this case, we have $\partial R\left(q_2, q^*, \lambda_2\right)/\partial q_2 = 0$, which in turn implies $\frac{dq_2}{d\lambda_2} = \infty$. Hence, in the neighborhood of $\lambda_2 = \lambda_2^{\min}$, the recession is deep, and asset prices and output are extremely sensitive to further changes in beliefs due to the destabilizing aggregate demand and growth forces that we discussed earlier. More generally, the term in the denominator of Eq. (30) can be calculated as,

$$-\partial R\left(q_2,q^*,\lambda_2\right)/\partial q_2=\lambda_2\exp\left(q_2-q^*\right)-\psi. \tag{31}$$

This expression is increasing in λ_2 (both because of the direct effect and the indirect effect via q_2). Hence, the destabilizing forces are stronger when optimism is lower. The intuition is that optimism increases the expected capital gains that counters the destabilizing forces. The lack

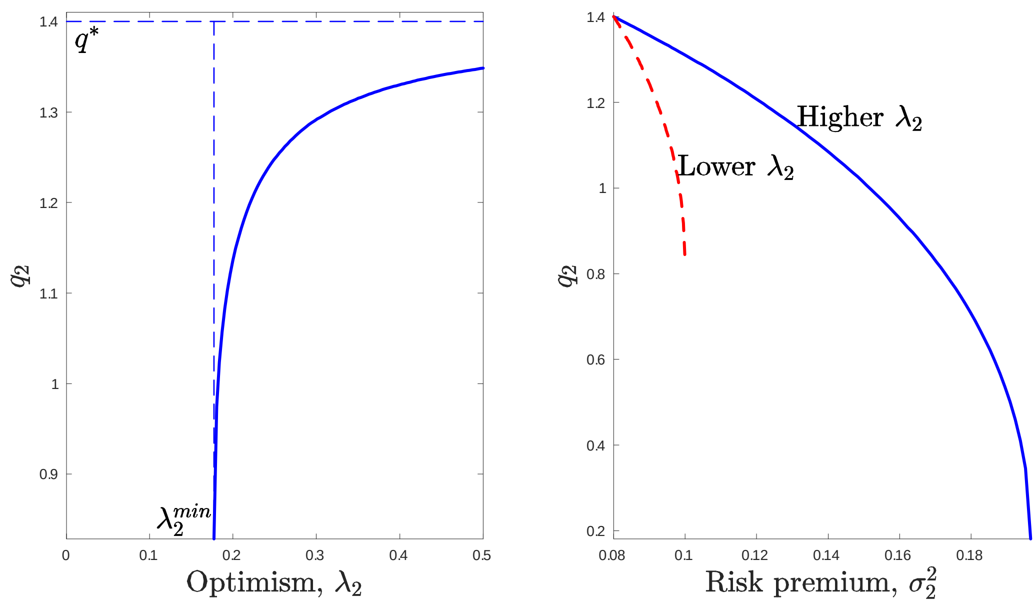

Figure 2: The left panel illustrates the effect of optimism on the asset price in state 2. The right panel illustrates the effect of risk premium on the asset price in state 2, when optimism is higher (solid line) and lower (dashed line).

of optimism unleashes these forces and makes the equilibrium prices and output very sensitive to exogenous changes in asset prices due to beliefs (as well as other factors). The left panel of Figure 2 illustrates these results for a particular parameterization.

Next consider the effect of an increase in the risk premium in the high-risk state. In our model, the risk premium is equal to the variance, σ_2^2 (Eq. (28)). Following the same steps as above, we obtain,

$$\frac{dq_2}{d\left(\sigma_2^2\right)} = \frac{-1}{-\partial R\left(q_2, q^*, \lambda_2\right)/\partial q_2} < 0. \tag{32}$$

Greater risk premium reduces the price due to its direct effect on the risk-adjusted return. As before, the effect is stronger when optimism is lower due to endogenous destabilizing forces. Formally, we have, $\frac{d}{d\lambda_2}\left|\frac{dq_2}{d\left(\sigma_2^2\right)}\right| < 0$. Combining Eqs. (31) and (32), we also obtain that the effect is stronger when the baseline level of the risk premium is higher (as this leads to a lower price level, q_2).[8] Formally, we have $\frac{d}{d\left(\sigma_2^2\right)}\left|\frac{dq_2}{d\left(\sigma_2^2\right)}\right| > 0$. The right panel of Figure 2 illustrates these results. Note that, for each level of optimism, there is a sufficiently high level of the risk premium that ensures Assumption 2 holds as equality and the economy experiences a deep recession. Beyond this level of the risk premium, there is no equilibrium with positive prices.

[8] To understand the intuition, note from Eq. (27) that the expected capital gains are given by $\lambda_2\left(1 - \exp\left(q_2 - q^*\right)\right)$. When q_2 is lower (due to higher σ_2^2), the capital gain *conditional on a transition* is already high and it is not very sensitive to further changes to q_2. On the other hand, the strength of the destabilizing aggregate demand and growth forces is controlled by the parameter, ψ, which is independent of the level of q_2.

Corollary 1. *(i) A decrease in optimism in state 2 reduces the price level, that is,* $\frac{dq_2}{d\lambda_2} > 0$.

(ii) An increase in the risk premium in state 2 reduces the price level, that is, $\frac{dq_2}{d(\sigma_2)^2} < 0$*; and by a larger magnitude when optimism is lower and the risk premium is higher, that is,* $\frac{d}{d\lambda_2}\left|\frac{dq_2}{d(\sigma_2^2)}\right| < 0$ *and* $\frac{d}{d(\sigma_2^2)}\left|\frac{dq_2}{d(\sigma_2^2)}\right| > 0$.

Note also that, as illustrated by Eq. (29), these changes that reduce the price in the high-volatility state, q_2, also reduce the interest rate in the low-volatility state, r_f^1. Intuitively, lower prices in state 2 also lower the asset prices and aggregate demand in state 1, which is countered by a lower interest rate.

4. Belief disagreements and speculation

We next consider the equilibrium with belief disagreements. As in the previous section, we first provide a general characterization. We then explicitly solve for the equilibrium for the special case with two states, $S = \{1, 2\}$.

The equilibrium depends on the wealth-weighted average transition probability,

$$\overline{\lambda}_{t,s,s'} = \sum_i \alpha_{t,s}^i \lambda_{s,s'}^i, \text{ where } \alpha_{t,s}^i = \frac{a_{t,s}^i}{k_{t,s} Q_{t,s}}.$$

Here, $\alpha_{t,s}^i$ denotes the wealth share of type i investors. Combining the optimality conditions (13) and (14) with the market clearing conditions (20) and (21), we obtain,

$$p_{t,s}^{s'} = \lambda_{s,s'}^i \frac{a_{t,s}^i}{a_{t,s'}^i} = \overline{\lambda}_{t,s,s'} \frac{Q_{t,s}}{Q_{t,s'}}, \tag{33}$$

$$\sigma_s + \sigma_{t,s}^Q = \frac{1}{\sigma_s + \sigma_{t,s}^Q} \left(r_{t,s}^k - r_{t,s}^f + \sum_{s'} \overline{\lambda}_{t,s,s'} \left(1 - \frac{Q_{t,s}}{Q_{t,s'}} \right) \right), \tag{34}$$

$$\omega_{t,s}^{k,i} = 1 \text{ for each } t, s, i.$$

The first equation says that the price of the Arrow-Debreu security is determined by the weighted average belief for the transition probability. The second equation says that the risk balance equation (25) in the benchmark case continues to hold in this setting as long as we calculate the transition probability with the weighted average belief. The last equation says that investors continue to allocate identical weights to capital. Intuitively, since their disagreements concern the jump probabilities, they use the Arrow-Debreu securities to speculate on these disagreements and do not distort their exposures to the diffusion risk.

It remains to characterize the evolution of the investors' wealth shares, $\alpha_{t,s}^i$. Plugging the evolution of wealth equation (10) into Eq. (33), and using $\omega_{t,s}^{k,i} = 1$, we obtain,

$$\omega_{t,s}^{s',i} = \lambda_{s,s'}^i - \overline{\lambda}_{t,s,s'}. \tag{35}$$

That is, the investor's wealth share in the Arrow-Debreu security is equal to her degree of optimism relative to the weighted average belief. Combining this with Eq. (10), we further obtain,

$$\begin{cases} \frac{d\alpha^i_{t,s}}{\alpha^i_{t,s}} = -\sum_{s' \neq s} \left(\lambda^i_{s,s'} - \overline{\lambda}_{t,s,s'}\right) dt, & \text{if there is no state change,} \\ \alpha^i_{t,s'} = \alpha^i_{t,s} \frac{\lambda^i_{s,s'}}{\overline{\lambda}_{t,s,s'}}, & \text{if there is a state change to } s'. \end{cases} \tag{36}$$

In particular, conditional on there not being a state change, the wealth shares evolve deterministically. If the investor is relatively optimistic about state transitions, then her wealth declines conditional on these transitions not being realized. However, when a state on which the investor is optimistic is eventually realized, the investor's wealth share makes a discrete upward jump. The symmetric opposite considerations apply to the wealth share of an investor that is relatively pessimistic.

The equilibrium is then characterized as follows. Regardless of the level of asset prices and output, Eq. (36) determines the evolution of investors' wealth shares. This in turn determines the weighted average belief, $\overline{\lambda}_{t,s,s'}$, as well as its evolution. Given the path of the weighted-average belief, $\left\{\overline{\lambda}_{t,s,s'}\right\}_t$, the equilibrium is determined by jointly solving the risk balance equation (34) and the goods market equilibrium condition (23). Solving these equations is slightly more involved than in the previous section since the weighted-average belief is generally not stationary, which implies the price of capital might also have a nonzero drift, $\mu^Q_{t,s}$ (although $\sigma^Q_{t,s}$ is zero as before).

Two-states special case. To characterize the equilibrium further, consider the special case with two states $S = \{1, 2\}$, with $\sigma_1 < \sigma_2$ from the previous section. Recall that we also use the shorthand notations, $\lambda_1 = \lambda_{1,2}$ and $\lambda_2 = \lambda_{2,1}$, to denote the transition rates respectively in states 1 and 2 (into the other state). Suppose there are two types of investors, $i \in \{o, p\}$, with beliefs denoted by, $\left\{\left(\lambda^i_1, \lambda^i_2\right)\right\}_{i \in \{o,p\}}$. Here, type o investors correspond to "optimists," and type p investors correspond to "pessimists." We denote optimists' transition probability relative to pessimists with the notation,

$$\Delta\lambda_s = \lambda^o_s - \lambda^p_s,$$

and assume beliefs satisfy the following.

Assumption 4. $\Delta\lambda_2 > 0$ and $\Delta\lambda_1 \leq 0$.

This assumption ensures that optimists are more optimistic than pessimists in either state. Specifically, when the economy is in the high-volatility state, optimists find the transition into the low-volatility state relatively likely ($\lambda^o_2 > \lambda^p_2$); when the economy is in the low-volatility state, optimists find the transition into the high-volatility state relatively unlikely ($\lambda^o_1 \leq \lambda^p_1$).

For notational simplicity, we use $\alpha_{t,s} = \alpha^o_{t,s} \in (0, 1)$ to represent optimists' wealth share.

Note that the weighted-average belief can be written as,

$$\overline{\lambda}_{t,s} = \lambda_s^p + \alpha_{t,s} \Delta \lambda_s. \tag{37}$$

Hence, optimists' wealth share denotes the appropriate state variable in this economy. By Eq. (35), optimists' investment in the contingent security is given by,

$$\omega_{t,s}^{s',o} = \Delta \lambda_s \left(1 - \alpha_{t,s}\right),$$

and by Eq. (36), their wealth share evolves according to,

$$\left\{ \begin{array}{ll} \dot{\alpha}_{t,s} = -\Delta \lambda_s \alpha_{t,s} \left(1 - \alpha_{t,s}\right), & \text{if there is no state change,} \\ \alpha_{t,s'} = \alpha_{t,s} \lambda_s^o / \left(\lambda_s^p + \alpha_{t,s} \Delta \lambda_s\right), & \text{if there is a state change to } s'. \end{array} \right. \tag{38}$$

Here, $\dot{\alpha}_{t,s} = \frac{d\alpha_{t,s}}{dt}$ denotes the derivative with respect to time. Recall that $\Delta\lambda_1 \leq 0$ and $\Delta\lambda_2 > 0$ (see Assumption 4). Hence, in the low-volatility state 1, optimists' wealth share drifts upwards, but it makes a downward jump if there is a transition into state 2. Intuitively, optimists sell put options on the aggregate state, which enables them to earn current profits at the expense of losses if the bad aggregate state is realized. Symmetrically, in the high-volatility state 2, optimists' wealth share drifts downwards but it makes an upward jump in case there is transition into state 1. Intuitively, optimists buy call options on the aggregate state, which reduces their current profits but generates gains if the good aggregate state is realized.

These observations also imply that the weighted-average belief in (37) (that determines asset prices) is effectively *extrapolative*. As the good (low-volatility) state persists longer, and optimists' wealth share increases, the aggregate belief becomes increasingly more optimistic. After a transition to a worse (high-volatility) state, the aggregate belief becomes more pessimistic. Conversely, the aggregate belief becomes more pessimistic as the bad state persists longer, and it becomes more optimistic after a transition into a better (low-volatility) state.

We next characterize the equilibrium (log) prices and factor utilizations within each state, $\left\{q_{t,s}, \eta_{t,s}\right\}_{s \in \{1,2\}}$. To this end, suppose Assumptions 1-3 hold according to optimists' as well as pessimists' beliefs. This ensures that, regardless of the wealth shares, state 1 features a positive interest rate, efficient price level, and full factor utilization, $r_{t,1}^f > 0, q_{t,1} = q^*$, and $\eta_{t,1} = 1$. We also conjecture that state 2 features a zero interest rate, a lower price level, and imperfect factor utilization, $r_{t,2}^f = 0, q_{t,2} < q^*$, and $\eta_{t,1} < 1$.

To characterize the price level in state 2, consider the risk balance equation (34). After substituting the return to capital from (24), and using $\mu_{t,2}^Q = \frac{dQ_{t,2}/dt}{Q_{t,2}} = \frac{dq_{t,2}}{dt}$, we obtain,

$$r_{t,2}^f = R\left(q_{t,2}, q^*, \overline{\lambda}_{t,2}\right) + \dot{q}_{t,2} - \sigma_2^2 = 0. \tag{39}$$

Here, $R\left(\cdot\right)$ is the function that characterizes the expected return to capital in the homogeneous

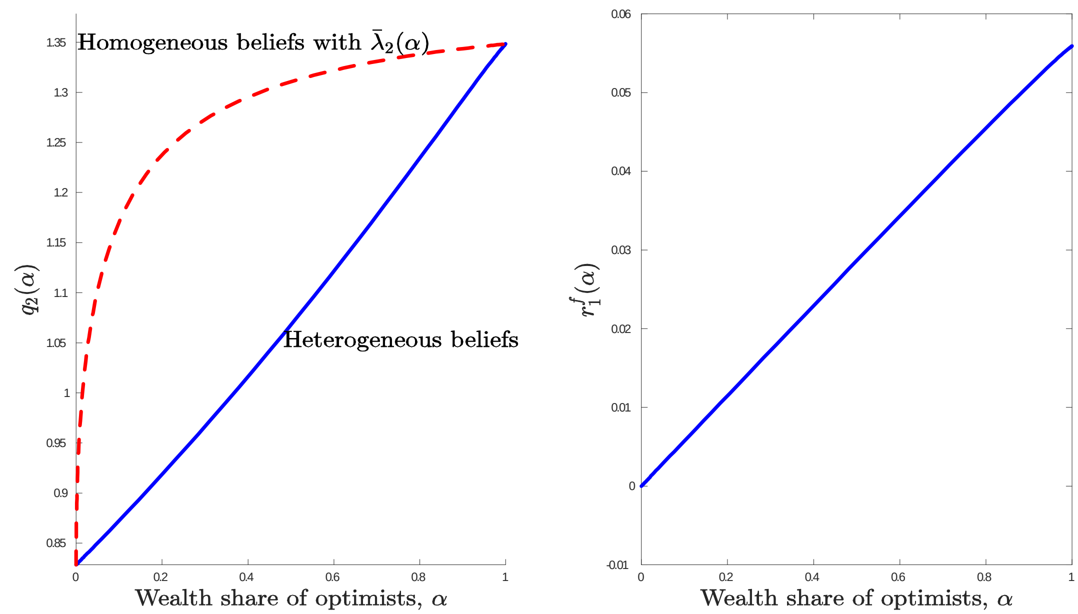

Figure 3: The solid lines illustrate the equilibrium price and interest rate functions under heterogeneous beliefs. The dashed line in the left panel illustrates the equilibrium price level that would obtain if all investors shared the weighted-average belief, $\overline{\lambda}_{t,2}$.

beliefs benchmark (cf. (27)), and $\dot{q}_{t,2} = \frac{dq_{t,2}}{dt}$ denotes the price drift conditional on there not being a transition. Eq. (39) illustrates that the expected return to capital is related to but not exactly the same as the return that would obtain in the benchmark in which all investors shared the weighted-average belief, $\overline{\lambda}_{t,2}$. In the present setting, the expected return (and thus, the equilibrium condition) is also affected by the wealth dynamics that change the weighted-average belief and ultimately introduce a drift into the asset price.

In particular, consider the joint evolution of optimists' wealth share and the price level conditional on there not being a state transition, denoted by $(\alpha_{t,2}, q_{t,2})$. Combining Eqs. $(38-39)$, we obtain a stationary differential equation,

$$\begin{aligned} \dot{q}_{t,2} &= -\left(R\left(q_{t,2}, q^*, \lambda_2^p + \alpha_{t,2}\Delta\lambda_2\right) - \sigma_2^2\right), \qquad (40) \\ \dot{\alpha}_{t,2} &= -\Delta\lambda_2\alpha_{t,2}\left(1-\alpha_{t,2}\right). \end{aligned}$$

In Appendix B, we show that this system is saddle path stable. In particular, for any initial wealth share, $\alpha_{t,2} \in (0,1)$, there exists a unique equilibrium price level, $q_{t,2} \in [q^p, q^o)$, such that the solution satisfies $\lim_{t\to\infty} \alpha_{t,2} = 0$ and $\lim_{t\to\infty} q_{t,2} = q_2^p$. When $\alpha_{t,2} = 1$, the solution satisfies $q_{t,2} = q_t^o$.

Since the system in (40) is stationary, the equilibrium price can be written as a function of optimists' wealth share, that is, $q_{t,2} = q_2\left(\alpha_{t,2}\right)$ for some function $q_2 : [0,1] \to [q^p, q^o]$. Eliminating time from the system , the price function solves the differential equation,

$$q_2'(\alpha)\,\Delta\lambda_2\alpha(1-\alpha) = R\left(q_2(\alpha), q^*, \lambda_2^p + \alpha\Delta\lambda_2\right) - \sigma_2^2, \tag{41}$$

together with the end-value conditions, $q_2(0) = q_2^p$ and $q_2(1) = q_2^o$. In Appendix B, we show that the price function, $q_2(\alpha)$, is strictly increasing in α. As in the previous section, greater optimism increases the asset price in state 2. We also show that $q_2(\alpha) < q_2^h(\alpha)$ for each $\alpha \in (0,1)$, where $q_2^h(\alpha)$ denotes the price level that would obtain in the homogeneous belief benchmark in which all investors share the belief, $\lambda_2^p + \alpha\Delta\lambda_2$. Intuitively, if the recession persists longer, pessimists will become more dominant and the price level will drift downward. The downward drift reduces the return to capital, which reduces the current price level to equilibrate the risk balance condition. This illustrates that, when output is demand constrained, speculation driven by belief disagreements reduces aggregate demand and output as well as asset prices.

The left panel of Figure 3 illustrates the price function, $q_2(\alpha)$, for a particular parameterization. We chose the parameters so that pessimists' belief in state 2, λ_2^p, satisfies Assumption 2 with equality. This implies that, when optimists' wealth share is low, asset prices and output are very low due to the destabilizing forces that we discussed in the previous section. The figure further illustrates that the equilibrium with heterogeneous beliefs differs sharply from the homogeneous beliefs benchmark in which all investors share the weighted average belief. In the benchmark, optimism greatly improves the outcomes by mitigating the destabilizing forces (see Section 3). With heterogeneous beliefs, optimism has a smaller effect since the investors recognize that, if the recession persists, pessimism will prevail and unleash the destabilizing forces. This suggests that it is enough to have one group of highly pessimistic agents to unleash the destabilizing aggregate demand and growth forces.

For completeness, we also characterize the equilibrium interest rate in state 1. Following similar steps, we obtain, $r_{t,1}^f = r_1^f(\alpha_{t,1})$ where $r_1^f : [0,1] \to \mathbb{R}_+$ denotes the function defined by,

$$r_1^f(\alpha) = R\left(q^*, q_2(\alpha'), \lambda_1^p + \alpha\Delta\lambda_1\right) - \sigma_1^2 \text{ where } \alpha' = \alpha\lambda_1^o / \left(\lambda_1^p + \alpha\Delta\lambda_1\right). \tag{42}$$

Here, $q_2(\alpha')$ captures the price that would obtain if there was an immediate transition into state 2. Since there is no price drift in state 1, the return to capital is characterized as in the homogeneous beliefs benchmark given the weighted average belief and the endogenous transition price. The risk-free interest rate is equal to the return to capital net of the risk premium. For our conjecture to be valid, we also require, $r_1^f(\alpha) > 0$ for each α. This condition holds because Assumptions 1-3 hold for pessimists (as well as optimists). The right panel of Figure 3 illustrates the interest-rate function. The following result summarizes the characterization of equilibrium.

Proposition 2. *Consider the model with two states, $s \in \{1,2\}$, and heterogeneous beliefs. Suppose Assumptions 1-3 hold for each belief type $i \in \{i, o\}$, and that beliefs are ranked according to Assumption 4. Then, optimists' wealth share evolves according to Eq.* (38). *The*

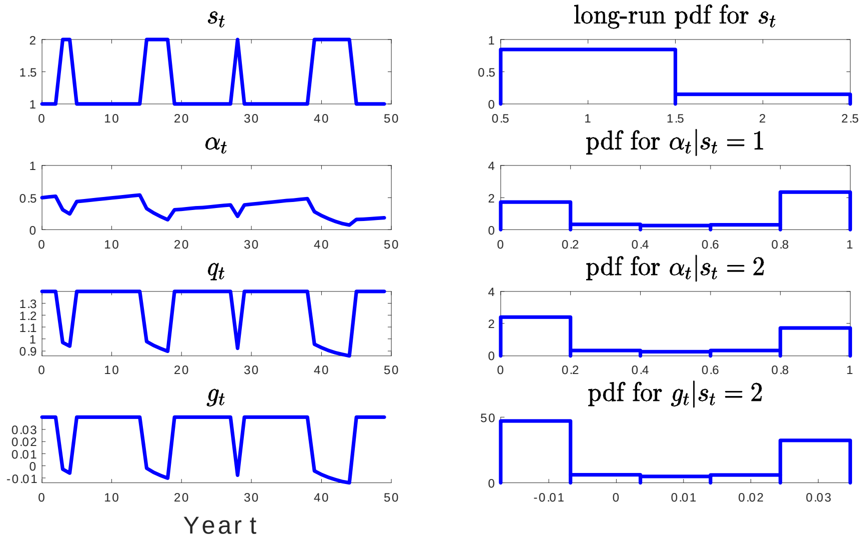

Figure 4: The left panels illustrate the evolution of the equilibrium variables over the medium run (50 years) for a particular realization of uncertainty. The right panels illustrate the long-run stationary distributions of equilibrium variables.

equilibrium prices and interest rates can be written as a function of optimists' wealth shares, $q_1(\alpha), r_1^f(\alpha), q_2(\alpha), r_2^f(\alpha)$. *At the high-volatility state,* $r_2^f(\alpha) = 0$ *and* $q_2(\alpha)$ *solves the differential equation* (39) *with* $q_2(0) = q_2^p$ *and* $q_2(1) = q_2^o$. *The price function satisfies* $\frac{dq_2(\alpha)}{d\alpha} > 0$ *and* $q_2(\alpha) < q_2^h(\alpha)$ *for each* $\alpha \in (0,1)$ *(where* $q_2^h(\alpha)$ *denotes the price in the homogeneous-belief benchmark with wealth-weighted average belief,* $\lambda_2^p + \alpha\Delta\lambda_2$*). At the low-volatility state,* $q_1(\alpha) = q^*$ *and* $r_1^f(\alpha)$ *is given by Eq.* (42)*. The interest rate function satisfies* $\frac{dr_1^f(\alpha)}{d\alpha} > 0$ *for each* $\alpha \in (0,1)$.

