Walrasian Equilibrium with Few Buyers

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Abstract. We study the existence and the properties of Walrasian equilibrium (WEQ) in combinatorial auctions, under two natural classes of valuation functions. The first class is based on additive capacities or weights, and the second on the influence in a social network. While neither class holds the gross substitutes condition, we show that in both classes the existence of WEQ is guaranteed under certain restrictions, and in particular when there are only two competing buyers.
For weight-based games, we complement the analysis with an empirical study, showing that WEQ very often exists even with three buyers.

1 Introduction

In a combinatorial auction, multiple items are for sale, and the utility of a buyer may depend on the particular set of acquired items in some composite manner.

In the most general case, a buyer may assign an arbitrary value to any group of items. However, typically there is some structure to the value function that is derived from the context. A trivial example is when the value for the buyer only depends on the number of acquired items, which are all identical.

In the following scenario, buyers' valuations can still be described by a relatively simple function. Consider a game where we can attribute some fixed capacity or weight to each item. The value to each buyer is then some function of the total (additive) capacity of its acquired items.

Concrete examples for such a scenario are when the items are storage devices, discrete time intervals for advertising, routers with certain throughput and so on. The utility of each buyer is increasing in the total storage/time/throughput, regardless of how it is divided among the purchased items.

Another interesting scenario is when buyers are firms recruiting influential nodes of a social network (such as news sites or popular blogs), trying to promote a product. The value for a firm in this case is proportional to the joint influence of the recruited nodes, which depends on the network structure and dynamics.

The most fundamental question in combinatorial auctions is regarding the expected outcomes, i.e. how will items be divided among buyers, and at what price. A standard solution is to ask whether there are prices such that if each buyer independently selects her optimal bundle of items, a valid allocation of items will arise. Such prices—if exist—are known as *Walrasian equilibrium* (WEQ). It is known that the allocation under Walrasian prices maximizes the social welfare (sum of buyers' utilities). A seemingly different combinatorial setting is that of a *labor market*, where buyers are firms competing on hiring workers, which are strategic agents rather than passive "items" [11]. However it is known that the models are in fact equivalent, and induce the same equilibrium outcomes. For details see [8], as well as the full version of this paper [15]. The labor market interpretation is very natural in both of the scenarios we consider: the "weight" of a worker corresponds to his productivity level (e.g., the number of images that an Amazon Turk user can tag in an hour). Similarly, the workers can be particularly influential members in a social network.

It is therefore of great interest to study the conditions under which Walrasian equilibria exist. Kelso and Crawford [11] provide sufficient conditions for the existence of a WEQ—namely, that all buyers' value functions hold a technical property called *gross substitutes* (See Appendix B.1 for a formal definition).

A characterization of all games that have a WEQ (see Section 3.2) was given in a classic paper by Bikhchandani and Mamer [2], which also commented that they *"...have been less successful at identifying sufficient conditions [for existence of a WEQ] on agents' preferences"* (p. 403).

Our contribution Our primary goal is to characterize the conditions under which a WEQ exists. In particular, we are interested in extending the results of Kelso and Crawford on existence of equilibrium for buyers with value functions based on capacities or social connections, two important cases that violate the gross-substitutes condition. We start by formally defining new classes of valuation functions inspired by the examples in the introduction.

In Section 4 we study games with additive capacities (weights) and show that an equilibrium always exists with two buyers. In games with arbitrary capacities and more than two buyers, a WEQ may not exist. We show through analysis and simulations that particular heuristic prices are usually quite stable, and that a WEQ exists in almost every instance. Further, simple heuristics can be applied to find prices that are usually almost market-clearing.

In the social network model, studied in Section 5, we prove the existence of equilibrium for two buyers when the network is sufficiently sparse. We also demonstrate that our conditions for existence are minimal, in the sense that by relaxing any of them we can construct a game with no WEQ. Results are summarized in Table 1 (page 12).

Due to space constraints and to allow continuous reading, some content has been deferred to the appendix. Omitted proofs are available in Appendix A. Examples of games used in the paper can be found in Appendix B. The details of our experimental setting appear in Appendix C.

2 Preliminaries

We denote vectors by bold lowercase letters, e.g. $\mathbf{a} = (a_1, a_2, \ldots)$. Sets are typically denoted by capital letters, e.g. $B = \{1, 2, \ldots\}$. When $\mathbf{a}$ is a vector of indexed elements and B is a set of indices, we use the shorthand notation $a(B) = \sum_{b \in B} a_b$.

