1 Introduction

Suppose we have \( n \in \mathbb{N} \) people in an office building, we know \( k \leq n \) of them are in the break room down the hall, and we would like to set up a model to predict who at some given time is in the room. We can occasionally hear people talking, indicating that some people outside the room have swapped with some who are in the room. We will call these mixing events. We also sometimes observe someone that is not in the room. We will call these observation events. We will consider the use of group actions, the Fourier Transform on the Symmetric group, and Isotypic decomposition to model this situation and compute probabilities according to our model.

Consider \( G = S_n \), the Symmetric group on \( \{1, \ldots, n\} \), acting on \( X \), \( k \)-sized subsets of \( \{1, \ldots, n\} \) in the natural way. We can think of probability distributions that reflect who we think is in the conference room as probability distributions on \( X \), which are elements of \( \mathbb{C}[X] \), the set of complex valued functions on \( X \). We can also think of these as polynomials in the elements of \( X \) with coefficients in \( \mathbb{C} \). For example we write
\[
0.4 \{1, 2, 3\} + 0.6 \{1, 3, 4\} \in \mathbb{C}[X]
\]
to mean the distribution where there is a 40% probability that persons 1, 2, and 3 are in the conference room, and a 60% probability that persons 1, 3, and 4 are in the conference room.

Mixing events correspond to multiplication by elements in the group ring \( \mathbb{C}G \). The irreducibles of our Fourier transform \( G \) correspond to the isotypic subspaces \( \mathbb{C}[X] \) such that \( \mathbb{C}[X] \) is a \( \mathbb{C}G \) module in the natural way. For example
\[
0.3 (12) + 0.7 (13) \in \mathbb{C}G
\]
is the distribution where there is a 30% probability that the event swaps persons 1 and 2 and a 70% probability that the event swaps persons 1 and 3. Then using the action of \( G \) on \( X \), we want to consider \( \mathbb{C}[X] \) as a \( \mathbb{C}G \) module in the natural way. For example
\[
(0.3 (12) + 0.7 (14))\{1, 2, 3\} = 0.3 (12) \cdot \{1, 2, 3\} + 0.6 (14) \cdot \{1, 2, 3\} = 0.3 \{1, 2, 3\} + 0.6 \{2, 3, 4\}.
\]

The case where \( X \) is also \( S_n \) instead of the set of \( k \) sets has been studied by Huang. In this case mixing events are convolution in the group ring. Huang uses the Fourier transform of \( S_n \) to both quickly calculate mixing events and to make approximations to distributions by keeping only lower order terms. He also explores calculating observation events in the Fourier domain.

In this paper, we use an isotypic decomposition of \( \mathbb{C}[X] \) such that the isotypic subspaces correspond to irreducible representations of the Fourier transform on \( S_n \), which are indexed by partitions of \( n \), to show that some of this methodology can be extended to the case where \( X \) is \( k \)-sized subsets of \( \{1, \ldots, n\} \).

2 Mixing Events

Mixing events correspond to multiplication by elements in the group ring \( \mathbb{C}G \). The irreducibles of our Fourier transform \( G \) correspond to the isotypic subspaces of \( \mathbb{C}[X] \) such that
multiplication by an element in the group ring can be computed as follows. The irreducible representations of $G$ and thus the corresponding isotypic subspaces are indexed by partitions of $n$. Let $\lambda$ be a partition of $n$ and let $\rho_{\lambda}$ be the irreducible partition of $n$ indexed by $\lambda$. Then if $f \in \mathbb{C}G$ and $g \in \mathbb{C}[X]$, $\hat{f}_{\rho_{\lambda}}$ is the Fourier transform of $f$ at the irreducible representation indexed by $\lambda$, and $\hat{g}_{\rho_{\lambda}}$ is the projection of $g$ onto the isotypic subspace indexed by $\lambda$, then for any $x \in X$

$$[\hat{f} \cdot g]_{\rho_{\lambda}}(x) = \hat{f}_{\rho_{\lambda}}(x)\hat{g}_{\rho_{\lambda}}(x) \in \mathbb{C}.$$ 

Thus as for Huang, if we work with the Fourier transform of elements of the group ring and isotypic decompositions of elements of $\mathbb{C}[X]$, instead of convolution, multiplication can be computed with a point-wise product.