Dynamics of equilibrium. Figure 4 illustrates the dynamics of the equilibrium. The panels on the left illustrates the evolution of the equilibrium variables over a 50-year horizon. Note that optimists' wealth share grows in the low-volatility state but it declines when the economy switches to the high-volatility state. The asset price is below its efficient level in the high-volatility state, and more so when optimists' wealth share is lower. The panels on the right illustrate the simulated long-run distributions of equilibrium variables. To obtain non-degenerate long-run wealth distribution, in which neither optimists nor pessimists permanently dominate, we simulate the economy with beliefs that are in the "middle" of optimists' and pessimists' beliefs in terms of the relative entropy distance.[9] Note that the economy spends relatively little time in the range in which optimists' wealth share is in the intermediate range. Intuitively, this

[9] Given two probability distributions $(p(\tilde{s}))_{\tilde{s}\in S}$ and $(q(\tilde{s}))_{\tilde{s}\in S}$, relative entropy of p with respect to q is defined as $\sum_{\tilde{s}} p(\tilde{s}) \log\left(\frac{p(\tilde{s})}{q(\tilde{s})}\right)$. Blume and Easley (2006) show that, in a setting with independent and identically distributed shocks (and identical discount factors), only investors whose beliefs have the maximal relative entropy distance to the true distribution survive. Since our setting features Markov shocks, we apply their result state-by-state to

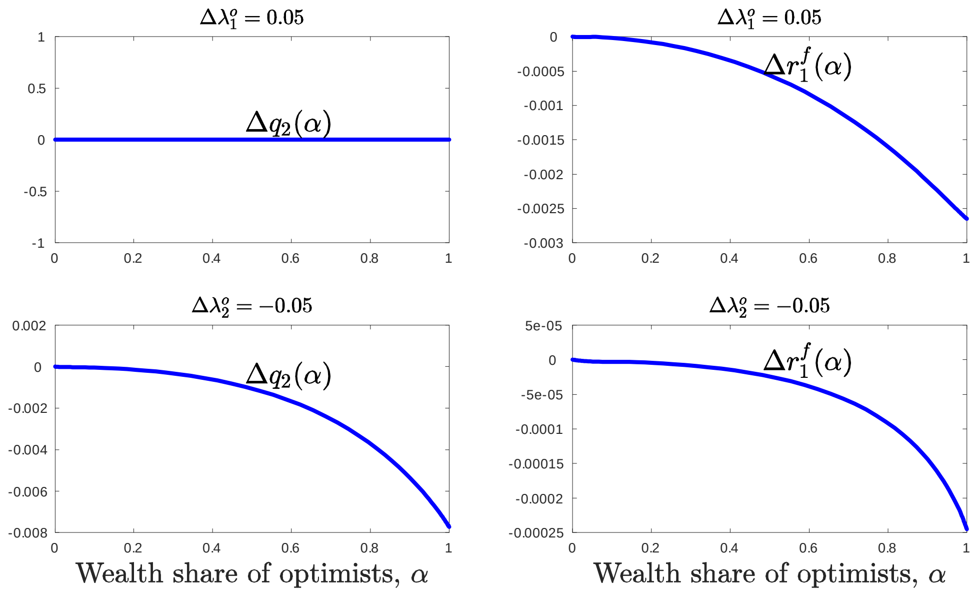

Figure 5: The effect of making optimists less optimistic in state 1 (top panels) and in state 2 (bottom panels).

range features substantial speculation, which is resolved only when one or the other belief type temporarily dominates. Note also that optimists tend to dominate more in the low-volatility state 1, whereas pessimists tend to dominate more in the high-volatility state 2. In the next section, we will investigate whether and how macroprudential policy can affect the evolution of investors' wealth shares.

Comparative statics of equilibrium. To facilitate our analysis of macroprudential policy, we also establish the comparative statics of reducing optimists' optimism. First consider a decline in their optimism in state 1 (captured by an increase in the transition probability, λ_1^o). As illustrated by the top panels of Figure 5, this leaves the price function in state 2 unchanged (since the beliefs in that state are unchanged) but it reduces the risk-free rate in state 1. Intuitively, lower optimism reduces the demand for risky assets in state 1 but this effect is countered by a reduction in the interest rate. Next consider a decline in optimists' optimism in state 2 (captured by a decrease in the transition probability, λ_2^o). As illustrated by the bottom panels of Figure 5, this reduces the price function in state 2 as well as the interest rate function in state 1. These results are similar to the effect of reducing optimism in state 2 in the common beliefs benchmark (cf. Corollary 1). Note also that, as expected, reducing an investor's optimism has a greater impact on prices when their wealth share is higher.

ensure that conditional probabilities satisfy the necessary survival condition. Specifically, for each state $s \in \{1, 2\}$, we choose λ_s^{sim} so that the relative entropy of the conditional probability distribution for the next state with respect to optimists' beliefs (in the discrete-time approximation of the model) is the same as the relative entropy with respect to pessimists' beliefs.

5. Welfare analysis and macroprudential policy

In this section, we establish our normative results on macroprudential policy. To this end, we first characterize investors' value functions in equilibrium. This establishes the determinants of welfare in this setting and illustrates the aggregate demand externalities. We then show that, when investors have heterogeneous beliefs, the equilibrium can be Pareto improved by macroprudential policy that restricts optimists' risk taking. Throughout, we focus on the two-state special case to simplify the notation.

5.1. Equilibrium value functions

Recall that the value function has the functional form in (11), where $v^i_{t,s}$ denotes the normalized value function per unit of capital stock. Eq. $(A.12)$ in Appendix A.2 characterizes $v^i_{t,s}$ as a solution to a differential equation. In the two-state special case, the equation becomes,

$$\rho v^i_{t,s} - \frac{\partial v^i_{t,s}}{\partial t} = \log \rho + q_{t,s} + \frac{1}{\rho}\left(\begin{array}{c} \psi q_{t,s} - \delta - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda^i_s - \overline{\lambda}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^i_s}{\overline{\lambda}_{t,s}}\right) \end{array} \right) + \lambda^i_s \left(v^i_{t,s'} - v^i_{t,s}\right). \tag{43}$$

The equilibrium value functions for type i are characterized by jointly solving the differential equations for states $s \in \{1,2\}$.

Eq. (43) illustrates the determinants of welfare. When there is a demand-driven recession ($q_{t,s} \leq q^*$), a lower equilibrium price, $q_{t,s}$, reduces investors' welfare since it is associated with lower factor utilization, $\eta_{t,s}$. Note that welfare declines due to a decline in current consumption (captured by the term, $\log \rho + q_{t,s}$) as well as a decline in investment and consumption growth (captured by the term, $\psi q_{t,s} - \delta = g_{t,s}$). The risk premium, σ_s^2, also affects the welfare through its influence on the risk-adjusted consumption growth. Finally, speculation among investors with heterogeneous beliefs also affects (perceived) welfare. This is captured by the term, $-\left(\lambda^i_s - \overline{\lambda}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^i_s}{\overline{\lambda}_{t,s}}\right)$, which is zero with common beliefs, and strictly positive with heterogeneous beliefs.

To facilitate our analysis of macroprudential policy, we also break down the value function into two components,

$$v_{t,s} = v^*_{t,s} + w_{t,s}, \tag{44}$$

where $v^*_{t,s}$ denotes *the first-best value function* that would obtain if there were no interest rate rigidities, and $w_{t,s} = v_{t,s} - v^*_{t,s}$ denotes *the gap value function* relative to the first best. Recall that the first-best equilibrium features $q_{t,s} = q^*$ for each t and s. Hence, the value function, $v^*_{t,s}$, is characterized as the solution to Eq. (43) with the efficient price level, $q_{t,s} = q^*$. Using linearity, the gap value function is then characterized as the solution to the following differential equation,

$$\rho w^i_{t,s} - \frac{\partial w^i_{t,s}}{\partial t} = \left(1 + \frac{\psi}{\rho}\right)(q_{t,s} - q^*) + \lambda^i_s \left(w^i_{t,s'} - w^i_{t,s}\right). \tag{45}$$

This illustrates that the gap value captures the loss of welfare due to the price deviations from the first best. The first best value, $v_{t,s}^{i,*}$, captures the remaining components of value including the perceived welfare from speculation. As we will see, the gap value functions are useful to understand the marginal effect of macroprudential policy on social welfare.

We next characterize the value function and its components in the equilibria we analyzed in the previous sections.

Value functions in the common beliefs benchmark. With common beliefs, the price level is stationary, $q_{t,s} = q_s$ for each s (see Section 3). Eq. (43) then implies that the value function is also stationary, $v_{t,s} = v_s$ for each s. In Appendix A.2, we show that the stationary values can be solved in closed form as,

$$\rho v_s = \log \rho + \overline{q}_s + \frac{1}{\rho}\left(\psi \overline{q}_s - \delta - \frac{1}{2}\overline{\sigma}_s^2\right), \tag{46}$$
$$\text{where } \overline{q}_s = \beta_s q_s + (1-\beta_s) q_{s'} \text{ and } \overline{\sigma}_s^2 = \beta_s \sigma_s^2 + (1-\beta_s)\sigma_{s'}^2,$$
$$\text{and } \beta_s = \frac{\rho + \lambda_{s'}}{\rho + \lambda_{s'} + \lambda_s}.$$

Here, the weights β_s and $1-\beta_s$ can be thought of as capturing the expected amount of "discounted time" the investor spends in each state (the weights reflect the fact that the investor starts in state s and discounts the future at rate ρ). The value in a state is the sum of the utility from (the discounted average of) current consumption and the present value of the risk-adjusted growth rate. All else equal, the value is decreasing in the weighted average volatility, $\overline{\sigma}_s$, but it is increasing in the weighted-average price level, $\overline{q}_s$.

It can also be seen that the first-best and the gap value components are stationary, $v_{t,s}^* = v_s^*$ and $w_{t,s}^* = w_s^*$. Using Eq. (45), we further obtain,

$$\rho w_s = (\overline{q}_s - q^*)\left(1 + \frac{\psi}{\rho}\right). \tag{47}$$

That is, the gap value is proportional to the weighted-average price gap relative to the first best. In Appendix A.2, we further show that $w_2 < w_1 < 0$, that is, the gap value is negative for both states but more negative in the high-volatility state.

Value functions with two belief types and aggregate demand externalities. With heterogeneous beliefs, the solution to Eq. (43) is not necessarily stationary since the price might have a drift. Consider the equilibrium with two belief types that we analyzed in Section 4. Recall that the equilibrium price in the high-volatility state is a function of optimists' wealth share, $q_2(\alpha)$. The equilibrium values and its components can also be written as a function of optimists' wealth share, $\left\{v_s^i(\alpha), v_s^{i,*}(\alpha), w_s(\alpha)\right\}_{s,i}$. Appendix A.2 characterizes these value functions as solutions to differential equations in α-domain (by combining Eqs. (43) and (45)

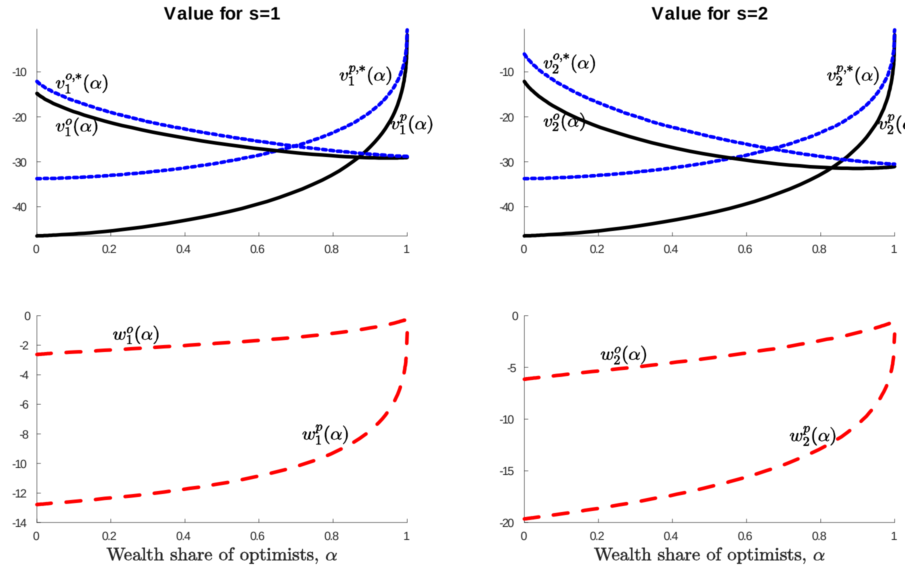

Figure 6: The plots illustrate the equilibrium value functions for each state and belief type. The solid lines are the actual value functions, $v_s^i(\alpha)$, the dotted lines are the first-best value functions, $v_s^{i,*}(\alpha)$, and the dashed lines (in the bottom panels) are the gap value functions, $w_s^i(\alpha)$.

with the evolution of $\alpha_{t,s}$ from (38)). Figure 6 illustrates the solution to these differential equations for the equilibrium plotted in the earlier Figure 3.

The bottom panels of Figure 6 show that the gap value functions are increasing in the wealth share of optimists, α, which illustrates the *aggregate demand externalities*. Greater α increases the effective optimism, which in turn leads to a greater equilibrium asset price in the high-volatility state (see Figure 3). This improves the gap value function in this state by raising the aggregate demand and bringing the economy closer to the first-best benchmark (see Eq. (45)). It also improves the gap value function in the low-volatility state, because the economy can always transition into the high-volatility state, and these transitions are less costly when α is greater. Hence, increasing optimists' wealth share is always associated with positive aggregate demand externalities. Individual optimists that take risks (or pessimists that take the other side of these trades) do not internalize their effects on asset prices, which leads to inefficiencies and generates scope for macroprudential policy.

The top panels of Figure 6 illustrate that the first-best value functions are increasing in α for pessimists but it is decreasing in α for optimists. These effects can be understood via pecuniary externalities in contingent security markets. Increasing the mass of optimists increases the price of contingent securities that optimists purchase, while decreasing the price of contingent securities that pessimists purchase. This creates negative pecuniary externalities (or crowd-out effects) on optimists, and positive pecuniary externalities on pessimists.

Finally, note that the actual value function is the sum of the first-best and the gap value functions. For pessimists, the actual value is always increasing in α, since the two components move in the same direction. For optimists, this is not necessarily the case since the gap value is increasing in α whereas the first-best value is decreasing.

5.2. Macroprudential policy

We capture macroprudential policy as risk limits on optimists. Consider the case with two belief types. Suppose, at any state s and time t, the planner can induce optimists to choose (instantaneous) allocations as if they have less optimistic beliefs. Specifically, optimists are constrained to choose allocations as if they have beliefs, $\left(\lambda_{t,1}^{o,pl}, \lambda_{t,2}^{o,pl}\right)$, that satisfy, $\lambda_{t,1}^{o,pl} \geq \lambda_1^o$ and $\lambda_{t,2}^{o,pl} \leq \lambda_2^o$. In Appendix A.3, we show that the planner can implement this policy by imposing inequality restrictions on optimists' portfolio weights (see Eq. $(A.18)$), while allowing them to make unconstrained consumption-savings decisions. Specifically, the policy constrains optimists from taking too low a position on the contingent security that pays in the high-volatility state, $\omega_{t,1}^{2,o} \geq \underline{\omega}_{t,1}^{2,o}$ (restrictions on selling "put options"). It also constrains optimists from taking too high a position on the contingent security that pays in the low-volatility state, $\omega_{t,2}^{1,o} \leq \overline{\omega}_{t,2}^{1,o}$ (restrictions on buying "call options"). Finally, the policy also constrains optimists' position on capital not to exceed the market average, $\omega_t^{k,o} \leq 1$ (since otherwise optimists start to speculate by holding too much capital).

For simplicity, we restrict attention to time-invariant policies. Specifically, the planner commits to a policy at time zero, $\lambda^{o,pl} \equiv \left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$, and implements it throughout. We assume the planner respects investors individual beliefs, that is, optimists' as well as pessimists' expected values in equilibrium are calculated according to their own beliefs. Finally, to trace the Pareto frontier, we also allow the planner to do a one-time wealth transfer among the agents at time zero.

Formally, let $V_{t,s}^i\left(a_{t,s}^i|\lambda^{o,pl}\right)$ denote type i investors' expected value in equilibrium when she starts with wealth $a_{t,s}^i$ and the policy is $\lambda^{o,pl}$. The planner's Pareto problem can then be written as,

$$\max_{\tilde{\lambda}^{o,pl},\tilde{\alpha}_{0,s}} \gamma^o V_{0,s}^o\left(\tilde{\alpha}_{0,s}Q_{0,s}k_{0,s}|\tilde{\lambda}^{o,pl}\right) + \gamma^p V_{0,s}^p\left(\left(1-\tilde{\alpha}_{0,s}\right)Q_{0,s}k_{0,s}|\tilde{\lambda}^{o,pl}\right). \tag{48}$$

Here, $\gamma^o, \gamma^p \geq 0$ (with at least one strict inequality) denote the Pareto weights, and $Q_{0,s}$ denotes the endogenous equilibrium price that obtains under the planner's policy.

The characterization of equilibrium with policy is the same as in Section 4. In particular, Eqs. (38) and (39) continue to hold with the only difference that optimists' beliefs are replaced by their "as-if" beliefs, $\left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$. The main difference in this case concerns the calculation of value functions, which are determined according to investors' actual beliefs. We start with a number of observations that simplify the welfare analysis.

First, the value function has the same functional form as before with appropriate normalized value functions, $v_{t,s}^i$. After substituting $a_{t,s}^i = \alpha_{t,s}^i k_{t,s} Q_{t,s}$, the functional form in (11) can be written as,

$$V_{t,s}^{i} = v_{t,s}^{i} + \frac{\log\left(\alpha_{t,s}^{i}\right) + \log\left(k_{t,s}\right)}{\rho}. \tag{49}$$

With macroprudential policy, the normalized value function, $v_{t,s}^{i}$, solves Eq. $(A.20)$ in Appendix A.3. This is the analogue of Eq. (43) with the only difference that the probability of a transition is calculated according to investors' *actual* beliefs, λ^{i}, as opposed to as-if beliefs, $\lambda^{i,pl}$ (for pessimists, the two beliefs coincide). As before, we also decompose the value function into first-best and gap value components, $v_{t,s}^{i} = v_{t,s}^{i,*} + w_{t,s}^{i}$.

Second, using Eq. (49), the planner's optimization problem (48) can be reduced to,[10]

$$\max_{\tilde{\lambda}^{o,pl}} \alpha_{0,s} v_{0,s}^{o} + \left(1 - \alpha_{0,s}\right) v_{0,s}^{p}, \tag{50}$$

where $\alpha_{0,s} \in [0,1]$ is the unique solution to $\frac{\gamma^{o}}{\gamma^{p}} = \frac{\alpha_{0,s}}{1-\alpha_{0,s}}$. Thus, the planner maximizes a wealth-weighted average of normalized utilities, where the relative wealths reflect the relative Pareto weights. We therefore define the planner's value function as a wealth-weighted average of individuals' value functions, $v_{t,s}^{pl} = \alpha_{t,s} v_{t,s}^{o} + \left(1-\alpha_{t,s}\right) v_{t,s}^{p}$. We also decompose the planner's value function into first-best and gap components, $v_{t,s}^{pl} = v_{t,s}^{pl,*} + w_{t,s}^{pl}$.

Third, the equilibrium with macroprudential policy is stationary. In particular, all equilibrium variables can be written as a function of optimists' wealth shares. As in the case without policy, we denote the equilibrium price functions with $\left\{q_s\left(\alpha\right)\right\}_s$, individuals' value functions with $\left\{v_s^{i}\left(\alpha\right), v_s^{i,*}\left(\alpha\right), w_s^{i}\left(\alpha\right)\right\}_{s,i\in\{o,p\}}$.[11] The planner's value function is a wealth-weighted average of individual value functions, $v_s^{pl}\left(\alpha\right) = \alpha v_s^{o}\left(\alpha\right) + \left(1-\alpha\right) v_s^{p}$. Similar expressions also hold for $v_s^{pl,*}\left(\alpha\right)$ and $w_s^{pl}\left(\alpha\right)$.

Fourth, note that the model features complete markets and no frictions other than interest rate rigidities. Hence, the First Welfare Theorem applies to the first-best allocations that also correct for these rigidities (and features efficient output). This in turn implies the derivative of the first-best value with respect to a marginal policy change is zero (otherwise, the first-best allocations could be Pareto improved). Since $v_s^{pl}\left(\alpha\right) = v_s^{pl,*}\left(\alpha\right) + w_s^{pl}\left(\alpha\right)$, we further obtain,

$$\left.\frac{\partial v_s^{pl}\left(\alpha\right)}{\partial \lambda^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^{o}} = \left.\frac{\partial w_s^{pl}\left(\alpha\right)}{\partial \lambda^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^{o}}. \tag{51}$$

[10] To see this, note that the planner's problem is to maximize,

$$\max_{\tilde{\lambda}^{o,pl},\tilde{\alpha}_{0,s}} \left(\gamma^{o} v_{0,s}^{o} + \gamma^{p} v_{0,s}^{p}\right) + \frac{\gamma^{o}\log\left(\tilde{\alpha}_{0,s}^{o}\right) + \gamma^{p}\log\left(1-\tilde{\alpha}_{0,s}^{o}\right)}{\rho} + \frac{\left(\gamma^{o}+\gamma^{p}\right)\log\left(k_{0,s}\right)}{\rho}.$$

Here, the last term (that features capital) is a constant that doesn't affect optimization. The second term links the planner's choice of wealth redistribution, $\alpha_{0,s}^{o}, \alpha_{0,s}^{p}$, to her Pareto weights, γ^{o}, γ^{p}. Specifically, the first order condition with respect to optimists' wealth share implies $\frac{\gamma^{o}}{\gamma^{p}} = \frac{\alpha_{0,s}}{1-\alpha_{0,s}}$. Thus, the planner effectively maximizes the first term after substituting γ^{o} and γ^{p} respectively with the optimal choice of $\alpha_{0,s}$ and $1-\alpha_{0,s}$.

[11] These functions also depend on the planner's policy choice, $\lambda^{o,pl}$. We suppress this dependence to simplify the notation.

The upshot of these observations is that the marginal effect of policy is determined by its marginal effect on the planner's gap value function, which can be written as,

$$w_s^{pl}(\alpha) = \alpha w_s^o(\alpha) + (1-\alpha) w_s^p(\alpha).$$

In Appendix A.3, we show that the gap value function is characterized as the solution to the same HJB equation (45) as before (determined by investors' actual beliefs, λ_s^i). Intuitively, macroprudential policy affects the path of prices, $q_{t,s}$, but it does not change the characterization of the gap values conditional on these prices. Combining Eq. (45) with the evolution of optimists' wealth share in (38) (which is determined by the as-if beliefs, $\lambda_s^{i,pl}$), we obtain the differential equation system,

$$\rho w_s^i(\alpha) = \left(1+\frac{\psi}{\rho}\right)(q_s(\alpha)-q^*) - \alpha(1-\alpha)\Delta\lambda_s^{pl}\frac{\partial w_s^i(\alpha)}{\partial \alpha} + \lambda_s^i\left(w_{s'}^i(\alpha') - w_s^i(\alpha)\right). \tag{52}$$

Here, $\alpha' = \alpha\frac{\lambda_s^{o,pl}}{\lambda_s^p + \alpha\Delta\lambda_s^{pl}}$ denotes optimists' wealth share after a transition evaluated under the as-if beliefs. It remains to characterize how macroprudential policy affects the solution to the system in (52). We start by analyzing the effect of macroprudential policy in the low-volatility state $s=1$, assuming that there is no intervention in the other state. We then analyze the polar opposite case in which there is intervention in the high-volatility state $s=2$ but not the other state.

5.2.1. Macroprudential policy during the boom

Suppose the planner is constrained to set $\lambda_2^{o,pl} = \lambda_2^o$, but she can choose $\lambda_1^{o,pl} \geq \lambda_1^o$, which induces optimists to act more pessimistically in the low-volatility state. First consider the special case, $\Delta\lambda_1 = 0$ (so investors disagree only in state $s=2$). We obtain a sharp result for this case, and we show in numerical simulations that the result also applies when $\Delta\lambda_1 < 0$.

Proposition 3. *Consider the equilibrium with heterogeneous beliefs that satisfy $\Delta\lambda_1 = 0$ (so disagreements only concern transitions out of state $s=2$), together with macroprudential policy that satisfies $\lambda_2^{o,pl} = \lambda_2^o$ (so policy affects only $\lambda_1^{o,pl}$). Macroprudential policy strictly increases the gap value according to each belief, that is,*

$$\left.\frac{\partial w_s^i(\alpha)}{\partial \lambda_1^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^o} > 0 \text{ for each } i \in \{o,p\}, s \in \{1,2\} \text{ and } \alpha \in (0,1).$$

Macroprudential policy also increases the planner's value, $\left.\frac{\partial v_s^{pl}(\alpha)}{\partial \lambda^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^o} = \left.\frac{\partial w_s^{pl}(\alpha)}{\partial \lambda^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^o} > 0$. In particular, regardless of the planner's Pareto weight and the current state, there exists a Pareto improving macroprudential policy, $\lambda_1^{o,pl}$.

The result shows that macroprudential policy improves the gap value function according to optimists as well as pessimists. Therefore, it also increases the wealth-weighted average gap value. In view of Eq. (51), it also increases the social welfare and leads to a Pareto improvement.

To obtain a sketch proof for the result, consider the differential equation (52) for state $s = 1$ and an arbitrary belief type $i \in \{o, p\}$. Differentiating this expression with respect to policy, $\lambda_1^{o,pl}$, and evaluating at the no-policy equilibrium, $\lambda_1^{o,pl} = \lambda_1^o$, we obtain,

$$\begin{aligned}(\rho + \lambda_1) \frac{\partial w_1^i(\alpha)}{\partial \lambda_1^{o,pl}} &= \left[-\alpha(1-\alpha) \frac{\partial w_1^i(\alpha)}{\partial \alpha} + \lambda_1 \frac{\partial \alpha'}{\partial \lambda_1^{o,pl}} \frac{\partial w_2^i(\alpha')}{\partial \alpha'} \right] + \lambda_1 \frac{\partial w_2^i(\alpha')}{\partial \lambda_1^{o,pl}}, \\ &= \alpha(1-\alpha) \left[-\frac{\partial w_1^i(\alpha)}{\partial \alpha} + \frac{\partial w_2^i(\alpha)}{\partial \alpha} \right] + \lambda_1 \frac{\partial w_2^i(\alpha)}{\partial \lambda_1^{o,pl}}. \end{aligned} \quad (53)$$

Here, $\lambda_1 \equiv \lambda_1^o = \lambda_1^p$ denotes investors' common belief in state 1. The second line substitutes $\frac{\partial \alpha'}{\partial \lambda_1^{o,pl}} = \frac{\alpha(1-\alpha)}{\lambda_1}$ (which follows from $\alpha' = \alpha \frac{\lambda_1^{o,pl}}{\lambda_1^p + \alpha \Delta \lambda_1^{pl}}$ and $\Delta\lambda_1 = 0$) as well as $\alpha' = \alpha$ (which follows from $\Delta\lambda_1 = 0$). The two terms inside the brackets capture the direct effects of macroprudential policy on social welfare. Macroprudential policy effectively induces optimists to purchase more insurance. This reduces optimists' relative wealth share in state 1 but improves their relative wealth share in state 2. Moreover, using the equilibrium prices, one unit of decline in wealth share in state 1 is associated with one unit of increase in expected wealth share in state 2.