Combinatorial auctions We consider a set K of k *items*, and a set N of n *buyers*, where $n \geq 2$. Every buyer $i \in N$ is associated with a non-decreasing value function $v_i : 2^K \to \mathbb{R}_+$, where $v_i(\emptyset) = 0$.

Given an auction $G = \langle K, N, (v_i)_{i \in N} \rangle$, a *valid outcome* is a pair $(P, \mathbf{x})$, where $P = (S_0, S_1, S_2, \ldots, S_n)$ is a partition of K among the buyers, where S_0 contains unallocated items. $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is a price vector.

$(P, \mathbf{x})$ is a *Walrasian equilibrium* (WEQ), if for every buyer, $S_i = \text{argmax}_{S \subseteq K}(v(S) - x(S))$, and unsold items have price 0. That is, market is cleared and the bundle of every buyer maximizes her utility under prices $\mathbf{x}$. The profit (utility) of buyer i is denoted by $r_i(S_i) = v_i(S_i) - x(S_i)$. The *social welfare* of a partition P is the sum of buyers values, i.e. $\sum_{i \in N} v_i(S_i)$. Note that the social welfare does not depend on prices.

2.1 Value functions

We use the notation m_i for the marginal value of an item to buyer i. For every $S \subseteq K$ and $j \notin S$, $m_i(j, S) = v_i(S \cup \{j\}) - v_i(S)$.

v_i is *submodular* (a.k.a. *concave*), if $v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + v_i(T)$ for all $S, T \subseteq K$. It is *strictly submodular* if the inequality is strict. Equivalently, v_i is submodular if the marginal contribution is nonincreasing. That is, for all $T \subseteq S$ and $j \notin S$, $m_i(j, T) \geq m_i(j, S)$. If the [strict] inequality holds only when S, T are disjoint, then we say that v_i is *[strictly] subadditive*. All value functions studied in this paper are submodular, i.e. have decreasing marginal returns. This assumption is standard in the economics literature [4,13].

Types We say that items j and j' are of the same *type* if all buyers are indifferent between them. That is, if for all $i \in N$, $S \subseteq K \setminus \{j, j'\}$, $v_i(S \cup \{j\}) = v_i(S \cup \{j'\})$. Similarly, we say that buyers i and i' have the same type, if $v_i \equiv v_{i'}$. Games where all buyers are of the same type are called *symmetric* games. In the simplest form of games, called *homogeneous games*, all items are of the same type. In such games each $v_i : [k] \to \mathbb{R}_+$ is a function of the *number* of the items used by buyer i, i.e. $v_i(S) = v_i(|S|)$.

Weighted games The primary type of value functions we consider in this work is based on capacities, or *weights*. Every item has some predefined nonnegative integer weight w_j, and the value of a set S depends only on its total weight. Thus each v_i is a function $v_i : \mathbb{N} \to \mathbb{R}_+$, where $v_i(S) = v_i\left(\sum_{j \in S} w_j\right)$.

A game where all value functions are weight-based is called a *weighted* game. Homogeneous games are a special case of weighted games, where all items have the same weight (w.l.o.g. weight 1).

A partition $P = (S_0, S_1, \ldots, S_n)$ of items in a weighted game is *balanced*, if the total weight of items that each buyer gets is the same, i.e. $w(S_i) = w(S_{i'})$ for all $i, i' \in N$. A partition is *almost balanced* if the total weight of any S_i and $S_{i'}$ (except S_0) differ by at most 1.

Influence in social networks Another value function we consider is inspired by social networks. Every social network then induces a game, where the items are some particular set of influential nodes (news sites, blogs, influential writers, etc.), and buyers are firms trying to purchase or hire nodes with *maximal influence* in the network (following [7,12]).

Synergy graphs Simple synergies between items can be represented by a weighted undirected graph, where every vertex is an item, and the value of a set is the total weight of edges linked to vertices in the set. This includes edges between vertices in the set, and edges between these vertices and external vertices.

All classes described above—except homogenous valuations—may violate the gross-substitute condition. See Appendix B.1 for details.

Fair pricing We also consider other natural requirements from prices. A price vector $\mathbf{x} = (x_1, \ldots, x_k)$ is *fair* if for every pair j, j' of the same type, it holds that $x_j = x_{j'}$. In *weighted games*, a price vector $\mathbf{x}$ is *proportional*, if for all j, j' it holds that $\frac{x_j}{x_{j'}} = \frac{w_j}{w_{j'}}$. Any proportional price vector is fair.