Then even if we approximate our probability distribution by only keeping the projections onto some subset of the isotypic subspaces, mixing events can be calculated without adding any error.

### 3 Observation

Let $G$ be a finite group that acts on finite set $X$. Let $Z$ be a random variable that takes on values of $X$. Let $f : X \to \mathbb{R}$ be defined by

$$f(x) = P(Z = x) \quad \text{for } x \in X.$$ 

Then $f$ can be a probability distribution on $X$. Suppose we observe event $A$ and we want to update our probability distribution. Then we want to calculate

$$P(Z = x \mid A) = \frac{P(A \mid Z = x)P(Z = x)}{P(A)} = cg(x)f(x),$$ 

where $c = 1/P(A) \in \mathbb{R}$ is a constant and $g : X \to \mathbb{R}$ is defined by $g(x) = P(A \mid Z = x)$.

If we calculate $gf$, re-normalizing to get a probability distribution on $X$ will give us $P(Z = x \mid A)$.

Therefore to update $f$ after observing event $A$, we need only consider calculating the point-wise product of two real valued functions on $X$.

We can calculate this by simply taking the point-wise product of two functions, but then we would need to project the product onto the isotypic subspaces again. Unlike for mixing events, the projections into each isotypic subspace does not only depend on the projection onto that isotypic subspace of the original distribution. Thus if we only store some of the projections, we can introduce extra error into our distribution after every observation. Suppose we only store lower order projections, which we will define for our next result. Then after an observation, our new distribution can have nonzero higher order projections. The following result gives a bound for how many higher order terms we introduce, but first we need to define our notion of order.

**Definition.** Let $f \in \mathbb{C}G$ or $f \in \mathbb{C}[X]$. Then we say $f$ is $i$th order if $\hat{f}_{\rho_{\lambda}} = 0$ for any partition $\lambda = (r, \ldots)$ such that $r < n - i$. 

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Note that we can use this definition for both $f \in \mathbb{C}G$ and $f \in \mathbb{C}[X]$ because each isotypic subspace corresponds to an irreducible representations of $G$. If we know the order of two elements of $\mathbb{C}[X]$, we can bound the order of their product.

**Proposition.** If $f, g \in \mathbb{C}[X]$ are $i$-th order and $j$-th order, respectively, then the point-wise product $f \cdot g$ is $(i + j)$-th order.

To prove this, we will first prove an analogous result for $\mathbb{C}G$. We will then construct a homomorphism that embed our elements of $\mathbb{C}[X]$ as elements of $\mathbb{C}G$, and using this we will extend the result about $\mathbb{C}G$ to the above proposition.

To prove the analogous result for $\mathbb{C}G$, we will be using results from harmonic analysis on $S_n$. We will need the following definitions.

**Definition (Dominance Ordering).** Let $\lambda = (\lambda_1, \ldots), \mu = (\mu_1, \ldots)$ be partitions of $n$. Then $\lambda \trianglerighteq \mu$ (we say $\lambda$ dominates $\mu$), if for each $i$,

$$\sum_{k=1}^{i} \lambda_k \geq \sum_{k=1}^{i} \mu_k.$$  

For example, $(4, 2) \trianglerighteq (3, 2, 1)$. Note that high dominance ordering corresponds to low frequency.

**Definition (Clebsch-Gordon Series).** Let $\lambda, \mu$ be partitions of $n$. Then $z_{\lambda \mu \nu}$ is the number of copies of $\rho_\lambda$ in $\rho_\mu \otimes \rho_\nu$, where $\otimes$ is the Kronecker product.