Next recall that the gap value function in either state is increasing in optimists' wealth share $\frac{\partial w_1^i(\alpha)}{\partial \alpha}, \frac{\partial w_2^i(\alpha)}{\partial \alpha} > 0$ (see Figure 6). Hence, macroprudential policy always involves a trade-off. Intuitively, optimism is a scarce resource that could also be utilized immediately as well as in the future. Moving optimism across states via macroprudential policy is always associated with costs as well as benefits. However, the typical situation is such that optimism increases the social welfare more in state 2, where it provides immediate benefits, as opposed to state 1, where its benefits are realized in case there is a future transition into state 2. For the special case with $\Delta\lambda_1 = 0$, we in fact have $\frac{\partial w_1^i(\alpha)}{\partial \alpha} = \frac{\lambda_1}{\rho + \lambda_1} \frac{\partial w_2^i(\alpha)}{\partial \alpha} < \frac{\partial w_2^i(\alpha)}{\partial \alpha}$. Combining this with Eq. (53) provides a sketch-proof of Proposition 3. The actual proof in Appendix B relies on the same idea but uses recursive techniques to establish the result formally.

The left panel of Figure 7 illustrates the result by plotting the change in the planners' value functions (in state $s = 1$) resulting from a small macroprudential policy change in state $s = 1$ (specifically, we start with the equilibrium with $\lambda_1^o = 0.03$ and set $\lambda_1^{o,pl} = 0.0305$). Note that the policy reduces the planner's first-best value function, since it distorts investors' allocations according to their own beliefs. However, the magnitude of this decline is small, illustrating the First Welfare Theorem (cf. Eq. (51)). Note also that the policy generates a relatively sizeable increase in the wealth-weighted average gap value function. Moreover, this increase is sufficiently large that the policy also increases the actual value function and generates a Pareto improvement, illustrating Proposition 3.

Macroprudential policy improves welfare by internalizing the aggregate demand externalities. In the demand-constrained state $s = 2$, optimists improve asset prices, which in turn increases

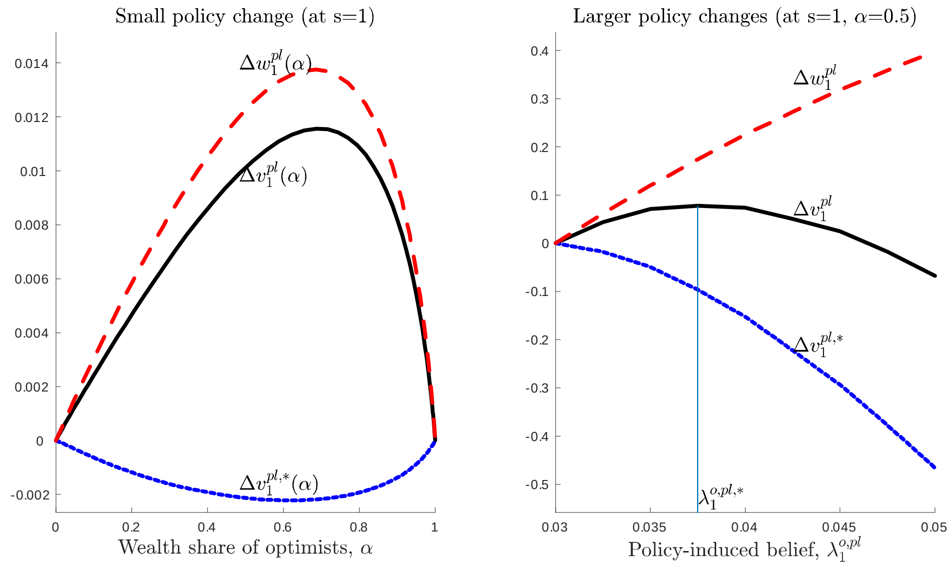

Figure 7: The left panel illustrates the effect of a small change in macroprudential policy in state 1 ($\lambda_1^{o,pl} = 0.0305$ where $\lambda_1^o = 0.03$) on the planner's value functions. The right panel illustrates the effect of larger changes as a function of policy.

aggregate demand and brings output closer to the first-best level. Individual optimists do not internalize that they would improve asset prices and output by bringing more wealth into state 2. Macroprudential policy works by increasing optimists' insurance purchases, which increases their wealth share in state 2. The result is reminiscent of the analysis in Korinek and Simsek (2016), in which macroprudential policy improves outcomes by inducing agents that have a high marginal propensity to consume (MPC) to bring more wealth into states in which there is a demand-driven recession. However, the mechanism here is different and operates via asset prices. In fact, in our setting, all investors have the same MPC equal to ρ. Optimists improve aggregate demand not because they spend more than pessimists, but because they increase asset prices and induce *all investors* to spend more, while also increasing aggregate investment and hence growth.

As this discussion suggests, the parametric restriction, $\Delta\lambda_1 = 0$, is useful to obtain an analytical result but it does not play a central role. We suspect that Proposition 3 also holds absent this assumption, even though we are unable to provide a proof. In our numerical simulations, we have not yet encountered a counterexample. The results displayed in Figure 7 actually correspond to a parameterization with $\Delta\lambda_1 < 0$ (specifically, we have $\lambda_1^o = 0.03 < \lambda_1^p = 0.075$).

Proposition 3 concerns a small policy change. The right panel of Figure 7 illustrates the effect of larger policies by plotting the changes in the planner's value as a function of the size of the policy (starting from, $\lambda_1^{o,pl} = \lambda_1^o$). For this exercise, we fix the optimists' wealth share at a particular level, $\alpha = 1/2$. Note that, as the policy becomes larger, the gap value continues to increase whereas the first-best value decreases. Moreover, the decline in the first-best value

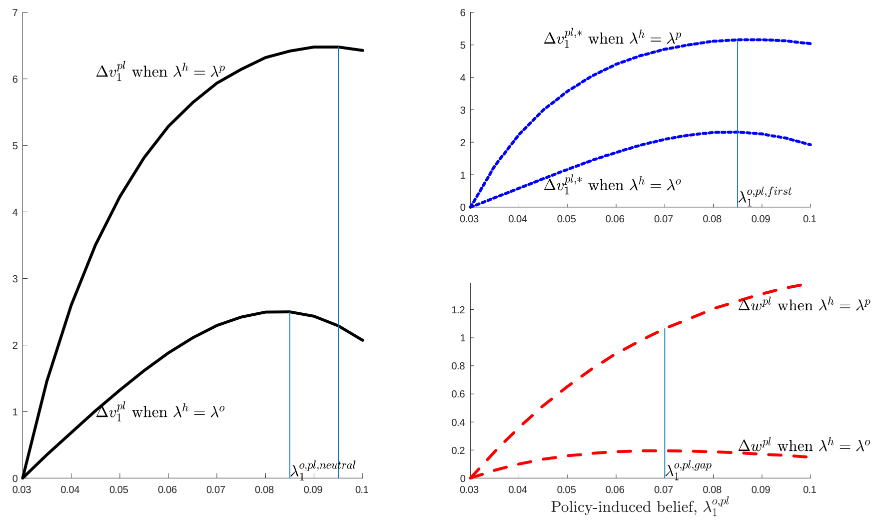

Figure 8: The left panel illustrates the effect of macroprudential policy on social welfare when all investors' value is calculated according to respectively optimists' or pessimists' belief. The panels on the right illustrate the effects on respectively the first-best and the gap value functions.

is negligible for small policy changes but it becomes sizeable for large policy changes. The (constrained) optimal macroprudential policy obtains at an intermediate level, $\lambda_1^{o,pl,*} > \lambda_1^o$.

The figure also illustrates that the constrained optimal policy intervention is not too large (specifically, we have $\lambda_1^{o,pl,*} = 0.0375$ where $\lambda_1^o = 0.03$). This is typically the case in our numerical simulations. The reason is that speculation generates high perceived utility for investors. Since macroprudential policy restricts speculation, the perceived costs quickly rise with the degree of the policy intervention, which implies that the optimal intervention is not too large.

Macroprudential policy according to a belief-neutral criterion. However, whether the utility from speculation should be counted towards social welfare is questionable. A recent literature argues that the Pareto criterion is not the appropriate notion of welfare for environments with belief disagreements. If investors' beliefs are different due to mistakes (say, in Bayesian updating), then it is arguably more appropriate to evaluate investors' utility according to the objective belief—which is common across the agents. Doing so would remove the speculative utility from welfare calculations, and it could lead to a constrained optimal policy that is much larger in magnitude. While reasonable, this approach faces a major challenge in implementation: whose belief should the policymaker use?

In recent work, Brunnermeier et al. (2014) offer a *belief-neutral welfare criterion* that circumvents this problem. The basic idea is to require the planner to evaluate social welfare according to a single belief, but also to make the welfare comparisons robust to the choice of the single belief. Specifically, their baseline criterion says that an allocation is belief-neutral superior to another allocation if it increases social welfare under every belief in the convex hull of investors' beliefs. Proposition 3 suggests their criterion can also be useful in this context since macroprudential policy increases the gap value according to each belief—that is, the gap-reducing welfare gains are belief neutral.

For a formal analysis, fix some $h \in [0,1]$ and let $v_s^i\left(\alpha; \lambda_1^{o,pl}, \lambda^h\right)$ denote the value function for an individual when the planner implements policy, $\lambda_1^{o,pl}$, and evaluates utility under the belief $\lambda^h = \lambda^p + h(\lambda^o - \lambda^p)$ (calculated by replacing λ_s^i in Eq. $(A.20)$ with λ_s^h). As before, define the planner's value function, $v_s^{pl}\left(\alpha; \lambda_1^{o,pl}, \lambda^h\right)$, as the wealth-weighted average of individual's value functions. Then, given the wealth share α (that corresponds to a particular Pareto weight), the policy, $\lambda_1^{o,pl}$, is a belief-neutral improvement over some other policy, $\tilde{\lambda}_1^{o,pl}$, as long as,

$$v_1^{pl}\left(\alpha; \lambda_1^{o,pl}, \lambda^h\right) > v_1^{pl}\left(\alpha; \tilde{\lambda}_1^{o,pl}, \lambda^h\right) \text{ for each } h \in [0,1]. \tag{54}$$

Figure 8 illustrates the belief-neutral optimal policy in the earlier example. The left panel plots the effect of the policy on the social welfare (given $\alpha = 1/2$) when the planner evaluates all investor's values under respectively pessimists' belief ($h = 0$) and optimists' belief ($h = 1$). The social welfare evaluated under intermediate beliefs lie in between these two curves. As the figure suggests, tightening the policy towards $\lambda_1^{o,pl,neutral} = 0.085$ constitutes a belief-neutral improvement. In particular, the belief-neutral criterion supports a much larger policy intervention than the Pareto criterion (cf. Figure 7).

The right panel provides further intuition by breaking the social welfare into its two components, $v_1^{pl} = v_1^{pl,*} + w_1^{pl}$. The top right panel shows that tightening macroprudential policy towards the belief, $\lambda_1^{o,pl,first} = 0.085$, generates a belief-neutral improvement in the "first best" social welfare, $v_1^{pl,*}$. Speculation induces investors to deviate from the optimal risk sharing benchmark in pursuit of perceived speculative gains. However, these speculative gains are transfers from other investors, and they do not count towards social welfare when investors' values are evaluated under a common belief (regardless of whose belief is used). Hence, if there were no interest rate rigidities, a belief-neutral planner would eliminate almost all speculation.[12]

The bottom right panel shows the effects of policy on the gap value, w_1^{pl}, which captures the reduction in social welfare due to interest rate rigidities. Tightening the macroprudential policy towards the belief, $\lambda_1^{o,pl,gap} = 0.07$ increases the gap value according to both optimists and pessimists (illustrating Proposition 3). Beyond this level, tightening the policy improves the gap value according to pessimists but not according to optimists—who perceive smaller benefits from macroprudential policy since they find the transition into state 2 unlikely.

It follows that, up to the level, $\lambda_1^{o,pl,gap} = 0.07$—which constitutes a sizeable policy intervention—there is no conflict in belief-neutral policy objectives. Tightening the policy helps to rein in speculation while also improving the gap value, according to any belief. This might be

[12] An unconstrained planner that uses a common belief for welfare calculations would set, $\lambda_1^{o,pl} = \lambda_1^p = 0.075$, so as to eliminate all speculation. Our constrained planner slightly overshoots this benchmark since she also corrects for the fact that she does not have access to macroprudential policy in state 2.

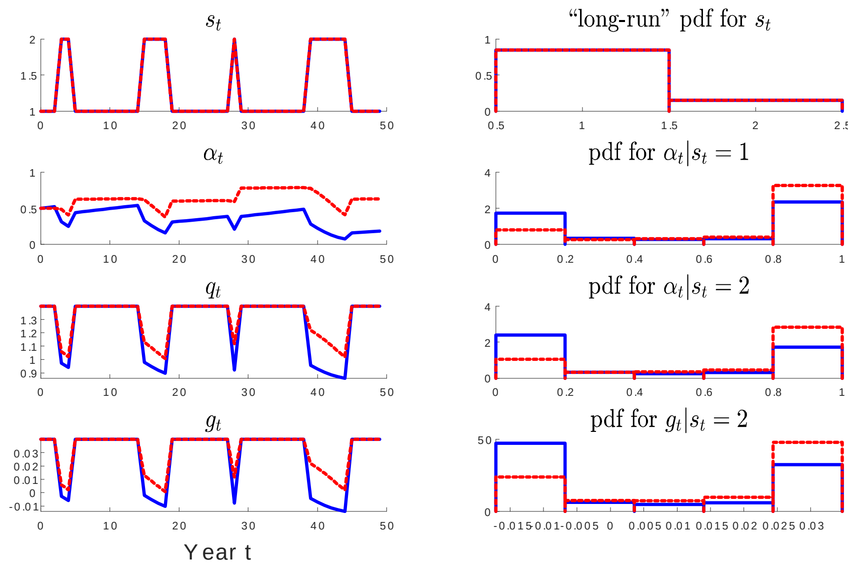

Figure 9: The left panels illustrate the evolution of the equilibrium without policy (solid line) and with macroprudential policy (dotted line) in the low volatility state over the medium run (50 years) for a particular realization of uncertainty. The right panels illustrate the realized distributions over a 1000-year horizon.

a natural choice for a planner who focuses exclusively on closing the output gaps relative to the first best while remaining agnostic about whether speculation improves or reduces social welfare. Beyond this level, tightening the policy continues to generate belief-neutral welfare gains by reducing speculation and improving risk sharing, but it also reduces the gap value according to optimists.

Dynamics of equilibrium with policy. We next consider how macroprudential policy affects the dynamics of equilibrium variables. The left panel of Figure 9 illustrates the evolution of equilibrium over a 50-year horizon when the planner implements the (belief–neutral) gap-value maximizing policy, $\lambda_1^{o,pl,gap} = 0.07$. For comparison, the figure also replicates the evolution of the equilibrium variables without policy from Figure 4. Note that the policy ensures optimists' wealth share drops relatively less when there is a transition into state 2. This in turn leads to greater asset prices and higher growth rate in state 2. The policy, however, is not without its drawbacks. As the period between years 5-15 illustrates, the policy slows down the growth of optimists' wealth share when the economy remains in state 1.

The right panel of the figure illustrates the probability distribution over a very long horizon (100000 years). As before, the economy is simulated with beliefs that are in the "middle" of optimists' and pessimists' beliefs in terms of the relative entropy distance. The figure suggests that optimists eventually tend to dominate the scene since their policy-induced beliefs are now closer to the middle belief than pessimists' beliefs. Whether or not this will actually happen depends on the objective belief.[13] The figure also suggests that the economy spends relatively more time in the range in which optimists' wealth share is in the intermediate range. This is intuitive, since the policy ensures that optimists' wealth grows more slowly in state 1 but that it is also preserved in state 2. As we argued earlier, the latter feature brings the output in state 2 closer to the first-best level, which generates a belief-neutral increase in expected gap values (regardless of which belief dominates in the long run).

5.2.2. Macroprudential policy during the bust

The analysis so far concerns macroprudential policy in the low-volatility state and maintains the assumption that $\lambda_2^{o,pl} = \lambda_2^o$. We next consider the polar opposite case in which the planner can reduce optimists' transition probability in the high-volatility state, $\lambda_2^{o,pl}$, while maintaining $\lambda_1^{o,pl} = \lambda_1^o$. We obtain a sharp result for the special case in which optimists' wealth share is sufficiently large.

Proposition 4. *Consider the equilibrium with heterogeneous beliefs, together with macroprudential policy that satisfies $\lambda_1^{o,pl} = \lambda_1^o$ (so policy affects only $\lambda_2^{o,pl}$). Then, there exists a threshold wealth-share level, $\overline{\alpha} < 1$, such that,*

$$\left.\frac{\partial w_s^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda^{o,pl}=\lambda^o} < 0 \text{ for each } i \in \{o,p\}, s \in \{1,2\} \text{ and } \alpha \in (\overline{\alpha}, 1).$$

Thus, over this range of wealth shares, macroprudential policy also reduces the planner's value, $\left.\frac{\partial v_s^{pl}(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda^{o,pl}=\lambda^o} = \left.\frac{\partial w_s^{pl}(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda^{o,pl}=\lambda^o} < 0.$

Thus, in contrast to Proposition 3, macroprudential policy in the bust phase can actually reduce the social welfare. To understand this result, consider the differential equation (52) for state $s = 2$ and an arbitrary belief type $i \in \{o,p\}$. Differentiating this expression with respect to policy, $\left(-\lambda_2^{o,pl}\right)$, (since we now consider a decrease in $\lambda_2^{o,pl}$) and evaluating at the no-policy equilibrium, $\lambda_2^{o,pl} = \lambda_2^o$, we obtain,

$$\left(\rho + \lambda_2^i\right)\frac{\partial w_2^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)} = \left(1 + \frac{\psi}{\rho}\right)\frac{\partial q_2(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)} + \alpha(1-\alpha)\left[\frac{\partial w_2^i(\alpha)}{\partial \alpha} - \lambda_2^i \frac{\partial \alpha'}{\partial \lambda_2^{o,pl}}\frac{\partial w_1^i(\alpha')}{\partial \alpha'}\right] + x^{\text{indirect}}, \tag{55}$$

where $\alpha' = a\frac{\lambda_2^{o,pl}}{\lambda_2^p + \alpha\Delta\lambda_2^{pl}}$. Here, x^{indirect} captures the induced effects of macroprudential policy on the value function that typically do not drive the comparative statics. The remaining terms capture the direct effects.

The bracketed terms in (55) are the analogues of the corresponding terms in Eq. (53). As before, these terms capture the potential benefits of macroprudential policy from reallocating

[13] In any case, the long-run survival analysis does not seem practically relevant in our context. With our simulation, pessimists do not disappear even after 100000 years.

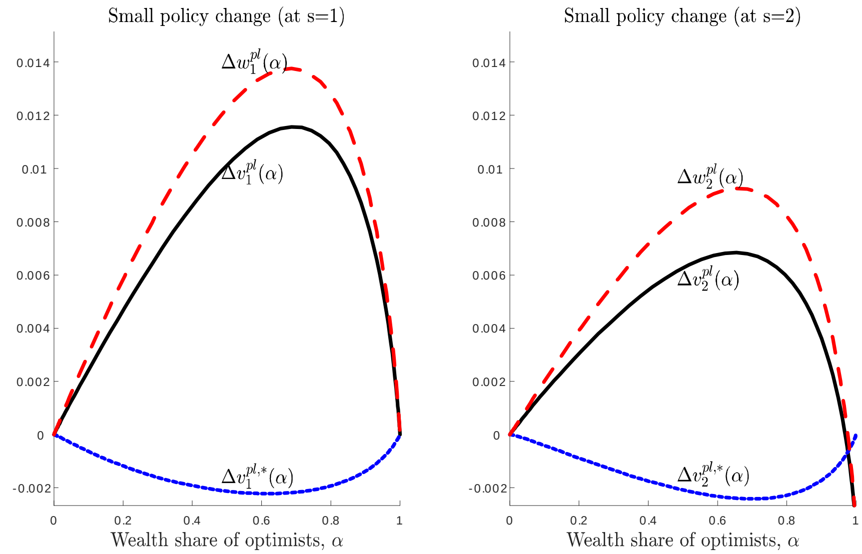

Figure 10: The left panel illustrates the effect of a small change in macroprudential policy in state 1 ($\lambda_1^{o,pl} = 0.0305$ where $\lambda_1^o = 0.03$) and the right pannel illustrates the effect of a comparable policy change in state 2.

optimists' wealth from state 1 to state 2. The net effect from these wealth changes is typically positive (in our simulations, this is always the case). Intuitively, optimists purchase too many call options that pay if there is a transition to the low-volatility state 1. They do not internalize that, if they were to keep their wealth in state 2, they would improve asset prices in subsequent times and bring aggregate output closer to the first-best level. Hence, the earlier benefits of macroprudential policy continue to apply in this setting.

Eq. (55) also illustrates that—unlike the earlier case—macroprudential policy in the bust state affects the current price level, with potential implications for social welfare. As we argued in the previous section, making optimists less optimistic in state 2 shifts the price function downward, $\frac{\partial q_2(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)} < 0$ (see Figure 5). Hence, the price impact of macroprudential policy is welfare reducing. Moreover, as optimists dominate the economy, $\alpha \to 1$, the price impact of the policy is still first order, whereas the beneficial effect from reshuffling optimists' wealth is second order (as captured by $\alpha(1-\alpha)$ in Eq. (55)). Thus, when optimists' wealth share is sufficiently large, the net effect of macroprudential policy is negative, illustrating Proposition 4.

This analysis also suggests that, even when the policy in the bust state exerts a net positive effect, it would typically increase the welfare by a smaller amount than a comparable policy in the boom state. Figure 10 illustrates this by plotting side-by-side the effects of a small policy change in either state. The left panel replicates the value functions from the earlier Figure 7, whereas the right panel illustrates the results from changing optimists' belief in the bust state by an amount that would generate a similar distortion in the first-best equilibrium as in our earlier analysis.[14] Note that a small macroprudential policy in state 2 has a smaller positive impact when optimists' wealth share is small, and it has a negative impact when optimists' wealth share is sufficiently large.

It is useful to emphasize that macroprudential policy does not have an adverse price impact in the boom state due to the interest rate response. As we argued in the previous section, reducing optimists' optimism in state 1 leaves the price level unchanged at $q_1(\alpha) = q^*$, but it shifts the interest rate function downward (see Figure 5). Intuitively, as macroprudential policy reduces the demand for risky assets, the interest rate policy lowers the rate to dampen its effect on asset prices and aggregate demand. In state 2, the interest rate is already at zero, so the interest rate policy cannot neutralize the adverse effects of macroprudential policy.

Taken together, our analysis in this section provides support for procyclical macroprudential policy. In states in which output is not demand constrained (in our model, state 1), macroprudential policy that restricts high valuation investors' (in our model optimists') risk taking is desirable. This policy improves welfare by ensuring that high valuation investors bring more wealth to the demand-constrained states, which in turn increases asset prices and output. Its adverse price effects are countered by a reduction in the interest rate. In contrast, in states in which output is demand constrained (in our model, state 2), macroprudential policy has counteracting effects on social welfare. While the policy has the same beneficial effects as before, it also lowers asset prices and aggregate demand, which cannot be countered by the interest rate. The latter effect reduces the overall usefulness of macroprudential policy, and it could even reduce social welfare.

6. Forward guidance

Macroprudential policy reduces the perverse interactions between speculation and interest-rate constraints, but it does not relax the latter. We next consider the possibility of forward-guidance type policies that lower interest rates in the future to stimulate aggregate output in earlier periods. Formally, we drop the policy rule in (18) and allow the planner to commit to setting an interest rate path subject to the lower bound constraint, (17), as well as the remaining equilibrium conditions.

To allow forward guidance to stimulate the economy, we let factor utilization, $\eta_{t,s}$, exceed one. However, excess utilization is costly and induces a faster depreciation of capital. In particular, we now replace the capital evolution equation (1) with

$$\frac{dk_{t,s}}{k_{t,s}} = \left(\varphi\left(\iota_{t,s}\right) - \delta\left(\eta_{t,s}\right)\right) dt + \sigma_s dZ_t, \text{ where } \delta\left(\eta_{t,s}\right) = \delta + \delta^{\eta} \max\left(0, \eta_{t,s} - 1\right).$$

Here, $\delta^{\eta}(\eta - 1)$ captures the additional depreciation rate caused by overutilization (which we

[14] Specifically, we calibrate the belief change in state 2 so that the maximum decline in the planner's first-best value function is the same in both cases, $\max_{\alpha} \left|\Delta v_2^{pl,*}(\alpha)\right| = \max_{\alpha} \left|\Delta v_1^{pl,*}(\alpha)\right|$.

take to be a linear function of η for simplicity). Appendix A.4.1 describes the remaining changes we make to the New Keynesian supply block of the model to accommodate excess factor utilization. With appropriate assumptions, equations (15) and (16) continue to apply in this setting: that is, output is determined by aggregate demand, and the rental rate of capital (which now also includes the overutilization costs) ensures all output accrues to the agents in the form of return to capital.