Note that we do not externally enforce fairness nor proportionality.

3 Properties of equilibrium outcomes

A WEQ has many desired properties, which motivate the search for such outcomes. In addition, some of these properties will be used as tools in the next sections to prove existence and non-existence of WEQ in various games.

The first property, which follows directly from the definition of WEQ, means that a buyer does not prefer a bundle with one additional item or one item less.

Lemma 1 *Let $(P, \mathbf{x})$ be a WEQ outcome in game G. Then (1) for all $i \in N, j \in S_i$, $x_j \leq m_i(j, S_i \setminus \{j\})$; and (2) for any $i' \neq i \in N, j \in S_i$, $x_j \geq m_{i'}(j, S_{i'})$.*

3.1 Individual rationality, Fairness and Envy freeness

We next present three simple observations (phrased as lemmas), showing that a WEQ outcome is always individually rational, envy free, w.l.o.g. fair.

Lemma 2 (Individual rationality) *Let $(P, \mathbf{x})$ be a WEQ outcome in game G, then (1) $x_j \geq 0$ for all $j \in K$; and (2) $v(S_i) - x(S_i) \geq 0$ for all $i \in N$.*

We say that buyer i *envies* buyer t in an outcome $(P, \mathbf{x})$, if i wants to trade items and payments. That is, if $v_i(S_t) - \sum_{j \in S_t} x_j > v_i(S_i) - \sum_{j \in S_i} x_j = r_i(P, \mathbf{x})$. An outcome is *envy-free* if no buyer envies any other buyer.

Lemma 3 (Envy freeness) *Let $(P, \mathbf{x})$ be a WEQ outcome in game G. Then $(P, \mathbf{x})$ is envy-free.*

The above holds because an envious buyer i can always forgo its current items S_i and buy S_t instead. The proof of Lemma 4 appears in the appendix.

Lemma 4 (Fairness) *If $(P, \mathbf{x})$ is a WEQ in game G, then there is a* fair *outcome $(P, \mathbf{x}^*)$ that is a WEQ in G, where the profit of each buyer remains the same.*

3.2 LP formulation and the welfare theorems

Computational schemes for representing combinatorial markets and to solve them, as well as the properties of Walrasian equilibria, have been thoroughly studied (see Blumrosen and Nisan [3] for a detailed review). Briefly, there is a standard Integer Linear Program, denoted $ILP(G)$, whose solutions describe the optimal partitions in the game. The *Linear Program Relaxation* of $ILP(G)$ is denoted by $LPR(G)$.

Two fundamental properties state that every WEQ is efficient, and characterize the conditions for existence [2,3].

First welfare theorem (FWT). *Every Walrasian equilibrium, if exists, is optimal in terms of the social welfare.*

Second welfare theorem (SWT). *A Walrasian equilibrium exists if and only if the integrality gap of $ILP(G)$ is zero, i.e. if the solution quality of $ILP(G)$ and $LPR(G)$ is the same.*

Moreover, in such cases it is known that the solutions to the dual linear program of $LPR(G)$ yield the market clearing prices $\mathbf{x}$, which in the labor market interpretation represent workers' salaries under WEQ.

4 Weighted games

The first class of value functions we study is based on capacities, or weights. Recall that a weight based value function $v : \mathbb{N} \to \mathbb{R}_+$ is a subadditive function, which maps the total weight of a set of items $w(S)$ to utility. This implies submodularity of $v(S)$ (see Lemma 1 in appendix). We sometimes write v as a vector of $w(K) + 1$ entries $(v(0), v(1), \ldots, v(w(K)))$, where by convention, the first entry $v(0) = 0$.

Before continuing to our existence results, we observe that without the subadditivity assumptions, weighted games may not posses a WEQ even in a most simple scenario (see Appendix B.2 for details).

Homogeneous games Suppose that all items have unit weight. Kelso and Crawford [11] show that in such games the core is always non-empty. It follows that a WEQ always exists. Moreover, when buyers are symmetric, then such a WEQ has a particularly simple form.

Let $q = \lfloor k/n \rfloor$ and $\delta = v(q + 1) - v(q)$. By FWT, in any WEQ every buyer has either q or $q + 1$ items. Also, by Lemma 1, there is a WEQ where the price of every item is δ.