The following result about the Clebsch-Gordon Series will also allow us to prove our result.

**Theorem (Murnaghan’s formulas).** (page 65 in pdf) Let $\lambda = (n-p, \lambda_2, \ldots), \mu = (n-q, \mu_2, \ldots)$ be partitions of $n$. Then the product $\rho_\lambda \otimes \rho_\mu$ does not contain any irreducibles corresponding to a partition whose first term is less than $n-p-q$. Therefore

$$z_{\lambda \mu} = 0.$$  

These concepts allow us to express the point-wise product of two elements of $\mathbb{C}G$ as follows.

**Theorem.** Let $f, g \in \mathbb{C}G$ and $\nu$ be a partition of $n$. Then there exist projection operators $P_{\lambda \mu}^{(\nu, l)}$ such that

$$[\hat{f} \cdot \hat{g}]_{\rho_\nu} = \sum_{\lambda \mu} \sum_{l=1}^{z_{\lambda \mu \nu}} (P_{\lambda \mu}^{(\nu, l)})^T \cdot (\hat{f}_{\rho_\lambda} \otimes \hat{g}_{\rho_\lambda}) \cdot (P_{\lambda \mu}^{(\nu, l)})^*.$$  

From this, we can see that if $z_{\lambda \mu \nu} = 0$ for all $\lambda, \mu$ such that $\hat{f}_{\rho_\lambda} \neq 0$ and $\hat{g}_{\rho_\lambda} \neq 0$, then

$$[\hat{f} \cdot \hat{g}]_{\rho_\nu} = 0.$$  

Now we can prove the following result for $\mathbb{C}G$ that is analogous to our proposition.
Theorem. If \( f, g \in \mathbb{C}G \) are \( i \)-th order and \( j \)-th order, respectively, then the point-wise product \( f \cdot g \) is \((i + j)\)-th order.

Proof. Consider any partition \( \nu \) of the form \((r, \ldots)\) such that \( r < n - i - j \).

Because \( f, g \) are \( i, j \)-th order, respectively, if \( \hat{f}_{\rho_\lambda} \neq 0 \) and \( \hat{g}_{\rho_\mu} \neq 0 \), then \( z_{\lambda\mu\nu} = 0 \).

Thus \( \hat{f}g_{\rho_{\nu}} = 0 \). Therefore \( fg \) is \((i + j)\)-th order. \( \square \)

Now we want to embed our elements of \( \mathbb{C}[X] \) as elements of \( \mathbb{C}G \). Let \( H = S_{n-k} \times S_k \subset S_n \). Every \( k \)-set of \( \{1, \ldots, n\} \) can be thought of as the coset of permutations of \( \mathbb{C}G \) that map 1 through \( k \) to the elements of the \( k \) set. Thus \( \mathbb{C}[X] \) is isomorphic as a \( \mathbb{C}G \) module to \( \mathbb{C}[G/H] \). Because \( H, G \) are a Gelfand pair, the irreducible representations of \( G \) each show up once as irreducible representations of \( G/H \). In fact the irreducible representations of \( G/H \) correspond to the isotypic subspaces of \( \mathbb{C}[X] \) and are indexed by the same partitions as the isotypic subspaces to which they correspond. Therefore to prove our proposition, we need to prove it for elements of \( \mathbb{C}[G/H] \).

Theorem (Shur’s Lemma). Let \( T : U \to U' \) be a \( \mathbb{C}G \)-module homomorphism, and suppose \( U, U' \) are irreducibles. Then,

\[
T = 0 \text{ if } U \not\cong U',
\]

or \( T \) is an isomorphism if \( T \neq 0 \).