We also focus on the benchmark model without disagreement and with two states developed in Section 3, with three additional assumptions. First, for analytical tractability, we assume the planner sets policies within a restricted class. We start by analyzing the case in which the planner is restricted to set a fixed level of interest rate for each state, r_s^f, and to implement a fixed level of factor utilization, η_s. Second, we assume the parameters satisfy the following.

Assumption 5. $\delta^{\eta} \exp(q^*) > A(\rho\psi + 1)$ (where $\exp(q^*) = \frac{A+\psi}{\rho+\psi}$).

This condition says the marginal loss of value from excess depreciation is greater than the marginal gain of value from increasing factor utilization beyond $\eta_1 = 1$. It ensures that, in the first-best benchmark without interest rate rigidities, the planner would not implement excess factor utilization (see Appendix A.4.2). Third, we also restrict attention to the case in which there is no investment, $\psi = 0$, which leads to particularly simple expressions. The case with $\psi > 0$ generates qualitatively similar results.

With these assumptions, it easy to see that the goods market equilibrium condition (19) continues to hold in the high-volatility state $s = 2$, but it is modified in the low-volatility state $s = 1$. We first characterize the equilibrium given a particular forward guidance policy, $\left(r_1^f, \eta_1 \geq 1\right)$, and illustrate that our model generates a version of the forward guidance puzzle. We then characterize the (constrained) optimal policy.

Effectiveness of forward guidance. As before, there is an increasing relationship between asset prices and factor utilization (cf. Eq. (22)),

$$q_1 = q(\eta_1) \equiv \log\left(\frac{A\eta_1}{\rho}\right).$$

This illustrates that providing greater stimulus in state 1 also increases the asset price in that state. Applying the risk balance condition (26) for state $s = 1$, we obtain,

$$r_1^f = \rho - \delta + \lambda_1 (1 - \exp(q(\eta_1) - q_2)) - \delta^{\eta}(\eta_1 - 1) - \sigma_1^2 \geq 0, \tag{56}$$

as long as $\eta_1 \geq 1$. This illustrates that providing greater stimulus in state $s = 1$ requires setting a lower interest rate. Likewise, using the condition for state $s = 2$, we obtain,

$$r_2^f = \rho - \delta + \lambda_2 (1 - \exp(q_2 - q(\eta_1))) - \sigma_2^2 = 0, \tag{57}$$

as long as $q_2 \leq q^*$. This illustrates that providing greater stimulus in state 1 also increases the asset price in state 2 by increasing the expected capital gains.

Given some $\eta_1 \geq 1$, the equilibrium pair, $\left(r_1^f, q_2\right)$ (with $q_2 \leq q_2^*$), is characterized as the solution to Eqs. (56) and (57). Note also that, applying these equations for $\eta_1 = 1$ results in the benchmark equilibrium described in Section 3, which we denote by $\left(r_1^{f,b}, q_1^b, q_2^b\right)$. Using these benchmark variables, Eqs. (56) and (57) can be respectively replaced with

$$\begin{aligned} r_1^f &= r_1^{f,b} - \delta^\eta \left(\eta_1 - 1\right), \\ q_2 &= q\left(\eta_1\right) + q_2^b - q_1^b. \end{aligned}$$

The planner can then be thought of as targeting some $\eta_1 \geq 1$, which requires setting the interest rate according to the first equation, and which determines the asset prices according to the second equation.

A surprising aspect of the equilibrium is that the prices in both states move in tandem *regardless of transition probabilities.* For instance, when state 2 is persistent (low λ_2), one could have expected the forward guidance policy that affects the outcomes in state 1 to have a small impact on q_2. Eq. (57) illustrates that this is *not* the case: forward guidance policy that increases $q_1\left(\eta_1\right)$ by some amount also increases q_2 by the same amount.

This is a manifestation of the "forward guidance puzzle": the phenomenon that, in standard New Keynesian models, interest rate announcements far in the future have large effects on current output as well as inflation (see, for instance, Del Negro et al. (2012); McKay et al. (2016); Werning (2015)). The puzzle obtains largely because of strong general equilibrium effects. Specifically, interest rate cuts in the distant future increase output in future periods. These output increases are anticipated by economic agents, which then increases output also in the current period. Our model is slightly different than the models analyzed by the recent literature since the transition out of the liquidity trap is probabilistic as opposed to deterministic. Nonetheless, the model features strong general equilibrium effects that lead to a similar forward guidance puzzle.

To illustrate the role of future policy commitments (and the associated general equilibrium effects), consider the following alternative policy exercise in which the strength of the policy is decreasing in the amount of time the economy spends in state $s = 2$ before transitioning into state $s = 1$. For concreteness, let t denote the time spent in state 2, and suppose the planner implements an after-transition price level, $q_1(t)$, that satisfies, $q_1(t) - q_1^b = \exp(-\zeta t)\left(q_1(0) - q_1^b\right)$, for some $q_1(0) \geq q_1^b$ and $\zeta \geq 0$. Once implemented, the price in state 1 is kept constant at $q_1(t)$ until a further state transition. The parameter, ζ, captures the rate at which the strength of the policy declines as state 2 persists longer. The earlier analysis is a special case with $q_1(0) = q\left(\eta_1\right)$ and $\zeta = 0$. When $\zeta > 0$, the equilibrium condition (57) is replaced with,

$$\rho - \delta + \lambda_2 \left(1 - \exp\left(q_2(t) - q_1(t)\right)\right) + \frac{dq_2(t)}{dt} - \sigma_2^2 = 0. \tag{58}$$

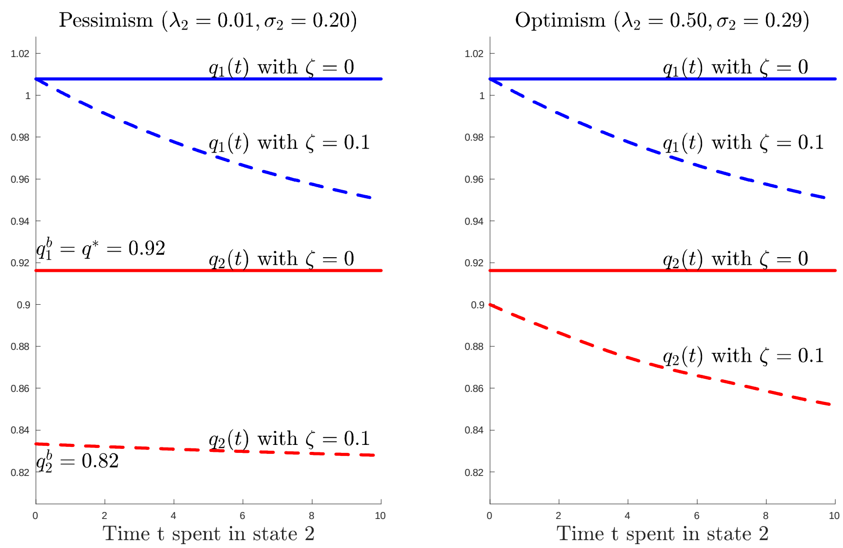

Figure 11: The effects of foward guidance policy on prices when investors are pessimistic (left panel) and optimistic (right panel), and when the policy is persistent (solid lines) and somewhat transient (dashed lines). When we change λ_2, we also change σ_2, so as to keep the benchmak price level, q_2^b, constant across policy exercises.

The equilibrium price function, $q_2(t)$, solves this differential equation with the limit condition, $\lim_{t\to\infty} q_2(t) = q_2^b$.

The left panel of Figure 11 illustrates the solution for different levels of ζ when investors are pessimistic (low λ_2). The solid lines illustrate the earlier results with persistent policy, $\zeta = 0$. Note that the policy increases $q_1(t)$ and $q_2(t)$ by the same amount. We chose $q_1(0)$ sufficiently high so that the policy completely eliminates the recession in the high volatility state, $q_2(0) = q^*$. The dashed lines illustrate the same example with slightly less persistent policy, $\zeta = 0.1$. In this case, $q_2(0)$ barely moves relative to its benchmark level, q_2^b. The intuition comes from inspecting Eq. (58). Conditional on there not being a transition, there will be a downward drift in prices as the policy will have weaker effects in the future, $\frac{dq_2(t)}{dt} < 0$. This downward drift is recognized by all investors and reduces the price also in earlier periods. This illustrates that, when investors are pessimistic, forward guidance increases the price largely because of its stimulative effects in future periods.

The right panel of Figure 11 repeats the same exercise when investors are optimistic (high λ_2). We also adjust the volatility parameter, σ_2, to ensure that the benchmark price, q_2^b, is the same as before (despite greater optimism). In this case, the policy generates a sizeable increase in $q_2(0)$ even when it is not very persistent. This illustrates that, when investors are optimistic, forward guidance policy relies much less on affecting the outcomes in the distant future. Rather, forward guidance in this case operates more directly by generating capital gains conditional on there being a transition in the near future.

These results relate to a growing literature that proposes to resolve the forward guidance puzzle by weakening the general equilibrium effects that obtain in the distant future. For instance, Angeletos and Lian (2016) illustrate that, when the policy is not common knowledge, the effectiveness of the forward policy declines with the time-horizon (since the general equilibrium effects rely on higher order beliefs). Farhi and Werning (2016a) illustrate that a similar outcome obtains when investors are boundedly rational (with level-k thinking) and the financial markets are incomplete. Gabaix (2017) obtains a related result by assuming that agents are inattentive to future changes to interest rates. While our model does not feature these ingredients, our analysis suggests that these types of frictions that weaken the effectiveness of policy in future periods have a stronger bite when investors are pessimistic about the transition probability. With optimism, there is scope for forward guidance policy even if the agents discount its effects in future periods.[15]

Optimal forward guidance. We next turn to the characterization of the (constrained) optimal forward guidance policy. For simplicity, consider the baseline case with $\zeta = 0$ so the planner chooses a time-invariant policy. Suppose the economy starts in state $s = 2$ and the planner chooses the policy, $\left(r_1^f, \eta_1\right)$, to maximize the welfare of the representative household in this state, v_2. Following the same steps as in Section 3, and using $\psi = 0$, the value function in state 2 is given by,

$$\begin{aligned} \rho v_2 &= \log \rho + \beta_2 \left(\frac{1}{\rho} q_2 - \frac{1}{\rho}\delta\right) + (1-\beta_2)\left(\frac{1}{\rho} q(\eta_1) - \frac{1}{\rho}\delta(\eta_1)\right) - \frac{1}{\rho}\overline{\sigma}_2^2, \qquad (59) \\ \text{where } \beta_2 &= \frac{\rho + \lambda_1}{\rho + \lambda_1 + \lambda_2}. \end{aligned}$$

As before, β_2 denotes the amount of discounting-adjusted time the economy spends in state 2. The main difference from Section 3 is that the depreciation rate in state 1 is replaced with $\delta(\eta_1) = \delta + \delta^\eta (\eta_1 - 1)$.

The planner effectively chooses $\eta_1 \geq 1$ to maximize Eq. (59) subject to the pricing relation, $q_2 = q(\eta_1) + q_2^b - q_1^b$, and the inequality constraints $q_2 \leq q^*$ and $r_1^f \geq 0$. Using the first order conditions, the planner utilizes some forward guidance ($\eta_1 > 1$) if and only if

$$\Delta q \equiv \log\left(\frac{1}{1-\beta_2}\frac{\rho}{\delta^\eta}\right) > 0. \qquad (60)$$

This expression illustrates that forward guidance is more likely to be utilized when it is cheaper (low δ^η), and when the current high-volatility state is likely to persist (high β_2). In the extreme case in which the high-volatility state is very transient, $\beta_2 = 0$, Assumption 5 implies that forward guidance is not used. In the other extreme in which the current state is very persistent,

[15] This discussion also connects with Caballero and Farhi (2017), who illustrate how forward guidance losses effectiveness in liquidity traps caused by an scarcity of safe assets. In their model, the marginal agent is concerned with further negative shocks and ascribes little value to promises to be delivered in better states of the world.

$\beta_2 \to 1$, the forward guidance is always used. It is also easy to check that, when condition (60) holds and there is an interior solution, the optimal forward guidance policy stimulates the prices by exactly Δq, that is, the equilibrium features $q_1 - q_1^b = q_2 - q_2^b = \Delta q$. If there is a corner solution, then the equilibrium with optimal policy features either $q_2^b = q^*$ or $r_1^f = 0$.

Note that the persistence parameter, β_2, is inversely related to the transition probability, λ_2. Hence, our analysis in this section suggests a tension in the determination of the optimal forward guidance policy. If investors are rationally optimistic, then forward guidance is not particularly desirable (since the recovery is imminent). However, if the policy becomes weaker with time horizon (perhaps due to the frictions emphasized by the recent literature), then optimism also increases the effectiveness of the policy. Hence, whether optimism increases or decreases the optimal provision of forward guidance is ambiguous and it is likely to depend on the details of the setting.

7. Endogenous Volatility and Incomplete Markets

So far, we emphasized how the interest rate rigidities, combined with risk premium shocks, can push the economy into a demand-driven recession, and how speculation exacerbates the recession. In this section, we illustrate how the interest rate rigidities and speculation can also generate *endogenous volatility* in asset prices. In our model, there is endogenous volatility even in the homogeneous beliefs benchmark, but the effects become particularly salient when investors speculate on their heterogeneous beliefs and their wealth shares fluctuate within each state due to incomplete markets. We develop this argument in two steps, first within the homogenous beliefs benchmark, and then within an extension of the framework to the incomplete markets case. The latter also helps to connect our framework to the related macrofinance literature summarized in Brunnermeier and Sannikov (2016b)

7.1. Endogenous Volatility from Interest Rate Rigidities

Given some $\Delta t > 0$, we define the proportional change in aggregate wealth over this time interval as,

$$\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}} \equiv \frac{\left(k_{t+\Delta t,s} Q_{t+\Delta t,s} - k_{t,s} Q_{t,s}\right)/\Delta t}{k_{t,s} Q_{t,s}}.$$

We will characterize the instantaneous volatility of this expression. First consider the homogenous belief benchmark that we analyzed in Section 3. In this model, while the instantaneous volatility conditional on there being no transition is exogenous (given by σ_s), *the unconditional volatility* that also incorporates the jump risk in asset prices is endogenous. In Appendix B, we show that the instantaneous variance is given by,

$$\lim_{\Delta t \to 0} Var_{t,s}\left(\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}}\right) = \sigma_s^2 + \sum_{s' \neq s} \lambda_{s,s'} \left(\frac{Q_{s'} - Q_s}{Q_s}\right)^2. \tag{61}$$

Here, $\{Q_s\}_s$ denote the equilibrium prices. The terms inside the summation capture the endogenous component of volatility. In the first best benchmark, we have $Q_s = Q^*$, and thus, the endogenous volatility is zero. In the equilibrium with interest rate rigidities, some states can feature, $Q_s < Q^*$, which leads to endogenous volatility. Our next result (corollary to Proposition 1) establishes this in the context of the two-state model that we analyzed in Section 3.

Corollary 2. *Consider the model with two states, $s \in \{1, 2\}$, with common beliefs and Assumptions 1-3 characterized in Proposition 1. For any $s \in \{1, 2\}$, the unconditional instantaneous variance of the proportional change in aggregate wealth is given by, $\sigma_s^2 + \lambda_s \left(\frac{Q_s - Q_{s'}}{Q_s}\right)^2$, and it is strictly greater than the instantaneous variance that would obtain in the first-best equilibrium without interest-rate frictions, σ_s^2.*

This result illustrates the main intuition for why interest rate rigidities generate endogenous volatility in asset prices. When there is a shock to the risk premium (which we capture with volatility shocks), the interest rate policy changes the rate to mitigate the impact of the shock on asset prices. Interest rate rigidities reduce the ability of the policy to lean against risk premium shocks, which leads to endogenous volatility.

While Corollary 2 focuses on risk premium shocks, "beliefs shocks" that induce investors to revise their expectations would also create endogenous volatility in asset prices through the same mechanism. Recall from our analysis in Section 4 that speculation among agents with heterogeneous beliefs leads to large swings in optimists wealth share, which in turn changes the wealth-weighted average belief that determines asset prices. Hence, speculation further exacerbates endogenous volatility via its impact on the average belief. This effect is already present in our earlier model with heterogeneous beliefs, in which case the endogenous volatility due to speculation translates into lower asset prices and output compared to the homogeneous beliefs benchmark (see Section 4). However, the effect of speculation on endogenous volatility becomes even more apparent when markets are incomplete, in which case speculation increases not only unconditional but also conditional (within-state) volatility. We turn to this case next.

7.2. Endogenous Volatility from Speculation and Incomplete Markets

Consider the model with heterogeneous beliefs analyzed in Section 4 with the only difference that investors *cannot* trade contingent securities. This implies that investors speculate on their different views by adjusting their holdings of capital. Consequently, investors' relative wealth shares become stochastic even in absence of state transitions, which makes endogenous volatility particularly salient.

Formally, investors solve problem (9) with the additional restriction that $\omega_s^{i,s'} = 0$ for each i and $s' \neq s$. The remaining equilibrium conditions are the same as before [cf. Definition 1]. In Appendix A.5, we show that a version of the risk balance condition (34) continues to apply in this case,

$$\sigma_s + \sigma_{t,s}^Q = \frac{1}{\sigma_s + \sigma_{t,s}^Q} \left(r_{t,s}^k - r_{t,s}^f + \sum_{s' \neq s} \overline{\lambda}_{s,s'} \left(\left\{ \omega_{t,s}^{k,i} \right\}_i \right) \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \right), \quad (62)$$

$$\text{where } \overline{\lambda}_{s,s'} \left(\left\{ \omega_{t,2}^{k,i} \right\}_i \right) = \sum_i \alpha_{t,s}^i \chi \left(\omega_{t,s}^{k,i} \right) \lambda_{s,s'}^i \text{ and } \chi \left(\omega_{t,s}^{k,i} \right) = \frac{Q_{t,s'}}{Q_{t,s} + \omega_{t,s}^{k,i} \left(Q_{t,s'} - Q_{t,s} \right)}.$$

As before, capital is priced according to an appropriately weighted average belief. The weights incorporate the fact that investors might have different marginal utilities after a jump since they place different portfolio weights on capital, that is, $\omega_{t,s}^{k,i}$ is not necessarily equal to one. In particular, investors' portfolio weights are characterized by jointly solving the following system of equations (over $i \in I$),

$$\left(\omega_{t,s}^{k,i} - 1 \right) \left(\sigma_s + \sigma_{t,s}^Q \right) = \frac{1}{\sigma_s + \sigma_{t,s}^Q} \sum_{s' \neq s} \left(\lambda_{s,s'}^i \chi \left(\omega_{t,s}^{k,i} \right) - \overline{\lambda}_2 \left(\left\{ \omega_{t,s}^{k,i} \right\}_i \right) \right) \left(\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \right). \quad (63)$$

Note that the investor holds more capital ($\omega_{t,s}^{k,i} > 1$), if she is optimistic about the expected capital gains relative to the weighted average belief. Unlike in Section 4, investors speculate on their different beliefs about transition probabilities by adjusting their portfolio weights on capital. Note also that, since capital is the only available risky investment, $\omega_{t,s}^{k,i}$ provides a measure of the investor's *leverage ratio:* the value of her risky assets divided by the value of her wealth.

To characterize the equilibrium further, consider the two-state special case. As before, we conjecture an equilibrium in which the prices and interest rates satisfy, $q_{t,1} = q^*, r_{t,1}^f > 0$ and $q_{t,2} < q^*, r_{t,2}^f = 0$. Moreover, the price in state 2 can be written as a function of optimists' wealth share, $q_{t,2} = q_2 \left(\alpha_{t,2} \right)$ for some function $q_2 : [0,1] \to [q_2^p, q_2^o]$. Likewise, the interest-rate in state 1 can be written as a function of optimists' wealth share, $r_1 \left(\alpha_{t,1} \right)$. We next characterize the equilibrium in state $s = 2$. The equilibrium in state $s = 1$ is characterized by similar steps.

Appendix A.5 shows that applying Eq. (63) in this case implies $\omega_{t,2}^{k,o} > 1 > \omega_{t,2}^{k,p}$: that is, optimists' leverage ratio exceeds one. Intuitively, optimists make a leveraged investment in capital since they assign a greater probability to transition into state $s = 1$ (that features higher asset prices). In view of their leveraged exposure to capital, optimists' wealth share fluctuates even without any state transition. In Appendix A.5, we show that within a state optimists' wealth share evolves according to,

$$\frac{d\alpha_{t,2}}{\alpha_{t,2}} = \left(\omega_{t,2}^{k,o} - 1 \right) \left(r_{t,2}^k - \left(\sigma_2 + \sigma_{t,2}^Q \right)^2 \right) dt + \left(\omega_{t,2}^{k,o} - 1 \right) \left(\sigma_2 + \sigma_{t,2}^Q \right) dZ_t, \quad (64)$$

where $r_{t,2}^k$ is given by Eq. (24) as before. Note that (since $\omega_{t,2}^{k,o} > 1$) a negative shock to the Brownian motion, $dZ_t < 0$, decreases optimists' wealth share. Since the asset price is a function

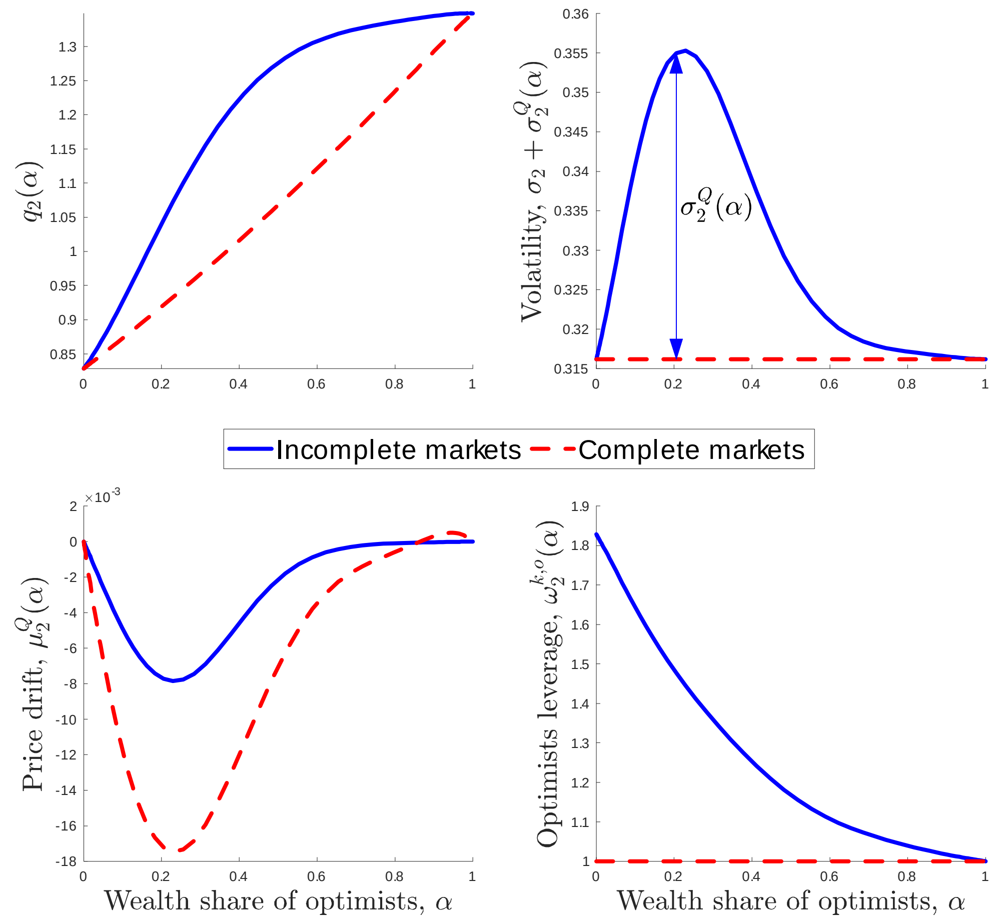

Figure 12: The solid lines (resp. the dashed) lines illustrate the equilibrium in state 2 with incomplete markets (resp. complete markets).

of optimists' wealth share, $q_{t,2} = q_2(\alpha_{t,2})$, these changes also translate into fluctuations in the asset price. In particular, the endogenous volatility term, $\sigma_{t,2}^Q$, is no longer zero and it is solved as part of the equilibrium.

Formally, our analysis in Appendix A.5 shows that Eq. (64) describes the drift and the volatility of asset prices, $\mu_{t,2}^Q$ and $\sigma_{t,2}^Q$, in terms of the price function as well as its derivatives, $q_2(\alpha), q_2'(\alpha), q_2''(\alpha)$. Combining these expressions with the risk balance condition (62) for state $s = 2$ provides a second order ordinary differential equation for the price function, $q_2(\alpha)$. The equilibrium is the solution to this differential equation together with the boundary conditions, $q_2(0) = q_2^p$ and $q_2(1) = q_2^o$, as well as $q_2'(0)$ (that we characterize in the appendix).

Figure 3 illustrates the equilibrium for the particular parameterization we analyzed in Section 4. The solid lines plot the equilibrium variables with incomplete markets, and the dashed lines plot the corresponding variables with complete markets. The top left panel shows that, as before, the price function is increasing in optimists' wealth share. Less obviously, the figure also illustrates that the price is greater than in the complete markets case. The bottom left panel shows that this outcome obtains because the complete markets case features a more negative price drift. Intuitively, completing the market (by adding the contingent securities) increases the scope for speculation, which implies that optimists' wealth share declines by more when the high-volatility state persists longer. This effect is related to the results in Simsek (2013b), which illustrate that financial innovation (that expands the set of traded financial assets) can increase investors' portfolio risks in view of greater speculation. In our model, greater portfolio risks translate into greater (unconditional) endogenous volatility, which in turn reduces asset prices and output during a demand recession.