4.1 The case of two buyers

Consider a symmetric weighted game with two buyers and only two items $K = \{h, l\}$, where $w_h \geq w_l$. This simple case can be solved as follows. Let $x_l = v(w_l + w_h) - v(w_h)$, and $x_h = v(w_l + w_h) - v(w_l)$, i.e. we set the payment of each item to be its own marginal contribution to the set K. Then, for a partition P where $|S_1| = |S_2| = 1$, $(P, \mathbf{x})$ is a WEQ.

However, the described prices are not necessarily proportional. A proportional outcome, which is also a WEQ, would be to pay $x_j = v(w_j)$ for each item $j \in \{l, h\}$. If v is strictly subadditive, then there are also other proportional WEQs where the buyers keep some of the profit. We next show that we can always find a proportional WEQ for two buyers and any number of items. Note that we only require that each value function will be subadditive.

Our main result for the weighted setting is as follows.

Theorem 5. *Let $G = \langle K, N, \mathbf{w}, v_1, v_2 \rangle$ be a weighted game with two buyers, then G admits a proportional WEQ.*

The proof hinges on the idea of computing the marginal value of a *unit of weight*. However in the general case this is an evasive notion that requires a nontrivial case analysis (see Appendix A.4). We bring here a simplified proof of the symmetric unbalanced case, and explain the main ideas of the general case.

Proof (sketch). Indeed, let $P = (S_1^*, S_2^*)$ be an optimal partition. For the symmetric case, denote $H = w(S_1^*)$, $L = w(S_2^*)$, and suppose that $L < H$. By optimality, the gap $H - L$ is minimal. We define $\delta = \frac{v(H)-v(L)}{H-L}$, and argue that it induces a WEQ $(P, \mathbf{x})$, where $\mathbf{x} = \delta \cdot \mathbf{w}$. Indeed, the profit of a buyer with items of total weight q is $r(q) = v(q) - \delta q$. Since $v(q)$ is concave, $r(q)$ is also concave, with maximum in $q^* \in \{\lfloor W/2 \rfloor, \lceil W/2 \rceil\}$ (since δ is between $m(\lfloor SW/2 \rfloor, 1)$ and $m(\lceil SW/2 \rceil, 1)$). Also, in P we have

$$\begin{aligned}(H-L)r_1 &= (H-L)(v(H) - H\delta) = (H-L)v(H) - H(v(H) - v(L)) \\ &= H \cdot v(L) - L \cdot v(H) = (H-L)(v(L) - L\delta) = (H-L)r_2\end{aligned}$$

This means that $r_1 = r_2$, and by minimality of the gap, buyers cannot get closer to the theoretical optimal profit $r(q^*)$: for any set of items S'_i, we have $|w(S'_i) - q^*| \geq |w(S_i^*) - q^*|$, and thus $r_i(q'_i) \leq r(q_i)$.

For the general case, assume that both S_1^*, S_2^* are non-empty (otherwise the solution is fairly easy). Let $q_i = w(S_i^*)$, $\check{w}_i = \min_{j \in S_i^*} w_j$. Since value functions are different, the marginal contribution of each unit of weight to each buyer may also be different. We therefore replace the "slope" δ with four different quantities y_1, z_1, y_2, z_2. We denote by y_i, z_i the *normalized* marginal value of the "lightest" item below and above the threshold q_i, respectively. Formally, $y_i = \frac{1}{\check{w}_i}(v_i(q_i) - v_i(q_i - \check{w}_i))$, and $z_i = \frac{1}{\check{w}_{-i}}(v_i(q_i + \check{w}_{-i}) - v_i(q_i))$.

Next, set $z_i^* = \max_{d \geq 1} \left\{ \frac{v_i(q_i+d) - v_i(q_i)}{d} \leq y_{-i} \right\}$. Intuitively, z_i^* is the *closest slope* to y_{-i} that we can get by adding weight above the threshold q_i. By its definition, $y_i, y_{-i} \geq z_i^* \geq z_i$.