Schur’s Lemma thus implies that a \( \mathbb{C}G \) module homomorphisms between \( \mathbb{C}[G] \) and \( \mathbb{C}[G/H] \) preserve order. Thus we want to find a module homomorphism that embeds elements of \( \mathbb{C}[G/H] \) into \( \mathbb{C}G \).

Define \( \phi : \mathbb{C}[G/H] \to \mathbb{C}[G] \) by

\[
\phi(f)(x) = f(xH),
\]

for \( f \in \mathbb{C}[G/H], x \in G \).

Theorem. \( \phi \) is a \( \mathbb{C}G \) module homomorphism.

Proof. Let \( f, g \in \mathbb{C}[G/H] \) and \( x \in G \). Then

\[
\phi(f + g)(x) = f(xH) + g(xH) = (\phi(f) + \phi(g))(x).
\]

Let \( a \in \mathbb{C} \). Then

\[
\phi(af)(x) = af(xH) = a\phi(f)(x).
\]

Let \( y \in G \subset \mathbb{C}G \). Then

\[
\phi(yf)(x) = yf(xH) = f(y^{-1}xH) = \phi(f)(y^{-1}x) = y\phi(f)(x).
\]

Because all elements of \( \mathbb{C}G \) are linear combinations of elements of \( G \), this shows that \( \phi \) is a module homomorphism. \( \square \)

Now we want a left sided inverse for \( \phi \). Define \( \psi : \mathbb{C}[G] \to \mathbb{C}[G/H] \) by

\[
\psi(f)(xH) = \frac{1}{|H|} \sum_{y \in xH} f(y),
\]

for \( f \in \mathbb{C}[G] \) and \( x \in G \).
Theorem. \( \psi \) is a \( \mathbb{C}G \) module homomorphism.

Proof. Let \( f, g \in \mathbb{C}[G] \) and \( x \in G \). Then

\[
\psi(f + g)(xH) = \frac{1}{|H|} \sum_{y \in xH} (f + g)(y) = (\psi(f) + \psi(g))(xH).
\]

Let \( a \in \mathbb{C} \). Then

\[
\psi(af)(xH) = \frac{1}{|H|} \sum_{y \in xH} af(y) = a\psi(f)(xH).
\]

Let \( z \in G \). Then

\[
\psi(zf)(xH) = \frac{1}{|H|} \sum_{y \in xH} zf(y) = \frac{1}{|H|} \sum_{y \in xH} f(z^{-1}y) = \frac{1}{|H|} \sum_{y \in z^{-1}xH} f(y) = \psi(f)(z^{-1}xH) = z\psi(f)(xH).
\]

Because all elements of \( \mathbb{C}G \) are linear combinations of elements of \( G \), this shows that \( \psi \) is a module homomorphism.

Theorem. Let \( f, g \in \mathbb{C}[G/H] \). Then

\[
f \cdot g = \psi(\phi(f) \cdot \phi(g)).
\]

Proof. This is true because

\[
\phi(fg)(xH) = (fg)(xH) = f(xH)g(xH) = \phi(f)\phi(g)(xH),
\]

and so

\[
\psi(\phi(f) \cdot \phi(g)) = \psi(\phi(fg)) = fg.
\]

Now we can return to our proposition which has been rephrased in terms of \( \mathbb{C}[G/H] \).

Proposition. If \( f, g \in \mathbb{C}[G/H] \) are \( i \)-th order and \( j \)-th order, respectively, then the pointwise product \( f \cdot g \) is \( (i + j) \)-th order.

Proof. Because \( \phi \) is a module homomorphism, and therefore preserves order, \( \phi(f), \phi(g) \in \mathbb{C}G \) are \( i \)-th and \( j \)-th order, respectively. Therefore by our analogous result for \( \mathbb{C}G \), \( \phi(f) \cdot \phi(g) \) is \( (i + j) \)-th order. Then because \( \psi \) is a module homomorphism, \( f \cdot g = \psi(\phi(f) \cdot \phi(g)) \) is \( (i + j) \)-th order.