The top right panel shows that, although the price level is higher, within-state volatility of capital is also higher in this case in view of the endogenous price volatility, $\sigma^Q_{t,2}$. The bottom right panel shows that this outcome obtains because optimists' leverage ratio exceeds one. The figure also illustrates that optimists' leverage ratio is greater when their wealth share is lower, as this leads to lower asset prices and greater capital gains from a transition into state $s = 1$, which optimists find more likely. Note also that $\sigma^Q_{t,2}$ obtains its highest level for an intermediate level of optimists' wealth share, $\alpha_{t,2}$. This result can be understood from Eq. (64), which illustrates that the magnitude of the fluctuations in $\alpha_{t,2}$ (that determines the price level, $q_2(\alpha_{t,2})$) is determined by the product of optimists' leverage ratio and the current level of their wealth share, $\left(\omega^{k,o}_{t,2} - 1\right)\alpha_{t,2}$. Intuitively, speculation is stronger, and leads to greater endogenous volatility, when both types of investors have a sizeable wealth share.

8. Final Remarks

We provide a macroeconomic framework where risk- and output–gaps are joint phenomena that feed into each other. The key tension in this framework is that asset prices have the dual role of equilibrating risk markets and supporting aggregate demand. When the dual role is inconsistent, the risk market equilibrium prevails. Interest rate policy works by taking over the role of equilibrating risk markets, which then leaves asset prices free to balance the goods markets. However, once interest rates reach a lower bound, the dual role problem reemerges and asset prices are driven primarily by risk markets equilibrium considerations. This reduces aggregate demand and triggers a recession, which then feeds back negatively into asset prices. The role of macroprudential regulation is to reduce the gap between the asset prices that equilibrate the risk and goods markets when interest rate policy is no longer available.

Interest rate cuts work in our model by improving the market's Sharpe ratio. From this perspective, any policy that reduces market volatility should have similar effects, which renders support to the many such policies implemented during the aftermath of the subprime and European crises. And it is also reasonable to expect that, over time and should the high riskawareness environment persist, the private sector may migrate to safer technologies and sectors, at the cost of lower growth. We are exploring these implications in concurrent work.

In the model we take the interest rate friction to be a stark zero lower bound constraint, which can be motivated with standard cash-substitutability arguments. In practice, this constraint is neither as tight nor as narrowly motivated: Central banks do have some space to bring rates into negative territory, especially when macroeconomic uncertainty is rampant, but there are also many other frictions besides cash substitutability that can motivate downward rigidity in rates once these are already low (see, e.g., Brunnermeier and Koby (2016) for a discussion of the "reversal rate", understood as a level of rates below which the financial system becomes impaired). The broader points of the dual role of asset prices and their interactions with aggregate demand constraints during recessions would survive many generalizations of the interest rate friction. Similarly, and as we hinted in the Forward Guidance section of the paper, one could also imagine situations that motivate ceilings on interest rates, in which case asset prices would overshoot and the productive capacity would become stretched.

We also chose to narrate the interactions between speculation and aggregate demand in terms of the economy's degree of optimism. Naturally, there is an entirely symmetric discussion in terms of the economy's degree of pessimism, in which case our paper connects with the literature that ascribes the depth of risk-based crises to Knightian uncertainty (see, e.g., Caballero and Krishnamurthy (2008); Caballero and Simsek (2013)). However, absent any direct mechanism to alleviate Knightian behavior during severe recessions, the key macroprudential point that optimists may need to be regulated during the boom survives this alternative motivation.

We deliberately kept our analysis of heterogeneity in asset valuation at an abstract level. With this ingredient our goal was to highlight the negative effect of (ex-ante) speculation on (ex-post) aggregate demand and growth. During booms, speculation boosts valuations and aggregate demand, but these effects can be offset by higher rates. In contrast, during severe recessions, speculation depresses valuations and aggregate demand, but these effects cannot be offset by interest rates cuts as these are constrained. In practice, this insight implies that (from a macroeconomic point of view) it doesn't matter whether banks are the optimists or pessimists: If the former, then their leverage during the boom (selling puts) is costly during recessions as they become impaired; if the latter, then their (fully collateralized) easy lending during booms (buying puts) implies that their optimistic borrowers will suffer losses during recessions.

As we noted earlier, our modeling approach belongs to the literature spurred by Brunnermeier and Sannikov (2014), although unlike that literature most of our analysis does not feature financial frictions. However, if we were to introduce these realistic frictions in our setting, many of the themes in that literature would reemerge and become exacerbated by aggregate demand feedbacks. For instance, as we show in Section 7.2 in an incomplete markets setting, optimists take leveraged positions on capital, and by doing so they induce endogenous volatility in asset prices and the possibility of tail events following a sequence of negative diffusion shocks that make the economy deeply pessimistic.

The model omits many realistic self-healing shocks that were arguably relevant for the Great Recession (as well as other deep recessions). For example, a financial crisis driven by a reduction in banks' net worth is typically mitigated over time as banks earn high returns and accumulate net worth (see Gertler et al. (2010); Brunnermeier and Sannikov (2014)). Likewise, household or firm deleveraging eventually loses its potency as debt is paid back (see Eggertsson and Krugman (2012); Guerrieri and Lorenzoni (2017)). Investment hangovers gradually dissipate as the excess capital is depleted (see Rognlie et al. (2017)). While these self-healing shocks are useful to understand the depth of the Great Recession, they raise the natural question of why the interest rates seem unusually low and the recovery (especially in investment) appears incomplete almost ten years after the start of the recession. Our paper illustrates how (objective and subjective) risk factors can drag the economy's recovery.

Conversely, the model also omits many sources of inertia that stem from financial markets. Throughout we have assumed that risk-markets clear instantly while goods markets are sluggish. In practice, risk-market have their own sources of inertia as financial institutions avoid or delay mark-to-market losses, liquidity evaporates, policy noise rises, and so on.

Finally, one feature of the aftermath of the subprime crisis is the present high valuation of risk-assets, that could appear to contradict the higher required equity risk premium observed in the data (see Figure 1). The model offers a natural interpretation for such a combination: While we focused exclusively on changes in the required risk-premium, there is also evidence that during this period both ρ and ψ have declined due to a variety of factors such as a worsening of the income distribution and an increase in monopoly rents (see, e.g., Gutiérrez and Philippon (2016)). Equation (22) shows that such declines require a higher valuation for any given level of output gap η, which is achieved via a drop in "rstar." In the interest rate constrained region the latter translates into a deeper recession, while once the economy has recovered the decline in"rstar" manifests in lower riskless interest rates and higher valuations. It may well be that this new high valuation full-employment equilibrium will bring about higher instability in the future by exacerbating speculation, which is a theme we intend to explore in future work.

References

Adrian, T., Boyarchenko, N., 2012. Intermediary leverage cycles and financial stability.

Angeletos, G.-M., Lian, C., 2016. Forward guidance without common knowledge. Tech. rep., National Bureau of Economic Research.

Bacchetta, P., Benhima, K., Kalantzis, Y., 2016. Money and capital in a persistent liquidity trap.

Basak, S., 2000. A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk. Journal of Economic Dynamics and Control 24 (1), 63–95.

Basak, S., 2005. Asset pricing with heterogeneous beliefs. Journal of Banking & Finance 29 (11), 2849–2881.

Basak, S., Cuoco, D., 1998. An equilibrium model with restricted stock market participation. Review of Financial Studies 11 (2), 309–341.

Bernanke, B. S., Gertler, M., Gilchrist, S., 1999. The financial accelerator in a quantitative business cycle framework. Handbook of macroeconomics 1, 1341–1393.

Bianchi, J., Mendoza, E. G., 2013. Optimal time-consistent macroprudential policy. Tech. rep., National Bureau of Economic Research.

Bloom, N., 2009. The impact of uncertainty shocks. econometrica 77 (3), 623–685.

Brunnermeier, M. K., Koby, Y., 2016. The reversal interest rate: An effective lower bound on monetary policy. Preparation). Print.

Brunnermeier, M. K., Sannikov, Y., 2014. A macroeconomic model with a financial sector. The American Economic Review 104 (2), 379–421.

Brunnermeier, M. K., Sannikov, Y., 2016a. The i theory of money. Tech. rep., National Bureau of Economic Research.

Brunnermeier, M. K., Sannikov, Y., 2016b. Macro, money, and finance: A continuous-time approach. Handbook of Macroeconomics 2, 1497–1545.

Brunnermeier, M. K., Simsek, A., Xiong, W., 2014. A welfare criterion for models with distorted beliefs. The Quarterly Journal of Economics 129 (4), 1753–1797.

Caballero, R. J., Farhi, E., 2017. The safety trap. Review of Economic Studies forthcoming.

Caballero, R. J., Farhi, E., Gourinchas, P.-O., 2017a. Rents, technical change, and risk premia accounting for secular trends in interest rates, returns on capital, earning yields, and factor shares. American Economic Review 107 (5), 614–620.

Caballero, R. J., Farhi, E., Gourinchas, P.-O., 2017b. The safe asset shortage conundrum. The Journal of Economic Perspectives 31 (3).

Caballero, R. J., Krishnamurthy, A., 2003. Excessive dollar debt: Financial development and underinsurance. The Journal of Finance 58 (2), 867–894.

Caballero, R. J., Krishnamurthy, A., 2008. Collective risk management in a flight to quality episode. The Journal of Finance 63 (5), 2195–2230.

Caballero, R. J., Simsek, A., 2013. Fire sales in a model of complexity. The Journal of Finance 68 (6), 2549–2587.

Chen, J., Hong, H., Stein, J. C., 2002. Breadth of ownership and stock returns. Journal of financial Economics 66 (2), 171–205.

Christiano, L. J., Eichenbaum, M. S., Trabandt, M., 2015. Understanding the great recession. American Economic Journal: Macroeconomics 7 (1), 110–167.

Christiano, L. J., Motto, R., Rostagno, M., 2014. Risk shocks. The American Economic Review 104 (1), 27–65.

Curdia, V., Woodford, M., 2010. Credit spreads and monetary policy. Journal of Money, Credit and Banking 42 (s1), 3–35.

Davila, E., Korinek, A., 2016. Pecuniary externalities in economies with financial frictions. The Review of Economic Studies.

Del Negro, M., Giannone, D., Giannoni, M. P., Tambalotti, A., Wu, J. C., 2017. Safety, liquidity, and the natural rate of interest.

Del Negro, M., Giannoni, M. P., Patterson, C., 2012. The forward guidance puzzle.

Detemple, J., Murthy, S., 1994. Intertemporal asset pricing with heterogeneous beliefs. Journal of Economic Theory 62 (2), 294–320.

Di Tella, S., 2012. Uncertainty shocks and balance sheet recessions. Job Market Paper.

Eggertsson, G. B., Krugman, P., 2012. Debt, deleveraging, and the liquidity trap: A fisher-minsky-koo approach. The Quarterly Journal of Economics 127 (3), 1469–1513.

Eggertsson, G. B., Mehrotra, N. R., Robbins, J. A., January 2017. A model of secular stagnation: Theory and quantitative evaluation. Working Paper 23093, National Bureau of Economic Research.
URL `http://www.nber.org/papers/w23093`

Eggertsson, G. B., Woodford, M., September 2006. Optimal Monetary and Fiscal Policy in a Liquidity Trap. The MIT Press, pp. 75–144.
URL `http://www.nber.org/chapters/c0076`

Farhi, E., Werning, I., 2016a. The power of forward guidance with bounded rationality and incomplete markets.

Farhi, E., Werning, I., 2016b. A theory of macroprudential policies in the presence of nominal rigidities. Econometrica 84 (5), 1645–1704.

Farhi, E., Werning, I., 2017. Fiscal unions. American Economic Review forthcoming.

Fostel, A., Geanakoplos, J., 2008. Leverage cycles and the anxious economy. The American Economic Review 98 (4), 1211–1244.

Gabaix, X., 2017. A behavioral new keynesian model. Tech. rep., National Bureau of Economic Research.

Geanakoplos, J., 2010. The leverage cycle. NBER macroeconomics annual 24 (1), 1–66.

Gertler, M., Karadi, P., 2011. A model of unconventional monetary policy. Journal of monetary Economics 58 (1), 17–34.

Gertler, M., Kiyotaki, N., et al., 2010. Financial intermediation and credit policy in business cycle analysis. Handbook of monetary economics 3 (3), 547–599.

Gilchrist, S., Zakrajšek, E., 2012. Credit spreads and business cycle fluctuations. The American Economic Review 102 (4), 1692–1720.

Guerrieri, V., Lorenzoni, G., 2017. Credit crises, precautionary savings, and the liquidity trap. Quarterly Journal of Economics forthcoming.

Gutiérrez, G., Philippon, T., 2016. Investment-less growth: An empirical investigation. Tech. rep., National Bureau of Economic Research.

Hall, R. E., 2011. The long slump. The American Economic Review 101 (2), 431–469.

Harris, M., Raviv, A., 1993. Differences of opinion make a horse race. Review of Financial studies 6 (3), 473–506.

Harrison, J. M., Kreps, D. M., 1978. Speculative investor behavior in a stock market with heterogeneous expectations. The Quarterly Journal of Economics 92 (2), 323–336.

He, Z., Krishnamurthy, A., 2012. A model of capital and crises. The Review of Economic Studies 79 (2), 735–777.

He, Z., Krishnamurthy, A., 2013. Intermediary asset pricing. The American Economic Review 103 (2), 732–770.

Iachan, F. S., Nenov, P. T., Simsek, A., 2015. The choice channel of financial innovation. Tech. rep., National Bureau of Economic Research.

Jeanne, O., Korinek, A., 2010. Managing credit booms and busts: A pigouvian taxation approach. Tech. rep., National Bureau of Economic Research.

Korinek, A., Nowak, M., 2016. Risk-taking dynamics and financial stability.

Korinek, A., Simsek, A., 2016. Liquidity trap and excessive leverage. American Economic Review 106 (3), 699–738.

Krugman, P., 1998. It's baaack: Japan's slump and the return of the liquidity trap. Brookings Papers on Economic Activity 1998 (2), 137–205.

Kubler, F., Schmedders, K., 2012. Financial innovation and asset price volatility. The American Economic Review 102 (3), 147–151.

Lintner, J., 1969. The aggregation of investor's diverse judgments and preferences in purely competitive security markets. Journal of financial and quantitative analysis 4 (04), 347–400.

Lorenzoni, G., 2008. Inefficient credit booms. The Review of Economic Studies 75 (3), 809–833.

McKay, A., Nakamura, E., Steinsson, J., 2016. The power of forward guidance revisited. The American Economic Review 106 (10), 3133–3158.

Midrigan, V., Philippon, T., Jones, C., et al., 2016. Beyond the liquidity trap: the secular stagnation of investment. In: 2016 Meeting Papers. No. 1429. Society for Economic Dynamics.

Miller, E. M., 1977. Risk, uncertainty, and divergence of opinion. The Journal of finance 32 (4), 1151–1168.

Moreira, A., Savov, A., 2017. The macroeconomics of shadow banking. The Journal of Finance forthcoming.
URL http://dx.doi.org/10.1111/jofi.12540

Rognlie, M., Shleifer, A., Simsek, A., 2017. Investment hangover and the great recession. American Economic Journal: Macroeconomics forthcoming.

Scheinkman, J. A., Xiong, W., 2003. Overconfidence and speculative bubbles. Journal of political Economy 111 (6), 1183–1220.

Silva, D. H., 2016. The risk channel of unconventional monetary policy.

Simsek, A., 2013a. Belief disagreements and collateral constraints. Econometrica 81 (1), 1–53.

Simsek, A., 2013b. Speculation and risk sharing with new financial assets. The Quarterly journal of economics 128 (3), 1365–1396.

Tobin, J., 1975. Keynesian models of recession and depression. The American Economic Review 65 (2), 195–202.

Varian, H. R., 1989. Differences of opinion in financial markets. In: financial Risk: Theory, Evidence and implications. Springer, pp. 3–37.

Werning, I., 2012. Managing a liquidity trap: Monetary and fiscal policy. Tech. rep., National Bureau of Economic Research.

Werning, I., 2015. Incomplete markets and aggregate demand. Tech. rep., National Bureau of Economic Research.

Xiong, W., Yan, H., 2010. Heterogeneous expectations and bond markets. Review of Financial Studies 23 (4), 1433–1466.

Zapatero, F., 1998. Effects of financial innovations on market volatility when beliefs are heterogeneous. Journal of Economic Dynamics and Control 22 (4), 597–626.

A. Appendix: Omitted Derivations

This appendix presents the derivations omitted from the main text. The next appendix presents the proofs.

A.1. Omitted derivations in Section 2

A.1.1. Recursive formulation of the portfolio problem

Note that the HJB equation corresponding to the portfolio problem (9) is given by,

$$\begin{aligned}\rho V_{t,s}^{i}\left(a_{t,s}^{i}\right) = & \max_{\tilde{\omega}^{k},\left\{\tilde{\omega}^{s'}\right\}_{s'\neq s},\tilde{c}} \log \tilde{c} + \frac{\partial V_{t,s}^{i}}{\partial a}\left(a_{t,s}^{i}\left(r_{t,s}^{f}+\tilde{\omega}^{k}\left(r_{t,s}^{k}-r_{t,s}^{f}\right)-\sum_{s'\neq s}\tilde{\omega}^{s'}\right)-\tilde{c}\right) \\ & +\frac{1}{2}\frac{\partial^{2} V_{t,s}^{i}}{\partial a^{2}}\left(\tilde{\omega}^{k} a_{t,s}^{i}\sigma_{t,s}^{k}\right)^{2}+\frac{\partial V_{t,s}\left(a_{t,s}\right)}{\partial t} \\ & +\sum_{s'\neq s}\lambda_{s,s'}^{i}\left(V_{t,s'}^{i}\left(a_{t,s}^{i}\left(1+\tilde{\omega}^{k}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}+\frac{\tilde{\omega}^{s'}}{p_{t,s}^{s'}}\right)\right)-V_{t,s}^{i}\left(a_{t}\right)\right)\end{aligned} \tag{A.1}$$

In view of the log utility, the solution has the form in (11). This implies, in particular, that

$$\frac{\partial V_{t,s}^{i}}{\partial a}=\frac{1}{\rho a_{t,s}} \text{ and } \frac{\partial^{2} V_{t,s}^{i}}{\partial a^{2}}=\frac{-1}{\rho\left(a_{t,s}\right)^{2}}.$$

The first order condition for $\tilde{c}$ then implies Eq. (12). The first order condition for $\tilde{\omega}^{k}$ implies,

$$\frac{\partial V_{t,s}^{i}}{\partial a} a_{t,s}\left(r_{t,s}^{k}-r_{t,s}^{f}\right)+\sum_{s'\neq s}\lambda_{s,s'}\frac{\partial V_{t,s'}^{i}\left(a_{t,s'}\right)}{\partial a} a_{t,s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}=-\frac{\partial^{2} V_{t,s}^{i}}{\partial a^{2}}\omega_{t,s}^{k}\left(a_{t,s}^{i}\sigma_{t,s}^{k}\right)^{2}.$$

After substituting for $\frac{\partial V_{t,s}^{i}}{\partial a}, \frac{\partial V_{t,s'}^{i}}{\partial a} \frac{\partial^{2} V_{t,s}^{i}}{\partial a^{2}}$ and rearranging terms, this also implies Eq. (13). Finally, the first order condition for $\tilde{\omega}^{s'}$ implies,

$$\frac{p_{t,s}^{s'}}{\lambda_{s,s'}^{i}}=\frac{\frac{\partial V_{t,s'}^{i}\left(a_{t,s'}\right)}{\partial a}}{\frac{\partial V_{t,s}^{i}\left(a_{t,s}\right)}{\partial a}}=\frac{a_{t,s}^{i}}{a_{t,s'}^{i}},$$

which is Eq. (14). This completes the characterization of the optimality conditions.

A.1.2. Description of the New Keynesian production firms

The supply side of our model features nominal rigidities similar to the standard New Keynesian setting. There is a continuum of measure one of production firms denoted by ν. These firms rent capital from the investment firms, $k_{t,s}(\nu)$, and produce differentiated goods, $y_{t,s}(\nu)$, subject to the technology,

$$y_{t,s}(\nu)=A\eta_{t,s}(\nu)\, k_{t,s}(\nu). \tag{A.2}$$

Here, $\eta_{t,s}(\nu)\in[0,1]$ denotes the firm's choice of capital utilization. In most of our analysis (except for Section 6) we assume utilization is free up to $\eta_{t,s}(\nu)=1$ and infinitely costly afterwards. The production firms sell their output to a competitive sector that produces the final output according to the CES technology, $y_{t,s} = \left(\int_0^1 y_{t,s}(\nu)^{\frac{\varepsilon-1}{\varepsilon}} d\nu\right)^{\varepsilon/(\varepsilon-1)}$, for some $\varepsilon > 1$. Thus, the demand for the firms' goods is given by,

$$y_{t,s}(\nu) = p_{t,s}(\nu)^{-\varepsilon} y_{t,s}, \text{ where } p_{t,s}(\nu) = P_{t,s}(\nu)/P. \tag{A.3}$$

Here, $p_{t,s}(\nu)$ denotes the firm's relative price, which depends on its nominal price, $P_{t,s}(\nu)$, as well as the ideal nominal price index, $P_{t,s} = \left(\int P_{t,s}(\nu)^{1-\varepsilon} d\nu\right)^{1/(1-\varepsilon)}$.

We also assume there are subsidies designed the correct the inefficiencies that stem from the firm's monopoly power and markups. In particular, the government taxes the firm's profits lump sum, and redistributes these profits to the firms in the form of a linear subsidy to capital. Formally, we let $\Pi_{t,s}(\nu)$ denote the equilibrium pre-tax profits of firm ν (that will be characterized below). We assume each firm is subject to the lump-sum tax determined by the average profit of all firms,

$$T_{t,s} = \int_\nu \Pi_{t,s}(\nu) d\nu. \tag{A.4}$$

We also let $R_{t,s} - \tau_{t,s}$ denote the after-subsidy cost of renting capital, where $R_{t,s}$ denotes the equilibrium rental rate paid to investment firms, and $\tau_{t,s}$ denotes a linear subsidy paid by the government. We assume the magnitude of the subsidy is determined by the government's break-even condition,

$$\tau_{t,s} \int_\nu k_{t,s}(\nu) d\nu = T_{t,s}. \tag{A.5}$$

Without price rigidities, the firm chooses $p_{t,s}(\nu), k_{t,s}(\nu), \eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu)$, to maximize its (pre-tax) profits,

$$\Pi_{t,s}(\nu) \equiv p_{t,s}(\nu) y_{t,s}(\nu) - (R_{t,s} - \tau_{t,s}) k_{t,s}(\nu), \tag{A.6}$$

subject to the supply constraint in $(A.2)$ and the demand constraint in $(A.3)$. The optimality conditions imply,

$$p_{t,s}(\nu) = \frac{\varepsilon}{\varepsilon - 1} \frac{R_{t,s} - \tau_{t,s}}{A} \text{ and } \eta_{t,s}(\nu) = 1.$$

That is, the firm charges a markup over its marginal costs, and utilizes its capital at full capacity. In a symmetric-price equilibrium, we further have, $p_{t,s}(\nu) = 1$. Using Eqs. $(A.2 - A.5)$, this further implies,

$$y_{t,s}(\nu) = y_{t,s} = A k_{t,s} \text{ and } R_{t,s} = \frac{\varepsilon - 1}{\varepsilon} A + \tau_{t,s} = A. \tag{A.7}$$

That is, output is equal to potential output, and capital earns its marginal contribution to potential output (in view of the linear subsidies).

We focus on the alternative setting in which the firms have a preset nominal price that is equal to one another, $P_{t,s}(\nu) = P$. In particular, the relative price of a firm is fixed and equal to one, $p_{t,s}(\nu) = 1$. The firm chooses the remaining variables, $k_{t,s}(\nu), \eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu)$, to maximize its (pre-tax) profits, $\Pi_{t,s}(\nu)$. We conjecture a symmetric equilibrium in which all firms choose the same allocation, $k_{t,s}, \eta_{t,s}, y_{t,s}$, output is determined by aggregate demand,

$$y_{t,s} = \eta_{t,s} A k_{t,s} = \int_I c^i_{t,s} di + k_{t,s} \iota_{t,s}, \text{ for } \eta_{t,s} \in [0,1], \tag{A.8}$$

and the rental rate of capital is given by,

$$R_{t,s} = A\eta_{t,s}. \tag{A.9}$$

To verify that the conjectured allocation is an equilibrium, first consider the case in which aggregate demand is below potential output, so that $y_{t,s} < Ak_{t,s}$ and $\eta_{t,s} < 1$. In this case, the firms can reduce their capital input, $k_{t,s}(\nu)$, and increase their factor utilization, $\eta_{t,s}(\nu)$, to obtain the same level of production. Since factor utilization is free (up to $\eta_{t,s}(\nu) = 1$), the after tax cost of capital must be zero, $R_{t,s} - \tau_{t,s} = 0$. Since its marginal cost is zero, and its relative price is one, it is optimal for each firm to produce according to the aggregate demand, which verifies Eq. $(A.8)$. Using Eqs. $(A.4)$ and $(A.5)$, we further obtain, $\tau_{t,s} = A\eta_{t,s}$. Combining this with the requirement that $R_{t,s} - \tau_{t,s} = 0$ verifies Eq. $(A.9)$.

Next consider the case in which aggregate demand is equal to potential output, so that $y_{t,s} = Ak_{t,s}$ and $\eta_{t,s} = 1$. In this case, a similar analysis implies there is a range of equilibria with $\frac{R_{t,s}-\tau_{t,s}}{A} \leq 1$ and $R_{t,s} = A$. Here, the first equation ensures it is optimal for the firm to meet the aggregate demand. The second equation follows from the subsidy and the tax scheme. In particular, the frictionless benchmark allocation $(A.7)$, that features $\frac{R_{t,s}-\tau_{t,s}}{A} = \frac{\varepsilon-1}{\varepsilon}$ and $R_{t,s} = A$, is also an equilibrium with nominal rigidities as long as the aggregate demand is equal to potential output.

A.2. Omitted derivations in Section 5.1 on equilibrium values

This subsection derives the HJB equation that describes the normalized value function in equilibrium. It then characterizes this equation further for various cases analyzed in Section 5.1.

Characterizing the normalized value function in equilibrium. Consider the recursive version of the portfolio problem in $(A.1)$. Recall that the value function has the functional form,

$$V^i_{t,s}\left(a^i_{t,s}\right) = \frac{\log\left(a^i_{t,s}/Q_{t,s}\right)}{\rho} + v^i_{t,s}.$$

Our goal is to characterize the value function per unit of capital, $v^i_{t,s}$ (corresponding to $a^i_{t,s} = Q_{t,s}$). To facilitate the analysis, we define,

$$\xi^i_{t,s} = v^i_{t,s} - \frac{\log Q_{t,s}}{\rho}. \tag{A.10}$$

Note that $\xi^i_{t,s}$ is the value function per unit wealth (corresponding to $a^i_{t,s} = 1$), and that the value function also satisfies

$$V^i_{t,s}\left(a^i_{t,s}\right) = \frac{\log\left(a^i_{t,s}\right)}{\rho} + \xi^i_{t,s}.$$

We first characterize $\xi^i_{t,s}$. We then combine this with Eq. $(A.10)$ to characterize our main object of interest, $v^i_{t,s}$.

After substituting the optimal consumption rule in (12) and Eq. (14) (as well as $a^i_{t,s} = 1$), we obtain the following version of the HJB equation in $(A.1)$,

$$\begin{aligned}\rho\xi^i_{t,s} = \log\rho + \frac{1}{\rho}\left(r^f_{t,s} + \omega^{k,i}_{t,s}\left(r^k_{t,s} - r^f_{t,s}\right) - \frac{1}{2}\left(\omega^{k,i}_{t,s}\right)^2\left(\sigma^k_{t,s}\right)^2 - \rho - \sum_{s'\neq s}\omega^{s',i}_{t,s}\right) \\ + \frac{\partial\xi^i_{t,s}}{\partial t} + \sum_{s'\neq s}\lambda^i_{s,s'}\left(\frac{\log\left(\frac{\lambda^i_{s,s'}}{p^{s'}_{t,s}}\right)}{\rho} + \xi^i_{t,s'} - \xi^i_{t,s}\right).\end{aligned}$$

As we describe in Section 4, the market clearing conditions imply the optimal investment in capital and contingent securities satisfies, $\omega^k = 1$ and $\tilde{\omega}_{t,s}^{s',i} = \lambda_{s,s'}^i - \overline{\lambda}_{t,s,s'}$ for each s', and the price of the contingent security is given by, $p_{t,s}^{s'} = \overline{\lambda}_{t,s,s'} \frac{Q_{t,s}}{Q_{t,s'}}$. Here, $\overline{\lambda}_{t,s,s'}$ denotes the weighted average belief defined in (37). Using these conditions, the HJB equation becomes,

$$\rho \xi_{t,s}^i = \log \rho + \frac{1}{\rho} \left(\begin{array}{c} r_{t,s}^k - \rho - \frac{1}{2}\left(\sigma_{t,s}^k\right)^2 \\ + \sum_{s' \neq s} \left(- \left(\lambda_{s,s'}^i - \overline{\lambda}_{t,s,s'}\right) + \lambda_{s,s'}^i \log\left(\frac{\lambda_{s,s'}^i}{\overline{\lambda}_{t,s,s'}}\right)\right) \end{array} \right) \tag{A.11}$$
$$+ \frac{\partial \xi_{t,s}^i}{\partial t} + \sum_{s' \neq s} \lambda_{s,s'}^i \left(\frac{1}{\rho} \log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right) + \xi_{t,s'}^i - \xi_{t,s}^i \right).$$

After substituting the return to capital from (24), and observing that $\sigma_{t,s}^Q = 0$ and $\sigma_{t,s}^k = \sigma_s$ in equilibrium, the HJB equation can be further simplified as,

$$\rho \xi_{t,s}^i = \left[\begin{array}{c} \log \rho + \frac{1}{\rho} \left(\begin{array}{c} \psi \log\left(Q_{t,s}\right) - \delta + \mu_{t,s}^Q - \frac{1}{2}\sigma_s^2 \\ + \sum_{s' \neq s} \left(- \left(\lambda_{s,s'}^i - \overline{\lambda}_{t,s,s'}\right) + \lambda_{s,s'}^i \log\left(\frac{\lambda_{s,s'}^i}{\overline{\lambda}_{t,s,s'}}\right)\right) \end{array} \right) \\ + \frac{\partial \xi_{t,s}^i}{\partial t} + \sum_{s' \neq s} \lambda_{s,s'}^i \left(\frac{1}{\rho} \log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right) + \xi_{t,s'}^i - \xi_{t,s}^i \right) \end{array} \right].$$

Here, the term inside the summation on the second line, $\sum_{s' \neq s} - \left(\lambda_{s,s'}^i - \overline{\lambda}_{t,s,s'}\right) + \lambda_{s,s'}^i \log\left(\frac{\lambda_{s,s'}^i}{\overline{\lambda}_{t,s,s'}}\right)$, is zero when there are no disagreements, and it is strictly positive when there are disagreements. It captures the intuition that speculation increases the expected value for optimists as well as pessimists.

We finally substitute $v_{t,s}^i = \xi_{t,s}^i + \frac{\log Q_{t,s}}{\rho}$ (cf. $(A.10)$) into the HJB equation to obtain the differential equation,

$$\rho v_{t,s}^i = \left[\begin{array}{c} \log \rho + \log\left(Q_{t,s}\right) + \frac{1}{\rho} \left(\begin{array}{c} \psi \log\left(Q_{t,s}\right) - \delta - \frac{1}{2}\sigma_s^2 \\ + \sum_{s' \neq s} \left(- \left(\lambda_{s,s'}^i - \overline{\lambda}_{t,s,s'}\right) + \lambda_{s,s'}^i \log\left(\frac{\lambda_{s,s'}^i}{\overline{\lambda}_{t,s,s'}}\right)\right) \end{array} \right) \\ + \frac{\partial v_{t,s}^i}{\partial t} + \sum_{s' \neq s} \lambda_{s,s'}^i \left(v_{t,s'} - v_{t,s}\right) \end{array} \right]. \tag{A.12}$$

Here, we have canceled terms by using the observation that $\frac{\partial \xi_{t,s}^i}{\partial t} = \frac{\partial v_{t,s}^i}{\partial t} - \frac{1}{\rho} \frac{\partial \log Q_{t,s}}{\partial t} = \frac{\partial v_{t,s}^i}{\partial t} - \frac{1}{\rho}\mu_{t,s}^Q$. In the two-states special case, the HJB equation can be further simplified to Eq. (43) in the main text.

Solving for the value function in the common beliefs benchmark. Next consider the benchmark with common beliefs. In this case, the HJB equation (43) implies the value functions are stationary, $v_{t,s} = v_s$, with values that satisfy,

$$\rho v_s = \log \rho + q_s + \frac{1}{\rho}\left(\psi q_s - \delta - \frac{1}{2}\sigma_s^2\right) + \lambda_s \left(v_{s'} - v_s\right). \tag{A.13}$$

Consider the same equation for $s' \neq s$. Multiplying that equation with λ_s and the above equation with $(\rho + \lambda_{s'})$, and adding up, we obtain,

$$\rho v_s = \log \rho + \beta_s \left(q_s + \frac{1}{\rho}\left(\psi q_s - \delta - \frac{1}{2}\sigma_s^2\right)\right) + (1 - \beta_s)\left(q_{s'} + \frac{1}{\rho}\left(\psi q_{s'} - \delta - \frac{1}{2}\sigma_{s'}^2\right)\right),$$

where $\beta_s = \frac{\rho + \lambda_s}{\rho + \lambda_s + \lambda_{s'}}$. After rearranging the terms, we obtain Eq. (46) in the main text.

Next note that $\{v_s^*\}_s$ is defined as the solution to the same equation system with $q_s = q^*$ for each s. Subtracting the corresponding equations, the gap value, $w_s = v_s - v_s^*$, satisfies,

$$\rho w_s = \left(1 + \frac{\psi}{\rho}\right)\left(\beta_s\left(q_s - q^*\right) + \left(1 - \beta_s\right)\left(q_{s'} - q^*\right)\right) = \left(1 + \frac{\psi}{\rho}\right)\overline{q}_s,$$

which gives Eq. (47) in the main text.

Note also that we have $q_1 - q^* = 0$ and $q_2 - q^* < 0$. Since $\beta_s \in (0,1)$, this implies $w_s < 0$ for each $s \in \{1,2\}$. Finally, using the inequality,

$$\beta_2 = \frac{\rho + \lambda_2}{\rho + \lambda_2 + \lambda_1} > 1 - \beta_1 = \frac{\lambda_2}{\rho + \lambda_2 + \lambda_1},$$

we further obtain the inequality, $w_2 < w_1 < 0$.

Solving the value function in the two-state model with belief heterogeneity. Next consider the case with two states and two belief types that we analyzed in Section 4. In this case, the value function and its components, $\left\{v_{t,s}^i, v_{t,s}^{i,*}, w_{t,s}\right\}_{s,i}$, can be written as functions of optimists' wealth share, $\left\{v_s^i\left(\alpha\right), v_s^{i,*}\left(\alpha\right), w_s\left(\alpha\right)\right\}_{s,i}$, that solve appropriate ordinary differential equations.

Recall that the price level in each state can be written as a function of optimists' wealth shares, $q_{t,s} = q_s\left(\alpha\right)$ (where we also have, $q_1\left(\alpha\right) = q^*$). Plugging in these price functions, and using the evolution of $\alpha_{t,s}$ from Eq. (38), the HJB equation (43) can be written as,

$$\rho v_s^i\left(\alpha\right) = \left[\begin{array}{c} \log\rho + q_s\left(\alpha\right) + \frac{1}{\rho}\left(\begin{array}{c} \psi q_s\left(\alpha\right) - \delta - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda_s^i - \left(\lambda_s^p + \alpha\Delta\lambda_s\right)\right) + \lambda_s^i \log\left(\frac{\lambda_s^i}{\lambda_s^p + \alpha\Delta\lambda_s}\right) \end{array}\right) \\ -\frac{\partial v_s^i}{\partial\alpha}\Delta\lambda_s\alpha\left(1-\alpha\right) + \lambda_s^i\left(v_{s'}^i\left(\alpha\frac{\lambda_s^o}{\lambda_s^p + \alpha\Delta\lambda_s}\right) - v_s^i\left(\alpha\right)\right) \end{array}\right].$$

For each $i \in \{o,p\}$, the value functions, $\left(v_s^i\left(\alpha\right)\right)_{s\in\{1,2\}}$, are found by solving this system of ODEs. For $i = 0$, the boundary conditions are that the values, $\{v_s^o\left(1\right)\}_s$, are the same as the values in the common belief benchmark characterized in Section 3 when all investors have the optimistic beliefs. For $i = p$, the boundary conditions are that the values, $\{v_s^p\left(0\right)\}_s$, are the same as the values in the common belief benchmark when all investors have the pessimistic beliefs.

Likewise, the first-best value functions, $\left(v_s^{i,*}\left(\alpha\right)\right)_{s\in\{1,2\}}$, are found by solving the analogous system after replacing $q_s\left(\alpha\right)$ with q^* (and changing the boundary conditions appropriately). Finally, after substituting the price functions into Eq. (45), the gap-value functions, $\left(w_s^i\left(\alpha\right)\right)_{s,i}$, are found by solving the following system (with appropriate boundary conditions),

$$\rho w_s^i\left(\alpha\right) = \left(1 + \frac{\psi}{\rho}\right)\left(q_s\left(\alpha\right) - q^*\right) - \frac{\partial w_s^i\left(\alpha\right)}{\partial\alpha}\Delta\lambda_s\alpha\left(1-\alpha\right) + \lambda_s^i\left(w_{s'}\left(\alpha\frac{\lambda_s^o}{\lambda_s^p + \alpha\Delta\lambda_s}\right) - w_s\left(\alpha\right)\right).$$

Figure 6 in the main text plots the solution to these differential equations for a particular parameterization.

A.3. Omitted derivations in Section 5.2 on macroprudential policy

Recall that macroprudential policy induces optimists to choose allocations as if they have different beliefs than their own. We next show that this allocation can be implemented with portfolio restrictions on optimists. We also derive the equilibrium value functions that result form this policy.

We first consider the general case in which there might be more than two states and the planner-induced beliefs might be time dependent. In this case, we let $\left(\lambda_{t,s,s'}^{o,pl}\right)_{t,s,s'}$ denote the planner-induced beliefs for optimists. For notational consistency, we also use $\lambda_{t,s,s'}^{i,pl} = \lambda_{s,s'}^{i}$ to denote the beliefs of other investors $i \neq o$, and we use $\overline{\lambda}_{t,s,s'}^{pl} = \sum_i \alpha_{t,s}^i \lambda_{t,s,s'}^{i,pl}$ to denote the planner-induced wealth-weighted average belief. We will first establish some general results under the assumption that the planner can perfectly control optimists' portfolio weights (see Eq. $(A.14)$). We will then restrict attention to the two-state special case and show that, in this case, the portfolio constraints can be relaxed to inequality restrictions (see Eq. $(A.18)$). Finally, we will restrict further attention to time-invariant policies, $\lambda^{o,pl} = \left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$, and derive the equilibrium value functions used in Section 5.2.

First consider the equilibrium that would obtain if optimists actually had the planner-induced beliefs, $\left(\lambda_{t,s,s'}^{o,pl}\right)_{t,s,s'}$. Using our analysis in Section 4 (with a slight extension to possibly time-variant beliefs), optimists' portfolios in this equilibrium would be given by,

$$\omega_{t,s}^{k,o,pl} = 1 \text{ and } \omega_{t,s}^{s',o,pl} = \lambda_{t,s,s'}^{o,pl} - \overline{\lambda}_{t,s,s'}^{pl} \text{ for each } t, s. \tag{A.14}$$

For now, suppose the planner requires optimists to hold portfolio weights that are exactly equal to these expressions. Note that macroprudential policy depends on the time and the state, (t, s), as well as investors' wealth shares. (Note also that the policy does not depend on the wealth of an individual optimists. Thus, optimists cannot act strategically to influence the policy.)

Formally, an optimist solves the HJB problem $(A.1)$ with the additional constraint $(A.14)$. In view of log utility, we conjecture that the value function has the same functional form (11) with potentially different normalized values, $\xi_{t,s}^o, v_{t,s}^o$, that reflect the constraints. Using this functional form, the optimality condition for consumption remains unchanged, $c_{t,s} = \rho a_{t,s}^o$ [cf. Eq. (12)]. Plugging this equation and the portfolio holdings in $(A.14)$ into the objective function in $(A.1)$ verifies that the value function has the conjectured functional form. It also implies that the unit-wealth value function satisfies [cf. Eq. $(A.10)$],

$$\begin{aligned}\xi_{t,s}^o &= \log\rho + \frac{1}{\rho}\left(r_{t,s}^f + \omega_{t,s}^{k,o,pl}\left(r_{t,s}^k - r_{t,s}^f\right) - \rho - \sum_{s'\neq s}\omega_{t,s}^{s',o,pl}\right) \\ &\quad -\frac{1}{2\rho}\left(\omega_{t,s}^{k,o,pl}\sigma_{t,s}^k\right)^2 + \frac{\partial \xi_{t,s}^o}{\partial t} + \sum_{s'\neq s}\lambda_{s,s'}^o\left(\frac{1}{\rho}\log\left(\frac{a_{t,s'}^o}{a_{t,s}}\right) + \xi_{t,s'}^o - \xi_{t,s}^o\right),\end{aligned} \tag{A.15}$$

where $\frac{a_{t,s'}^o}{a_{t,s}} = 1 + \omega_{t,s}^{k,o,pl}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} + \frac{\omega_{t,s}^{s',o,pl}}{p_{t,s}^{s'}}$ in view of the budget constraints of problem $(A.1)$.

Note also that since the remaining investors $i \neq o$ are unconstrained, their optimality conditions are unchanged. It follows that the equilibrium takes the form in Section 4 with the difference that optimists' beliefs are replaced by their as-if beliefs, $\lambda_{t,s,s'}^{o,pl}$. This verifies that the planner can implement the equilibrium with as-if beliefs, $\lambda_{t,s,s'}^{o,pl}$, using the portfolio restrictions in $(A.14)$.

Next consider the calculation of optimists' value functions in equilibrium. Since the analysis in Section 4 applies with as-if beliefs, we have,

$$\frac{a_{t,s'}^o}{a_{t,s}} = \frac{\alpha_{t,s'}^o}{\alpha_{t,s}^o}\frac{Q_{t,s'}}{Q_{t,s}} = \frac{\lambda_{t,s,s'}^{o,pl}}{\overline{\lambda}_{t,s,s'}^{pl}}\frac{Q_{t,s'}}{Q_{t,s}}. \tag{A.16}$$

Plugging this expression as well as Eq. $(A.14)$ into Eq. $(A.15)$, optimists' unit-wealth value function satisfies,

$$
\begin{aligned}
\xi^o_{t,s} &= \log\rho + \frac{1}{\rho}\left(\begin{array}{c} r^k_{t,s} - \rho - \frac{1}{2}\left(\sigma^k_{t,s}\right)^2 \\ \sum_{s'\neq s} -\left(\lambda^{o,pl}_{t,s,s'} - \overline{\lambda}^{pl}_{t,s,s'}\right) + \lambda^o_{s,s'}\log\left(\frac{\lambda^{o,pl}_{t,s,s'}}{\overline{\lambda}^{pl}_{t,s,s'}}\right)\end{array}\right) \\
&\quad + \frac{\partial \xi^o_{t,s}}{\partial t} + \sum_{s'\neq s}\lambda^o_{s,s'}\left(\frac{1}{\rho}\log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right) + \xi^o_{t,s'} - \xi^o_{t,s}\right),
\end{aligned}
$$

Note that this is the same as Eq. $(A.15)$ with the difference that the as-if beliefs, $\left(\lambda^{o,pl}_{t,s,s'}\right)$, are used to calculate the costs and returns from financial positions, whereas the actual beliefs, $\left(\lambda^o_{s,s'}\right)$, are used to calculate the transition probabilities. Using the steps after Eq. $(A.15)$, we also obtain the following generalization of Eq. $(A.12)$,

$$
\rho v^i_{t,s} = \left[\begin{array}{c} \log\rho + \log(Q_{t,s}) + \frac{1}{\rho}\left(\begin{array}{c} \psi\log(Q_{t,s}) - \delta - \frac{1}{2}\sigma^2_s \\ +\sum_{s'\neq s}\left(-\left(\lambda^{i,pl}_{t,s,s'} - \overline{\lambda}^{pl}_{t,s,s'}\right) + \lambda^i_{s,s'}\log\left(\frac{\lambda^{i,pl}_{t,s,s'}}{\overline{\lambda}^{pl}_{t,s,s'}}\right)\right)\end{array}\right) \\ +\frac{\partial v^i_{t,s}}{\partial t} + \sum_{s'\neq s}\lambda^i_{t,s,s'}\left(v_{t,s'} - v_{t,s}\right)\end{array}\right]. \quad \text{(A.17)}
$$

Two-states special case. Next consider the special case with two states, $s \in \{1,2\}$, and two belief types, $i \in \{o,p\}$. In this case, we claim that the portfolio constraints in $(A.14)$ can be relaxed to the following inequality restrictions,

$$
\begin{aligned}
\omega^{k,o,pl}_{t,s} &\leq 1 \text{ for each } s, \\
\omega^{2,o,pl}_{t,1} &\geq \underline{\omega}^{2,o}_{t,1} \equiv \lambda^{o,pl}_{t,1} - \overline{\lambda}^{pl}_{t,1} \text{ and } \omega^{1,o,pl}_{t,2} \leq \overline{\omega}^{1,o}_{t,2} \equiv \lambda^{o,pl}_{t,2} - \overline{\lambda}^{pl}_{t,2}.
\end{aligned} \quad \text{(A.18)}
$$

In particular, we will establish that all inequality constraints bind, which implies that optimists optimally choose the portfolio weights in Eq. $(A.14)$. Thus, our earlier analysis continues to apply when optimists are subject to the more relaxed constraints in $(A.18)$.

The result follows from the assumption that the planner-induced beliefs are more pessimistic, $\lambda^{o,pl}_{t,1} \geq \lambda^o_1$ and $\lambda^{o,pl}_{t,2} \leq \lambda^o_2$. To see this formally, note that the optimality condition for capital is given by the following generalization of Eq. (13),

$$
\omega^{k,o,pl}_{t,s}\sigma^k_{t,s} \leq \frac{1}{\sigma^k_{t,s}}\left(r^k_{t,s} - r^f_{t,s} + \lambda^o_s\frac{a^o_{t,s}}{a^o_{t,s'}}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right) \text{ and } \omega^{k,o,pl}_{t,s} \leq 1, \quad \text{(A.19)}
$$

with complementary slackness. Note also that,

$$
\lambda^o_s\frac{a^o_{t,s}}{a^o_{t,s'}}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} = \lambda^o_s\frac{\overline{\lambda}^{pl}_{t,s}}{\lambda^{o,pl}_{t,s}}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \geq \overline{\lambda}^{pl}_{t,s}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \text{ for each } s.
$$

Here, the equality follows from Eq. $(A.16)$ and the inequality follows by considering separately the two cases, $s \in \{1,2\}$. For $s = 2$, the inequality holds since $Q_{t,s'} - Q_{t,s} > 0$ and the beliefs satisfy, $\lambda^o_{t,s} \geq \lambda^{o,pl}_s$. For $s = 1$, the inequality holds since $Q_{t,s'} - Q_{t,s} < 0$ and the beliefs satisfy, $\lambda^{o,pl}_{t,s} \geq \lambda^o_s$. Note also that in equilibrium the return to capital satisfies the risk balance condition [cf. Eq. (34)],

$$\sigma^k_{t,s} = \frac{1}{\sigma^k_{t,s}}\left(r^k_{t,s} - r^f_{t,s} + \overline{\lambda}^{pl}_{t,s}\left(1 - \frac{Q_{t,s}}{Q_{t,s'}}\right)\right).$$

Combining these expressions implies, $\sigma^k_{t,s} \leq \frac{1}{\sigma^k_{t,s}}\left(r^k_{t,s} - r^f_{t,s} + \lambda^o_s \frac{a^o_{t,s}}{a^o_{t,s'}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right)$, which in turn implies the optimality condition $(A.19)$ is satisfied with $\omega^{k,o,pl}_{t,s} = 1$. A similar analysis shows that optimists also choose the corner allocations in contingent securities, $\omega^{2,o,pl}_{t,1} = \underline{\omega}^{2,o}_{t,1}$ and $\omega^{1,o,pl}_{t,2} = \overline{\omega}^{1,o}_{t,2}$, verifying that the portfolio constraints $(A.14)$ can be relaxed to the inequality constraints in $(A.18)$.

Value functions with time-invariant policies. Next suppose the planner sets time-invariant policies, $\left(\lambda^{o,pl}_1, \lambda^{o,pl}_2\right)$, which is the assumption we work with in the main text. In this case, the HJB equation $(A.17)$ for the value function can be further simplified to,

$$\rho v^i_{t,s} - \frac{\partial v^i_{t,s}}{\partial t} = \log\rho + q_{t,s} + \frac{1}{\rho}\begin{pmatrix} \psi q_{t,s} - \delta - \frac{1}{2}\sigma^2_s \\ -\left(\lambda^{i,pl}_s - \overline{\lambda}^{pl}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^{i,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\right)\end{pmatrix} + \lambda^i_s\left(v^i_{t,s'} - v^i_{t,s}\right). \tag{A.20}$$

We also characterize the first-best and the gap value functions, $v^{i,*}_{t,s}$ and $w^i_{t,s}$, that we use in the main text. By definition, the first-best value function solves the same differential equation $(A.20)$ after substituting $q_{t,s} = q^*$. It follows that the gap value function $w^i_{t,s} = v^i_{t,s} - v^{i,*}_{t,s}$, solves,

$$\rho w^i_{t,s} - \frac{\partial w^i_{t,s}}{\partial t} = \left(1 + \frac{\psi}{\rho}\right)(q_{t,s} - q^*) + \lambda^i_s\left(w^i_{t,s'} - w^i_{t,s}\right),$$

which is the same as the differential equation (45) without macroprudential policy. The latter affects the path of prices, $q_{t,s}$, but it does not affect how these prices translate into gap values.

Note also that, as before, the value functions can be written as functions of optimists' wealth share, $\left\{v^i_s(\alpha), v^{i,*}_s(\alpha), w_s(\alpha)\right\}_{s,i}$. For completeness, we also characterize the differential equations that these functions satisfy in equilibrium with macroprudential policy. Combining Eq. $(A.20)$ with the evolution of optimists' wealth share conditional on no transition, $\dot{\alpha}_{t,s} = -\Delta\lambda^{pl}_s \alpha_{t,s}(1 - \alpha_{t,s})$, the value functions, $\left(v^i_s(\alpha)\right)_{s,i}$, are found by solving,

$$\rho v^i_s(\alpha) = \begin{bmatrix} \log\rho + q_s(\alpha) + \frac{1}{\rho}\begin{pmatrix} \psi q_s(\alpha) - \delta - \frac{1}{2}\sigma^2_s \\ -\left(\lambda^{i,pl}_s - \overline{\lambda}^{pl}_{t,s}\right) + \lambda^i_s\log\left(\frac{\lambda^{i,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\right)\end{pmatrix} \\ -\frac{\partial v^i_s}{\partial\alpha}\Delta\lambda^{pl}_s\alpha(1-\alpha) + \lambda^i_s\left(v^i_{s'}\left(\alpha\frac{\lambda^{o,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\right) - v^i_s(\alpha)\right)\end{bmatrix},$$

with appropriate boundary conditions. Likewise, the first-best value functions, $\left(v^{i,*}_s(\alpha)\right)_{s\in\{1,2\}}$, are found by solving the analogous system after replacing $q_s(\alpha)$ with q^*. Finally, combining Eq. (45) with the evolution of optimists' wealth share, the gap-value functions, $\left(w^i_s(\alpha)\right)_{s,i}$, are found by solving Eq. (52) in the main text.

A.4. Omitted derivations in Section 6 on forward guidance

A.4.1. Description of the New Keynesian production firms with overutilization

Consider the New Keynesian block described in Appendix A.1.2 in which the firms have preset prices that are equal to one another, $P_{t,s}(\nu) = P$. We now change the technology so that the capital utilization rate can exceed one, $\eta_{t,s}(\nu) > 1$, but this also increases the depreciation rate of capital as described in Section 6. As before, production firms choose capital utilization rate, but they are required to compensate the investment firms for the cost of overutilization. Formally, the rental rate for a unit of capital that will be utilized at rate $\eta_{t,s}(\nu)$ is now given by

$$R_{t,s} = \underline{R}_{t,s} + \delta^{\eta} Q_{t,s} \max\left(0, \eta_{t,s}(\nu) - 1\right),$$

where $\underline{R}_{t,s}$ denotes the baseline rental rate and $\delta^{\eta} Q_{t,s}\left(\eta_{t,s}(\nu) - 1\right)$ denotes the competitive price for overutilization.

As before, the government collects the firms' profits in the form of lump-sum taxes, and redistributes these taxes in the form of linear subsidies to capital. To obtain overutilization as an equilibrium outcome, we assume the government also fully subsidizes the cost of overutilization. Formally, the lump-sum taxes are given by Eq. $(A.4)$ as before, and the capital subsidies are determined by,

$$\tau_{t,s} \int_{\nu} k_{t,s}(\nu)\, d\nu = T_{t,s} - \delta^{\eta} Q_{t,s} \int_{\nu} \max\left(0, \eta_{t,s}(\nu) - 1\right) k_{t,s}(\nu)\, d\nu. \tag{A.21}$$

This is a slightly modified version of $(A.21)$, which incorporates the government's cost from subsidizing the overutilization of capital.

Since overutilization is fully subsidized, a production firm's (pre-tax) profits are given by,

$$\Pi_{t,s}(\nu) \equiv y_{t,s}(\nu) - \left(\underline{R}_{t,s} - \tau_{t,s}\right) k_{t,s}(\nu).$$

Here, we also used the observation that the firm's relative price is fixed and equal to one, $p_{t,s}(\nu) = P_{t,s}(\nu)/P = 1$. The firm chooses $k_{t,s}(\nu), \eta_{t,s}(\nu), y_{t,s}(\nu)$ to maximize $\Pi_{t,s}(\nu)$ subject to the supply constraint in $(A.2)$ and the demand constraint in $(A.3)$. We conjecture a symmetric equilibrium in which all firms choose the same allocation, $k_{t,s}, \eta_{t,s}, y_{t,s}$, output is determined by aggregate demand according to Eq. $(A.8)$, and the baseline and the full rental rates of capital are given by respectively,

$$\underline{R}_{t,s} = A\eta_{t,s} - \delta^{\eta} Q_{t,s} \max\left(0, \eta_{t,s} - 1\right) \text{ and } R_{t,s} = A\eta_{t,s}. \tag{A.22}$$

This is a slight generalization of Eq. $(A.9)$ that also characterizes the baseline rental rate.

We next verify that the conjectured allocation is an equilibrium. The cases with $\eta_{t,s} \leq 1$ are identical to our analysis in Appendix A.1.2. Consider the case in which aggregate demand is sufficiently large so that Eq. $(A.8)$ implies $\eta_{t,s} > 1$. As before, the firm can always reduce its capital input, $k_{t,s}(\nu)$, and increase its factor utilization, $\eta_{t,s}(\nu)$, to obtain the same level of production. Since factor utilization is effectively free (due to the subsidies), the after tax cost of capital must be zero, $\underline{R}_{t,s} - \tau_{t,s} = 0$. Since its marginal cost is zero, and its relative price is one, it is optimal for each firm to produce according to the aggregate demand, which verifies Eq. $(A.8)$ for this case. Using Eqs. $(A.4)$ and $(A.21)$, we further obtain $\underline{R}_{t,s} = \tau_{t,s} = A\eta_{t,s} - \delta^{\eta} Q_{t,s} \max\left(0, \eta_{t,s} - 1\right)$, verifying Eq. $(A.22)$.

A.4.2. Characterizing the equilibrium with forward guidance

Most of the analysis is provided in the main text. We next characterize the value function, ρv_2, for the general case with investment, $\psi \geq 0$. We use this expression to illustrate that Assumption 5 ensures forward guidance policy would not be utilized in the first-best benchmark. We then derive the value function when $\psi = 0$ and characterize the optimal policy.

As in the main text, assume the planner follows condition (17) in state $s = 2$, and commits to implement the stationary policy, $\left(r_1^f, \eta_1\right)$ (with $\eta_1 \geq 1$), in state $s = 1$. The HJB equation $(A.13)$ continues to hold in this setting after replacing the depreciation rate in state 1 with $\delta_1(\eta_1) = \delta + \delta^{\eta}(\eta_1 - 1)$. Following the same steps as in Section 3, the value function in state 2 is given by,

$$\rho v_2 = \log \rho + \beta_2 \left(\left(\psi + \frac{1}{\rho} \right) q_2 - \frac{1}{\rho} \delta \right) + (1 - \beta_2) \left(\left(\psi + \frac{1}{\rho} \right) q_1(\eta_1) - \frac{1}{\rho} \delta_1(\eta_1) \right) - \frac{1}{\rho} \overline{\sigma}_2^2. \tag{A.23}$$

Here, $q_1(\eta_1) = \log \left(\frac{A\eta_1 + \psi}{\rho + \psi} \right)$ (see Eq. (22)), and β_2 and $\overline{\sigma}_2^2$ are defined in Eq. (46).

In the first-best benchmark, we have $q_2 = q^*$ and it does not depend on η_1. Hence, we have

$$\begin{aligned} \frac{dv_2}{d\eta_1}|_{\eta_1 \geq 1} &= (1 - \beta_2) \left(\left(\psi + \frac{1}{\rho} \right) q_1'(\eta_1) - \frac{1}{\rho} \delta^{\eta} \right) \\ &= \frac{1 - \beta_2}{\rho} \left((\psi\rho + 1) \frac{\rho + \psi}{A\eta_1 + \psi} A - \delta^{\eta} \right) \\ &= \frac{1 - \beta_2}{\rho} \left(\frac{(\psi\rho + 1) A}{\exp(q^*)} - \delta^{\eta} \right) \end{aligned}$$

Here, the last line uses $\exp(q^*) = \log \left(\frac{A + \psi}{\rho + \psi} \right)$, which follows from substituting $\eta_1 = 1$ into Eq. (22). In view of Assumption 5, we have $\frac{dv_2}{d\eta_1}|_{\eta_1 \geq 1} < 0$. This proves that the planner would not utilize the forward guidance policy in the first-best benchmark.

Next consider the case $\psi = 0$. In this case, Eq. $(A.23)$ implies Eq. (59) in the main text. Combining this expression with the constraints derived in the main text, the planner's problem can be written as,

$$\begin{aligned} & \max_{\eta_1 \geq 1} f(\eta_1) \equiv \beta_2 (q_2 - \delta) + (1 - \beta_2)(q_1(\eta_1) - \delta_1(\eta_1)) \\ & \text{s.t. } r_1^f = r_1^{f,b} - \delta^{\eta}(\eta_1 - 1), \\ & \text{and } q_2 = q(\eta_1) + q_2^b - q_1^b, \\ & \text{and } q_2 \leq q^* \text{ and } r_1^{f,b} \geq 0. \end{aligned}$$

The objective function is concave and the constraint sets are linear and nonempty (in view of Assumptions 1-3). Thus, the first order optimality conditions are necessary and sufficient for optimality.

Next note that we have,

$$\frac{df(\eta_1)}{d\eta_1} = \beta_2 \frac{dq_2}{d\eta_1} + (1 - \beta_2) \left(\frac{dq_1}{d\eta_1} - \frac{d\delta_1}{d\eta_1} \right) = \frac{\rho}{\eta_1} - (1 - \beta_2) \delta^{\eta}.$$

Here, the second equality uses the observation that $q_2 - q_1$ is constant, and $q_1(\eta_1) = \log \left(\frac{A\eta_1}{\rho} \right)$. This establishes that the forward guidance is used, $\eta_1 > 1$, if and only if $\Delta q \equiv \log \left(\frac{\rho}{\delta^{\eta}} \frac{1}{1 - \beta_2} \right) > 0$ [see Eq. (60)].

Next note that, when the solution is interior (no constraint binds), the optimal level of forward guidance is determined by, $\frac{df(\eta_1)}{d\eta_1} = 0$, which implies $\eta_1 = \frac{\rho}{\delta^\eta} \frac{1}{1-\beta_2} > 1$. The resulting capital price level satisfies,

$$q_1 = \log\left(\frac{A}{\rho}\right) + \log(\eta_1) = q^* + \Delta q.$$

This further implies $q_2 = q_2^b + \Delta q$. Using the constraints, we also obtain, $r_1^f = r_1^{f,b} - \delta^\eta (\exp(\Delta q) - 1)$. Consider the threshold,

$$\overline{\Delta q} = \min\left(q^* - q_2^b, \log\left(1 + r_1^{f,b}/\delta^\eta\right)\right) > 0.$$

When $\Delta q \in \left(0, \overline{\Delta q}\right)$, the price and the interest-rate levels implied by the optimal policy satisfy, $q_2 < q^*$ and $r_1^f > 0$. Thus, all inequality constraints are slack and there is an interior solution as described above. Otherwise, we have $q_2 = q^*$ or $r_1^f = 0$. This completes the derivations omitted from Section 6.

A.5. Omitted derivations in Section 7.2 on incomplete markets

We first derive Eqs. (62) and (63). When the investor cannot trade contingent securities, the optimality condition for consumption and capital are still given by respectively Eq. (12) and (13). Note also that Eq. (10) implies,

$$\frac{a_{t,s'}^i}{a_{t,s}^i} = 1 + \omega_{t,s}^{k,i} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}.$$

Substituting this into Eq. (13), the optimality condition for capital can be written as,

$$\begin{aligned} \omega_{t,s}^{k,i} \sigma_{t,s}^k &= \frac{1}{\sigma_{t,s}^k} \left(r_{t,s}^k - r_{t,s}^f + \sum_{s' \neq s} \lambda_{s,s'}^i \chi\left(\omega_{t,s}^{k,i}\right) \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \right), \qquad \text{(A.24)} \\ \text{where } \chi\left(\omega_{t,s}^{k,i}\right) &= \frac{Q_{t,s'}}{Q_{t,s} + \omega_{t,s}^{k,i} \left(Q_{t,s'} - Q_{t,s}\right)} \end{aligned}$$

Combining Eq. $(A.24)$ with the market clearing condition, $\sum_i \alpha_{t,2}^i \omega_{t,s}^{k,i} = 1$, we obtain Eq. (62). Subtracting this expression from Eq. $(A.24)$, we obtain Eq. (63).

Next consider the equilibrium in state $s = 2$. Let $q_2(\alpha)$ denote the price function given optimists' wealth share. Let $\sigma_2^Q(\alpha)$ and $\mu_2^Q(\alpha)$ denote functions that describe the volatility and the drift of prices given optimists' wealth share. Applying Eq. (62) for state $s = 2$, and substituting for $r_{t,2}^f = 0$ and $r_{t,2}^k$ from Eq. (24), we obtain,

$$\left(\sigma_2 + \sigma_2^Q(\alpha)\right)^2 = \rho + \psi q_2(\alpha) - \delta + \mu_2^Q(\alpha) + \sigma_2 \sigma_2^Q(\alpha) + \overline{\lambda}_2\left(\left\{\omega_{t,2}^{k,i}\right\}_i\right)\left(1 - \exp\left(q_2(\alpha) - q^*\right)\right). \quad \text{(A.25)}$$

This equation characterizes the price function given the endogenous volatility and drift of the price level.

Applying Eqs. (63) for state $s = 2$, we further obtain,

$$\begin{aligned} \left(\omega_{t,2}^{k,o} - 1\right)\left(\sigma_2 + \sigma_2^Q(\alpha)\right) &= \frac{1-\alpha}{\sigma_2 + \sigma_2^Q(\alpha)} \Delta\lambda_2\left(\omega_{t,2}^{k,o}\right)\left(1 - \exp\left(q_2(\alpha) - q^*\right)\right), \qquad \text{(A.26)} \\ \text{where} \qquad \Delta\lambda_2\left(\omega_{t,2}^{k,o}\right) &= \chi\left(\omega_{t,2}^{k,o}\right)\lambda_2^o - \chi\left(\omega_{t,2}^{k,p}\right)\lambda_2^p \text{ with } \omega_{t,2}^{k,p} = \frac{1 - \alpha\omega_{t,2}^{k,o}(\alpha)}{1-\alpha}. \end{aligned}$$

This equation implicitly characterizes the portfolio weights, $\omega_{t,2}^{k,o}, \omega_{t,2}^{k,p}$ given $\alpha, q_2(\alpha), \sigma_2^Q(\alpha)$. Note that

$\Delta\lambda_2\left(\omega_{t,2}^{k,o}\right)$ is a decreasing function of $\omega_{t,2}^{k,o}$, with $\Delta\lambda_2(1) = \lambda_2^o - \lambda_2^p > 0$ and $\lim_{\omega_{t,2}^{k,o} \to \overline{\omega}_{t,2}^{k,o}} \Delta\lambda_2\left(\omega_{t,2}^{k,o}\right) < 0$ for some upper bound, $\overline{\omega}_{t,2}^{k,o}$ (which is defined as the level that implies $\chi\left(\overline{\omega}_{t,2}^{k,p}\right) = \infty$). Hence, given $\alpha, q_2(\alpha), \sigma_2^Q(\alpha)$, Eq. $(A.26)$ has a unique solution that satisfies, $\omega_{t,2}^{k,o} > 1 > \omega_{t,2}^{k,p}$, verifying the claim in the main text.

We next derive the dynamics of optimists' wealth share in state $s = 2$. Conditional on no transition, the optimists' and the aggregate wealth respectively evolve according to,

$$\begin{aligned}\frac{da_{t,2}^o}{a_{t,2}^o} &= \left(r_{t,2}^f + \omega_{t,2}^{k,o}\left(r_{t,2}^k - r_{t,2}^f\right) - \rho\right)dt + \omega_{t,2}^o\left(\sigma_2 + \sigma_{t,2}^Q\right)dZ_t, \\ \frac{d\left(Q_{t,2}k_{t,2}\right)}{Q_{t,2}k_{t,2}} &= \left(r_{t,2}^f + \left(r_{t,2}^k - r_{t,2}^f\right) - \rho\right)dt + \left(\sigma_2 + \sigma_{t,2}^Q\right)dZ_t.\end{aligned}$$

Here, the first equation follows from combining Eqs. (10) and (12), and the second equation follows from combining (3) with (24). Applying Ito's lemma for quotients implies Eq. (64) in the main text. Combining this expression with Eq. $(A.25)$, we further obtain,

$$\frac{d\alpha_{t,2}}{\alpha_{t,2}} = \left(\omega_{t,2}^o - 1\right)\left[-\overline{\lambda}_2\left(\left\{\omega_{t,2}^{k,i}\right\}_i\right)\left(1 - \exp\left(q\left(\alpha_{t,2}\right) - q^*\right)\right)dt + \left(\sigma_2 + \sigma_{t,2}^Q\right)dZ_t\right].$$

It follows that the volatility and the drift of optimists' wealth share (defined as the coefficients in, $\frac{d\alpha_{t,2}}{\alpha_{t,2}} = \mu_2^\alpha\left(\alpha_{t,2}\right)dt + \sigma_2^\alpha\left(\alpha_{t,2}\right)dZ_t$) satisfy,

$$\begin{aligned}\sigma_2^\alpha(\alpha) &= \left(\omega_{t,2}^o - 1\right)\left(\sigma_2 + \sigma_2^Q(\alpha)\right), \\ \mu_2^\alpha(\alpha) &= -\left(\omega_{t,2}^o - 1\right)\overline{\lambda}_2\left(\left\{\omega_{t,2}^{k,i}\right\}_i\right)\left(1 - \exp\left(q_2(\alpha) - q^*\right)\right).\end{aligned} \tag{A.27}$$

Here, we have also written the volatility and the drift as a function of the state variable, α.

It remains to characterize the volatility and the drift of the price level. Next note that $Q_{t,2} = \exp\left(q_2\left(\alpha_{t,2}\right)\right)$. Using the dynamics of $\alpha_{t,2}$ together with Ito's Lemma, we obtain,

$$\frac{dQ_{t,2}}{Q_{t,2}} = \left(q_2'\left(\alpha_{t,2}\right)\alpha_{t,2}\mu_{t,2}^\alpha + \frac{1}{2}\left(q_2''\left(\alpha_{t,2}\right) + \left(q_2'\left(\alpha_{t,2}\right)\right)^2\right)\left(\alpha_{t,2}\sigma_{t,2}^\alpha\right)^2\right)dt + q_2'\left(\alpha_{t,2}\right)\alpha_{t,2}\sigma_{t,2}^\alpha dZ_t.$$

In particular, we have,

$$\begin{aligned}\sigma_2^Q(\alpha) &= q_2'(\alpha)\,\alpha\sigma_2^\alpha(\alpha), \\ \mu_2^Q(\alpha) &= q_2'(\alpha)\,\alpha\mu_2^\alpha(\alpha) + \frac{1}{2}\left(q_2''(\alpha) + \left(q_2'(\alpha)\right)^2\right)\alpha^2\sigma_2^\alpha(\alpha)^2.\end{aligned} \tag{A.28}$$

Eqs. $(A.27)$ and $(A.28)$ jointly characterize the volatility and drift terms, $\sigma_2^Q(\alpha), \mu_2^Q(\alpha), \sigma_2^\alpha(\alpha), \mu_2^\alpha(\alpha)$. To make further progress, note also that the expressions for the volatility imply,

$$\frac{\sigma_2^Q(\alpha)}{\sigma_2 + \sigma_2^Q(\alpha)} = q_2'(\alpha)\,\alpha\left(\omega_{t,2}^o - 1\right).$$

Since $\omega_{t,2}^o$ is an implicit function of $\alpha, q_2(\alpha), \sigma_2^Q(\alpha)$, this equation characterizes $\sigma_2^Q(\alpha)$ as an implicit function of $\alpha, q_2(\alpha), q_2'(\alpha)$. The two equations in $(A.27)$ then characterize $\sigma_2^\alpha(\alpha), \mu_2^\alpha(\alpha)$, as implicit functions of $\alpha, q_2(\alpha), q_2'(\alpha)$.

Finally, we substitute the expression for $\mu_2^Q(\alpha)$ into Eq. $(A.25)$, and rearrange terms, to obtain,

$$q_2^{''}(\alpha) = \frac{2}{\alpha^2 \sigma_2^\alpha(\alpha)^2}\left[\begin{array}{c}\left(\sigma_2 + \sigma_2^Q(\alpha)\right)^2 - \left(\begin{array}{c}\rho + \psi q_2(\alpha) - \delta + \sigma_2\sigma_2^Q(\alpha) \\ +\overline{\lambda}_2\left(\left\{\omega_{t,2}^{k,i}\right\}_i\right)(1-\exp(q_2(\alpha)-q^*))\end{array}\right) - q_2'(\alpha)\alpha\mu_2^\alpha(\alpha) \\ -\frac{1}{2}\left(q_2'(\alpha)\right)^2\alpha^2\sigma_2^\alpha(\alpha)^2\end{array}\right]. \tag{A.29}$$

Since the terms on the right hand side are implicit functions of $\alpha, q_2(\alpha), q_2'(\alpha)$, this expression provides a second order nonlinear ordinary differential equation in α. Below, we illustrate how the initial condition, $q_2'(0)$, can be calculated from this expression. The differential equation can then be solved forward by using the initial conditions, $q_2(0) = q_2^p$ and $q_2'(0)$ (as well as $q_2(1) = q_2^o$).

It remains to characterize the initial condition, $q_2'(0)$. To this end, we apply L'Hospital's Rule to Eq. $(A.29)$ to obtain $q_2''(0) =$

$$\lim_{\alpha\to 0}\frac{\frac{d}{d\alpha}\left\{\left(\sigma_2+\sigma_2^Q(\alpha)\right)^2 - \left(\begin{array}{c}\rho+\psi q_2(\alpha)-\delta+\sigma_2\sigma_2^Q(\alpha) \\ +\overline{\lambda}_2\left(\left\{\omega_{t,2}^{k,i}\right\}_i\right)(1-\exp(q_2(\alpha)-q^*))\end{array}\right) - q_2'(\alpha)\alpha\mu_2^\alpha(\alpha)\right\} - O(\alpha)}{O(\alpha)}.$$

Here, $O(\alpha)$ denotes terms that satisfy $\lim_{\alpha\to 0} O(\alpha) = 0$. For a stable solution, we require $q_2''(0)$ to be finite. This is the case only if the derivative of the set-bracketed term is zero, that is,

$$\begin{aligned}
&\frac{d}{d\alpha}\left\{\left(\sigma_2+\sigma_2^Q(\alpha)\right)^2 - \left(\begin{array}{c}\rho+\psi q_2(\alpha)-\delta+\sigma_2\sigma_2^Q(\alpha) \\ +\overline{\lambda}_2\left(\left\{\omega_{t,2}^{k,i}\right\}_i\right)(1-\exp(q_2(\alpha)-q^*))\end{array}\right) - q_2'(\alpha)\alpha\mu_2^\alpha(\alpha)\right\}|_{\alpha=0} \\
= \; &\sigma_2\frac{d\sigma_2^Q(0)}{d\alpha} - \left(\begin{array}{c}\psi q_2'(0) + \frac{d\overline{\lambda}_2}{d\alpha}|_{\alpha=0}(1-\exp(q_2(0)-q^*)) \\ -\overline{\lambda}_2|_{\alpha=0} q_2'(0)\exp(q_2(0)-q^*)\end{array}\right) - q_2'(0)\mu_2^\alpha(0) = 0.
\end{aligned} \tag{A.30}$$

Here, we used the observation that $\sigma_2^Q(0) = 0$ (cf. Eq. $(A.28)$). The initial condition, $q_2'(0)$, is the unique level of the derivative that ensures Eq. $(A.30)$ hold.

To characterize further, observe that Eqs. $(A.28)$ and $(A.27)$ imply,

$$\begin{aligned}
\frac{d\sigma_2^Q(0)}{d\alpha} &= q_2'(0)\sigma_2^\alpha(0) = q_2'(0)\left(\omega_{t,2}^o|_{\alpha=0}-1\right)\sigma_2, \\
\text{and } \mu_2^\alpha(0) &= -\left(\omega_{t,2}^o|_{\alpha=0}-1\right)\overline{\lambda}_2|_{\alpha=0}(1-\exp(q_2(0)-q^*)).
\end{aligned}$$

Observe also that our earlier characterization of the portfolio weights implies,

$$\begin{aligned}
\omega_{t,2}^{k,o}|_{\alpha=0} &> 1 \text{ and } \omega_{t,2}^{k,p}|_{\alpha=0} = 1 \\
\overline{\lambda}_2|_{\alpha=0} &= \lambda_2^p \text{ and } \Delta\lambda_2|_{\alpha=0} = \chi\left(\omega_{t,2}^{k,o}|_{\alpha=0}\right)\lambda_2^o - \lambda_2^p \\
\frac{d\overline{\lambda}_2}{d\alpha}|_{\alpha=0} &= \Delta\lambda_2|_{\alpha=0} - \lambda_2^p\frac{d\omega_{t,2}^{k,p}}{d\alpha}|_{\alpha=0}(1-\exp(q_2(0)-q^*)) \\
\text{and } \frac{d\omega_{t,2}^{k,p}}{d\alpha}|_{\alpha=0} &= -\frac{\Delta\lambda_2|_{\alpha=0}}{\sigma_2^2}(1-\exp(q_2(0)-q^*)).
\end{aligned}$$

Here, the first line follows directly by Eq. $(A.26)$, the second line follows by the definitions of $\overline{\lambda}_2$ and $\Delta\lambda_2|_{\alpha=0}$ (and $\chi(1) = 1$), the third line follows by differentiating $\overline{\lambda}_2$ and observing that

$\chi'(1) = -(1-\exp(q_2(0)-q^*))$ (see Eq. (62) for definitions of $\overline{\lambda}_2\left(\left\{\omega_{t,2}^{k,i}\right\}_i\right)$ and $\chi\left(\omega_{t,2}^{k,i}\right)$), and the last line follows by applying (63) for pessimists ($i=p$) and evaluating the derivative. Combining these observations with Eq. ($A.30$) enables us to solve for $q_2'(0)$ in terms of the parameters and completes the characterization.

Numerical solution to the ODE in ($A.29$)**.** Solving the ODE numerically poses numerical challenges due to the singularity at $\alpha=0$ (as well as $\alpha=1$). We need to start with $\alpha=\varepsilon$, for some small $\varepsilon>0$, which introduces numerical error since $q_2(\varepsilon), q_2'(\varepsilon)$ are not exactly the same as $q_2(0), q_2'(0)$. Moreover, the differential equation is such that small changes in the initial values around $\alpha\simeq 0$ introduce large deviations from the actual solution for larger levels of α. Therefore, finding the solution via the standard ODE solvers requires an exhaustive search for the initial values, $q_2(\varepsilon), q_2'(\varepsilon)$ (in the neighborhood of $q_2(0), q_2'(0)$). While this approach eventually works, it is computationally costly.

We therefore adopt an alternative "global" approach in which we approximate the price function with a polynomial, $q_2(\alpha)=\sum_{\tilde{n}=0}^{n} B_{\tilde{n}}\alpha^{\tilde{n}}$, where n is a large integer and $\{B_0,..,B_n\}$ are coefficients to be determined. We pick $n-2$ points on the interval, $\alpha\in(0,1)$, at which we require the differential equation to be exactly satisfied (with our polynomial approximation). Adding the three end-value conditions for $q_2(0), q_2'(0)$ and $q_2(1)$ gives us $n+1$ equations in $n+1$ unknowns, $\{B_0,..,B_n\}$. Solving this system is computationally feasible, and it provides an approximate solution, which is close to the solution obtained with the computationally intensive approach.

B. Appendix: Proofs

Proof of Proposition 1. Most of the proof is provided in the main text. It remains to show that Assumptions 1-3 ensure there exist a unique solution, $q_2<q^*$ and $r_1^f>0$, to Eqs. (28) and (29).

Eq. (24) illustrates that $R(q_2,q^*,\lambda_s)$ is concave in the current price, q_2. Taking the first order condition, it follows that $R(q_2,q^*,\lambda_s)$ is maximized at,

$$q_2^{\max}=q^*+\log(\psi/\lambda_2).$$

Moreover, the maximum value is given by

$$\begin{aligned}
R(q_2^{\max},q^*,\lambda_2) &= \rho-\delta+\psi(q^*+\log(\psi/\lambda_2))+\lambda_2(1-\exp(\log(\psi/\lambda_2))) \\
&= \rho-\delta+\psi q^*+\psi\log(\psi/\lambda_2)+\lambda_2-\psi.
\end{aligned}$$

Assumption 2 implies that $q_2^{\max}\leq q^*$, and that $R(q_2^{\max},q^*,\lambda_2)\geq\sigma_2^2$. Assumption 1 implies that $R(q^*,q^*,\lambda_2)<\sigma_2^2$. It follows that Eq. (28) has a unique solution that satisfies $q_2\in[q_2^{\max},q^*)$.

Next note that $R(q^*,q_2,\lambda_1)$ is increasing in q_2. Thus,

$$\begin{aligned}
R(q^*,q_2,\lambda_1)>R(q^*,q_2^{\max},\lambda_1) &= \rho-\delta+\psi q^*+\lambda_1(1-\exp(-\log(\psi/\lambda_2))), \\
&= \rho-\delta+\psi q^*+\lambda_1-\lambda_1\frac{\lambda_2}{\psi}.
\end{aligned}$$

Assumption 3 implies that $R(q^*,q_2^{\max},\lambda_1)>\sigma_1^2$. This in turn implies that Eq. (29) has a solution that satisfies $r_1^f>0$, completing the proof. □

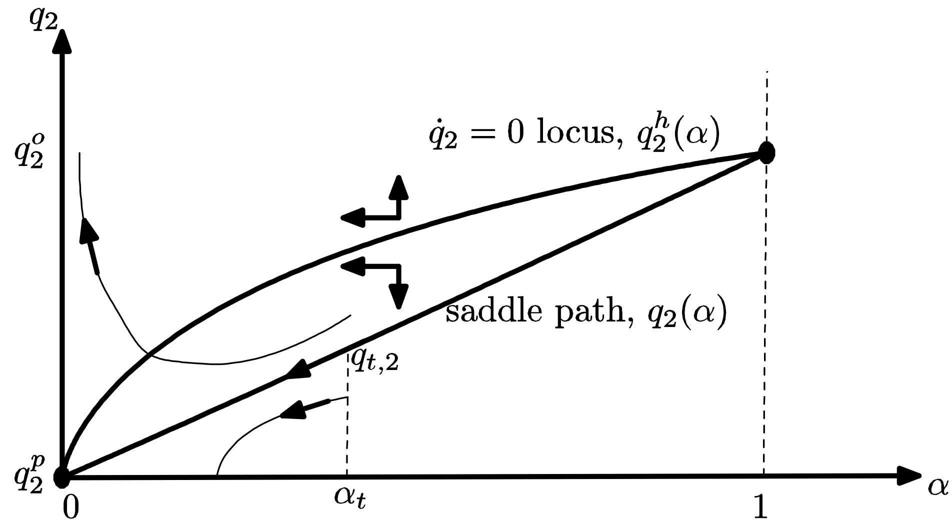

Figure 13: The phase diagram that describes the equilibrium with heterogeneous beliefs.

Proof of Corollary 1. Most of the proof is provided in the main text. We only note that combining Eqs. (31) and (32) implies $\left|\frac{dq_2}{d(\sigma_2^2)}\right| = \frac{1}{\lambda_2 \exp(q_2 - q^*) - \psi}$. This in turn implies $\frac{d}{d\lambda_2}\left|\frac{dq_2}{d(\sigma_2^2)}\right| < 0$ and $\frac{d}{d(\sigma_2^2)}\left|\frac{dq_2}{d(\sigma_2^2)}\right| > 0$. □

Proof of Proposition 2. We analyze the solution to the system in (40) using the phase diagram over the range $\alpha \in [0,1]$ and $q \in [q_2^p, q_2^o]$. First note that the system has two steady states given by, $(\alpha_{t,2} = 0, q_{t,2} = q_2^p)$, and $(\alpha_{t,2} = 1, q_{t,2} = q_2^o)$. Next note that the system satisfies the Lipschitz condition over the relevant range. Thus, the vector flows that describe the law of motion do not cross. Note also that the locus, $\dot{q}_2 = 0$, is given by a strictly increasing function, $q_2 = q_2^h(\alpha)$, where $q_2^h(\alpha)$ denotes the equilibrium in the homogeneous belief benchmark when all investors share the belief, $\lambda_2^p + \alpha\Delta\lambda_2$. Moreover, since the return is decreasing in q_2, $q_2 < q_2^h(\alpha)$ implies $\dot{q}_2 < 0$ and $q_2 > q_2^h(\alpha)$ implies $\dot{q}_2 > 0$. Finally, note that $\dot{\alpha} < 0$ for each $\alpha \in (0,1)$.

Combining these observations, the phase diagram has the shape in Figure 13. In particular, the system is saddle path stable. Given any $\alpha_{t,2} \in [0,1)$, there exists a unique solution, $q_{t,2}$, which ensures that $\lim_{t\to\infty} q_{t,2} = q_2^p$. We define the price function (the saddle path) as $q_2(\alpha)$. Note that the price function satisfies $q_2(\alpha) < q_2^h(\alpha)$ for each $\alpha \in (0,1)$, since the saddle path cannot cross the locus, $\dot{q}_2 = 0$. Note also that $q_2(1) = q_2^o$, since the saddle path crosses the other steady-state, $(\alpha_{t,2} = 1, q_{t,2} = q_2^o)$.

Next note that Eq. (40) implies the differential equation (41). Thus, the above analysis shows there exists a solution to the differential equation with $q_2(0) = q_2^p$ and $q_2(1) = q_2^o$. Note also that this solution is unique since the saddle path is unique. Hence, the price function is equivalently characterized as the unique solution to the differential equation (41). Note also that $q_2(\alpha) < q_2^h(\alpha)$ implies $R(q_2(\alpha), q^*, \lambda_2^p + \alpha\Delta\lambda_2) - \sigma_2^2 > 0$. Combining this with the differential equation (41), we further obtain, $\frac{dq_2(\alpha)}{d\alpha} > 0$ for each $\alpha \in (0,1)$.

Next consider Eq. (42) which characterizes the interest rate function, $r_1^f(\alpha)$. Note that $\frac{dr_1^f(\alpha)}{d\alpha} > 0$ since $\frac{dq_2(\alpha')}{d\alpha} > 0$ (recall that $\alpha' = \alpha\lambda_1^o/(\lambda_1^p + \alpha\Delta\lambda_1)$). Note also that $r_1^f(\alpha) > r_1^f(0) > 0$, where the latter inequality follows since Assumptions 1-3 holds for the pessimistic belief. Thus, the interest rate in state 1 is always positive, which verifies our conjecture and completes the proof. □

Proof of Proposition 3. For this proof, we find it useful to work with the transformed state variable,

$$b_{t,s} = \log\left(\frac{\alpha_{t,s}}{1-\alpha_{t,s}}\right), \text{ which implies } \alpha_{t,s} = \frac{1}{1+\exp(-b_{t,s})}. \tag{B.1}$$

The variable, $b_{t,s}$, varies between $(-\infty, \infty)$ and provides a different measure of optimism, which we refer to as "bullishness." Note that there is a one-to-one relation between optimists' wealth share, $\alpha_{t,s} \in (0,1)$, and the bullishness, $b_{t,s} \in \mathbb{R} = (-\infty, +\infty)$. Optimists' wealth dynamics in (38) become particularly simple when expressed in terms of the bullishness,

$$\begin{cases} \dot{b}_{t,s} = -\Delta\lambda_s, & \text{if there is no state change,} \\ b_{t,s'} = b_{t,s} + \log\lambda_s^o - \log\lambda_s^p, & \text{if there is a state change.} \end{cases} \tag{B.2}$$

With a slight abuse of notation, we also let $q_s(b)$ and $w_s^i(b)$ denote respectively the price function and the gap value function in terms of the bullishness.

Note also that, since $\frac{db}{d\alpha} = \frac{1}{\alpha(1-\alpha)}$, we have the identities,

$$\frac{\partial q_2(b)}{\partial b} = \alpha(1-\alpha)\frac{\partial q_2(\alpha)}{\partial b} \text{ and } \frac{\partial w_s^i(b)}{\partial b} = \alpha(1-\alpha)\frac{\partial w_s^i(\alpha)}{\partial \alpha}. \tag{B.3}$$

Using this observation, the differential equation for the price function, Eq. (41), can be written in terms of bullishness as,

$$\frac{\partial q_2(b)}{\partial b}\Delta\lambda_2^{pl} = R\left(q_2(b), q^*, \lambda_2^p + \alpha\Delta\lambda_2^{pl}\right) - \sigma_2^2. \tag{B.4}$$

Likewise, the differential equation for the gap value function, Eq. (52) can be written as bullishness,

$$\rho w_s^i(b) = \left(1+\frac{\psi}{\rho}\right)(q_s(b) - q^*) - \Delta\lambda_s^{pl}\frac{\partial w_s^i(b)}{\partial b} + \lambda_s^i\left(w_{s'}^i(b') - w_s^i(b)\right). \tag{B.5}$$

We next turn to the proof. To establish the comparative statics of the gap value function, we first describe it as a fixed point of a contraction mapping. Recall that, in the time domain, the gap value function solves the HJB equation (45). Integrating this equation forward, we obtain,

$$w_s^i(b_{0,s}) = \int_0^\infty e^{-(\rho+\lambda_s^i)t}\left(\left(1+\frac{\psi}{\rho}\right)(q_s(b_{t,s}) - q^*) + \lambda_s^i w_{s'}^i(b_{t,s'})\right)dt, \tag{B.6}$$

for each $s \in \{1,2\}$ and $b_{0,s} \in \mathbb{R}$. Here, $b_{t,s}$ denotes the bullishness conditional on there not being a transition before time t, whereas $b_{t,s'}$ denotes the bullishness if there is a transition at time t. Solving Eq. $(B.2)$ (given as-if beliefs, $\lambda^{i,pl}$) we further obtain,

$$b_{t,s} = b_{0,s} - t\Delta\lambda_s^{pl} \text{ and } b_{t,s'} = b_{0,s} - t\Delta\lambda_s^{pl} + \log\lambda_s^{o,pl} - \log\lambda_s^p. \tag{B.7}$$

Hence, Eq. $(B.6)$ describes the value function as a solution to an integral equation given the closed form solution for bullishness in $(B.7)$.

Let $B(\mathbb{R}^2)$ denote the set of bounded value functions over $\mathbb{R}^2$. Given some continuation value function, $\left(\tilde{w}_s^i(b)\right)_s \in B\left(\mathbb{R}^2\right)$, we define the function, $\left(T\tilde{w}_s^i(b)\right)_s \in B\left(\mathbb{R}^2\right)$, so that

$$T\tilde{w}_s^i(b_{0,s}) = \int_0^\infty e^{-\left(\rho+\lambda_s^i\right)t}\left(\left(1+\frac{\psi}{\rho}\right)(q_s(b_{t,s})-q^*)+\lambda_s^i\tilde{w}_{s'}^i(b_{t,s'})\right)dt, \tag{B.8}$$

for each s and $b_{0,s} \in \mathbb{R}$. Note that the resulting value function is bounded since the price function, $q_s(b_{t,s})$, is bounded (in particular, it lies between q_p and q^*). It can be checked that operator T is a contraction mapping with respect to the sup norm. In particular, it has a fixed point, which corresponds to the gap value function, $\left(w_s^i(b)\right)_s$.

We next show that the value function has strictly positive derivative with respect to bullishness or optimism. To this end, we first note that the value function is differentiable since it solves the differential equation (52). Next, we implicitly differentiate the integral equation $(B.6)$ with respect to $b_{0,s}$, and use Eq. $(B.7)$, to obtain,

$$\frac{\partial w_s^i(b_{0,s})}{\partial b} = \int_0^\infty e^{-\left(\rho+\lambda_s^i\right)t}\left(\left(1+\frac{\psi}{\rho}\right)\frac{\partial q_s(b_{t,s})}{\partial b}+\lambda_s^i\frac{\partial w_{s'}^i(b_{t,s'})}{\partial b}\right)dt. \tag{B.9}$$

Note from Eq. $(B.4)$ that the derivative of the price function, $\frac{\partial q_s(b)}{\partial b}$, is bounded. Thus, Eq. $(B.9)$ describes the derivative of the value function, $\frac{\partial w_s^i(b_{0,s})}{\partial b}$, as a fixed point of a corresponding operator T^{∂_b} over bounded functions (which is related to but different than the earlier operator, T). This operator is also a contraction mapping with respect to the sup norm. Since $\frac{\partial q_s(b_{t,s})}{\partial b} > 0$ for each b, and $\lambda_s^i > 0$ for each s, it can further be seen that the fixed point satisfies, $\frac{\partial w_s^i(b_{0,s})}{\partial b} > 0$ for each b and $s \in \{1,2\}$. Using Eq. $(B.3)$, we also obtain $\frac{\partial w_s^i(\alpha)}{\partial \alpha} > 0$ for each $\alpha \in (0,1)$ and $s \in \{1,2\}$.

Next consider the comparative statics of the fixed point with respect to macroprudential policy. We implicitly differentiate the integral equation $(B.6)$ with respect to $\lambda_1^{o,pl}$, and use Eq. $(B.7)$, to obtain,

$$\begin{aligned}
\frac{\partial w_1^i(b_{0,1})}{\partial \lambda_1^{o,pl}} &= \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_1^i\left(\frac{\partial w_2^i(b_{t,2})}{\partial \lambda_1^{o,pl}}+\frac{\partial w_2^i(b_{t,2})}{\partial b}\frac{db_{t,2}}{d\lambda_1^{o,pl}}\right)dt,\\
\frac{\partial w_2^i(b_{0,2})}{\partial \lambda_1^{o,pl}} &= \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_2^i\frac{\partial w_1^i(b_{t,1})}{\partial \lambda_1^{o,pl}}dt.
\end{aligned}$$

Note also that, using Eq. $(B.7)$ implies, $\frac{db_{t,2}}{d\lambda_1^{o,pl}} = -t + \frac{1}{\lambda_1^{o,pl}}$. Plugging this into the previous system, and evaluating the partial derivatives at $\lambda_1^{o,pl} = \lambda_1$, we obtain,

$$\begin{aligned}
\frac{\partial w_1^i(b_{0,1})}{\partial \lambda_1^{o,pl}} &= h(b_{0,1}) + \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_1^i\frac{\partial w_2^i(b_{t,2})}{\partial \lambda_1^{o,pl}}dt,\\
\frac{\partial w_2^i(b_{0,2})}{\partial \lambda_1^{o,pl}} &= \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_2^i\frac{\partial w_1^i(b_{t,1})}{\partial \lambda_1^{o,pl}}dt,\\
\text{where } h(b_{0,1}) &= \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_1^i\frac{\partial w_2^i(b_{t,2})}{\partial b}\left(-t+\frac{1}{\lambda_1^o}\right)dt.
\end{aligned} \tag{B.10}$$

Note that the function, $h(b)$, is bounded since the derivative function, $\frac{\partial w_2^i(b)}{\partial b}$, is bounded (see $(B.9)$). Hence, Eq. $(B.10)$ describes the partial derivative functions, $\left(\frac{\partial w_s^i(b)}{\partial \lambda_1^{o,pl}}|_{\lambda_1^{o,pl}=\lambda_1^o}\right)_s$, as a fixed point of a corresponding operator T^{∂_λ} over bounded functions (which is related to but different than the earlier operator, T). Since $h(b)$ is bounded, it can be checked that the operator T^{∂_λ} is also a contraction mapping with respect to the sup norm. In particular, it has a fixed point, which corresponds to the partial derivative functions.

The analysis so far applies generally. We next consider the special case, $\Delta\lambda_1 = 0$, and show that it implies the partial derivatives are strictly positive. In this case, $\lambda_1^i = \lambda_1$ for each $i \in \{o,p\}$. In addition, Eq. $(B.7)$ implies $b_{t,2} = b_{0,2}$. Using these observations, for each $b_{0,1}$, we have,

$$\begin{aligned} h(b_{0,1}) &= \frac{\partial w_2^i(b_{0,2})}{\partial b} \int_0^\infty e^{-(\rho+\lambda_1)t}\lambda_1 \left(-t + \frac{1}{\lambda_1}\right) dt \\ &= \frac{\partial w_2^i(b_{0,2})}{\partial b} \left(-\frac{\lambda_1}{\rho+\lambda_1}\frac{1}{\rho+\lambda_1} + \frac{1}{\rho+\lambda_1}\right) > 0. \end{aligned}$$

Here, the inequality follows from our earlier result that $\frac{\partial w_2^i(b_{0,2})}{\partial b} > 0$. Since $h(b) > 0$ for each b, and $\lambda_s^i > 0$, it can further be seen that the fixed point that solves $(B.10)$ satisfies $\frac{\partial w_s^i(b)}{\partial \lambda_s^{o,pl}} > 0$ for each b and $s \in \{1,2\}$. Using Eq. $(B.3)$, we also obtain $\frac{\partial w_s^i(\alpha)}{\partial \lambda_1^{o,pl}} > 0$ for each $\alpha \in (0,1)$ and $s \in \{1,2\}$. $\square$

Proof of Proposition 4. A similar analysis as in the proof of Proposition 3 implies that the partial derivative function, $\frac{\partial w_s^i(b)}{\partial\left(-\lambda_2^{o,pl}\right)}$, is characterized as the fixed point of a contraction mapping over bounded functions (the analogue of Eq. $(B.10)$ for state 2). In particular, the partial derivative exists and it is bounded. Moreover, since the corresponding contraction mapping takes continuous functions into continuous functions, the partial derivative is also continuous over $b \in \mathbb{R}$. Using Eq. $(B.3)$, we further obtain that the partial derivative, $\frac{\partial w_s^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}$, is continuous over $\alpha \in (0,1)$.

Next note that $w_s^i(1) \equiv \lim_{\alpha\to 1} w_s^i(\alpha)$ exists and is equal to the value function according to type i beliefs when all investors are optimistic. In particular, the asset prices are given by $q_1 = q^*$ and $q_2 = q^o$, and the transition probabilities are evaluated according to type i beliefs. Then, following the same steps as in our analysis of value functions in Section 3, we obtain,

$$\begin{aligned} w_s^i(1) &= \left(1 + \frac{\psi}{\rho}\right)\left(\beta_s^i q_s^o + \left(1-\beta_s^i\right) q_{s'}^o - q^*\right), \\ \text{where } \beta_s^i &= \frac{\rho + \lambda_{s'}^i}{\rho + \lambda_{s'}^i + \lambda_s^i}. \end{aligned}$$

Here, β_s^i denotes the expected amount of "discount time" the investor spends in state s according to type i beliefs. We consider this equation for $s = 2$ and take the derivative with respect to $\left(-\lambda_2^{o,pl}\right)$ to obtain,

$$\frac{\partial w_s^i(1)}{\partial\left(-\lambda_2^{o,pl}\right)} = \left(1 + \frac{\psi}{\rho}\right)\beta_s^2 \frac{dq_s^o}{d\left(-\lambda_2^{o,pl}\right)} < 0.$$

Here, the inequality follows since reducing optimists' optimism reduces the price level in the common belief benchmark (see Corollary 1).

Note that the inequality, $\frac{\partial w_s^i(1)}{\partial\left(-\lambda_2^{o,pl}\right)} < 0$, holds for each state s and each belief type i. Using the continuity of the partial derivative function, $\frac{\partial w_s^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}$, we conclude that there exists $\overline{\alpha}$ such that $\left.\frac{\partial w_s^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda^{o,pl}=\lambda^o} < 0$ for each i, s and $\alpha \in (\overline{\alpha}, 1)$, completing the proof. $\square$

Proof of Corollary 2. The corollary follows by combining Eq. (61) with Proposition 1. The remaining step is to prove Eq. (61). To this end, let $s' \in \{1, 0\}$ (a random variable) that indicates the first state after a state transition between t and $t + \Delta t$. If there is no state transition, then we use the convention $s' = s$. Using the law of total variance, we have,

$$Var_{t,s}\left(\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}}\right) = E^{s'}\left[Var_{t,s}\left(\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}}|s'\right)\right] + Var^{s'}\left(E_{t,s}\left[\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}}|s'\right]\right). \quad \text{(B.11)}$$

Here, $E^{s'}[\cdot]$ and $Var^{s'}[\cdot]$ denote, respectively, the expectations and the variance operator over the random variable, s'. We next calculate each component of variance.

For the first component, we have,

$$E^{s'}\left[Var_{t,s}\left(\frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}}|s'\right)\right] = e^{-\sum_{s' \neq s} \lambda_{s,s'} \Delta t} \sigma_s^2 \Delta t + \left(1 - e^{-\sum_{s' \neq s} \lambda_{s,s'} \Delta t}\right) O(\Delta t).$$

Here, the first term captures the variance conditional on there being no transition, $s' = s$. The variance in this case comes from the Brownian motion for $k_{t,s}$ (since $\sigma^Q = 0$). The second term captures the average variance conditional on there being a transition, $s' \neq s$. This term satisfies, $\lim_{\Delta t \to 0} O(\Delta t) = 0$. Dividing by Δt and evaluating the limit, we obtain,

$$\lim_{\Delta t \to 0} E^{s'}\left[Var_{t,s}\left(\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}}|s'\right)\right] = \sigma_s^2. \quad \text{(B.12)}$$

For the second component, we have,

$$\begin{aligned}
Var^{s'}\left(E_{t,s}\left[\frac{\Delta k_{t,s} Q_{t,s}}{k_{t,s} Q_{t,s}}|s'\right]\right) &= Var^{s'}\left(E_{t,s}\left[\frac{\Delta Q_{t,s}}{Q_{t,s}}|s'\right]\right) + O\left((\Delta t)^2\right), \\
&= \sum_{s'} \Pr(s') \left(\frac{Q_{t+\Delta t,s'} - Q_{t,s}}{Q_{t,s}} - E_{s'}\left[\frac{Q_{t+\Delta t,s'} - Q_{t,s}}{Q_{t,s}}\right]\right) + O\left((\Delta t)^2\right) \\
&= \sum_{s'} \Pr(s') \left(\frac{Q_{t+\Delta t,s'} - E_{s'}\left[Q_{t+\Delta t,s'}\right]}{Q_{t,s}}\right) + O\left((\Delta t)^2\right) \\
&= \left(1 - \sum_{s' \neq s} \lambda_{s,s'} \Delta t\right) \left(\frac{Q_s - \overline{Q}}{Q_s}\right)^2 + \sum_{s' \neq s} \lambda_{s,s'} \Delta t \left(\frac{Q_{s'} - \overline{Q}}{Q_s}\right)^2 + O\left((\Delta t)^2\right), \\
&\quad \text{where } \overline{Q} = \left(1 - \sum_{s' \neq s} \lambda_{s,s'} \Delta t\right) Q_s + \sum_{s' \neq s} \lambda_{s,s'} \Delta t Q_{s'}.
\end{aligned}$$

Here, $O\left((\Delta t)^2\right)$ denotes terms that satisfy, $\lim_{\Delta t \to 0} \frac{O\left((\Delta t)^2\right)}{\Delta t} = 0$. The first line uses the observation that, conditional on a path of state transitions, the expected level of investment is constant. Thus (for small Δt) the state transitions change the return only through their impact on the price level. The remaining lines calculate the variance of price changes by focusing on a single transition event (the events that have two or more state transitions are appended to the term, $O\left((\Delta t)^2\right)$). Dividing the last line by Δt and evaluating the limit, we obtain,

$$\lim_{\Delta t \to 0} Var^{s'}\left(E_{t,s}\left[\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}}|s'\right]\right) = \sum_{s' \neq s} \lambda_{s,s'} \left(\frac{Q_{s'} - Q_s}{Q_s}\right)^2. \quad \text{(B.13)}$$

Plugging Eqs. $(B.12)$ and $(B.13)$ into Eq. $(B.11)$, we obtain Eq. (61). $\square$