To complete the construction, we use z_i^* as a proxy for the marginal value of a unit. We define $\delta = \max\{z_1^*, z_2^*\}$, and set the (proportional) prices $x_j = \delta w_j$. The proof proceeds by showing that to improve the profit of a buyer, the total weight of acquired items changes by less than $\check{w}_i$. On the other hand, we show that if such a small change is possible, a better partition than P can be constructed, which is a contradiction to FWT. □

4.2 More than two buyers

A question that naturally arises is whether we can generalize Theorem 5, i.e. prove that a WEQ always exists for any number of buyers, perhaps even with the additional requirement of proportionality. A result by Gul and Staccheti [8] shows that whenever there is a buyer whose value function violates gross-substitutes, it is possible to construct an example (with additional unit-demand buyers) where an equilibrium does not exist. While we cannot apply their construction directly, it gives little hope that a WEQ exists in the general case of weighted games. -substitutes (in the general case), then we know by that there must be games where a Walrasian equilibrium (and thus WEQ) does not exist. However, the construction in [8] uses an unbounded number of buyers/firms. We may still hope that with few firms equilibrium (maybe even a proportional one) does exits.

Indeed, Proposition 6 below shows that a WEQ is not guaranteed for multiple buyers. We first show nonexistence of *proportional* WEQ.

Example 1. Consider a symmetric game G^* where $w_1 = 5, w_2 = 6, w_3 = 7$, and $v(5) = 5, v(6) = 6, v(7) = 6$. Clearly in the optimal partition each buyer has a single item. Suppose there is some proportional WEQ, where $x_j = \delta \cdot w_j$ for all j. Then by Envy-freeness (Lemma 3), all three buyers make the same profit, i.e.

$$r_1 = r_2 = r_3 \Rightarrow v(5) - 5\delta \overset{(\#1)}{=} v(6) - 6\delta \overset{(\#2)}{=} v(7) - 7\delta.$$

By Eq. (#1), $\delta = v(6) - v(5) = 1$, whereas by (#2), $\delta = v(7) - v(6) = 0$. A contradiction. ♢

Proposition 6 *For any $n \geq 3$, there is a symmetric weighted game with n buyers that does not have a WEQ at all.*

An example proving the proposition appears in Appendix B.2We hereby construct an example with $n = 4$ buyers.

Example 2. Our game G has 9 items in total, where the weights are $(2, 2, 2, 2, 2, 3, 3, 3, 5)$. We define $v(w) = \min\{w, 6\}$. We claim that the optimal partition must be either $P_1 = (\{5\}, \{3, 2, 2\}, \{3, 3\}, \{2, 2, 2\})$ or $P_2 = (\{3, 2\}, \{5, 2\}, \{3, 3\}, \{2, 2, 2\})$ (up to permutations of items of the same type). See Fig. 1.

By the shape of v, the optimal partition minimizes $|\{i \in N : w(S_i) < w'\}|$ for $w' = 1$, then for $w' = 2, 3, 4$, etc. Indeed, the total weight is 24, but it cannot be divided in a balanced way. Thus we get that every optimal partition has total weights of $(5, 7, 6, 6)$. P_1 and P_2 are the only ways to construct such a partition. Next, we want to assign payments.

By Lemma 4, we can set a uniform payment to each type of item, thus we should determine the values of x_2, x_3 and x_5. Now, by Envy-freeness (Lemma 3), $r_3 = r_4$, and thus $0 = r_3 - r_4 = (v(6) - 3x_2) - (v(6) - 2x_3) = 3x_2 - 2x_3$. Similarly, buyers 1 and 2 can trade the 5-item for the set $\{2, 3\}$. Thus $x_5 = x_2 + x_3$. We get that $\mathbf{x}$ must be a proportional payment vector, where $\mathbf{x} = \delta \mathbf{w}$ for some unit payoff δ. However, buyers 1,2 and 3 have a total weight of 5,7 and 6 respectively, exactly as in the game G^* in Example 1. As we show above, such a proportional payment vector cannot be stable.[3] $\diamondsuit$

We argue that while examples without WEQ exist even for symmetric buyers, proportional payoffs can be derived from an optimal partition using simple and heuristics. We show that the stability of these payoffs is inversely related to the gap between the total weight buyers acquire, supporting this claim with a formal argument and an experimental study. First, we prove that if there exists a partition that is *almost balanced*, then together with the proportional payoffs it forms a WEQ, regardless of the value function v. Note that when weights are small integers, there is typically an almost-balanced partition.[4] We then generate weighted games from a simple distribution, showing that the incentive of buyers to deviate grows as the partition becomes less balanced.

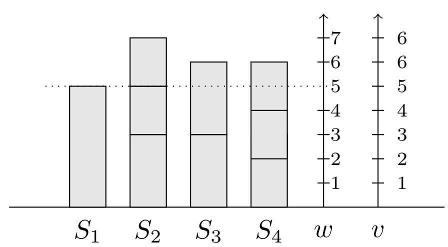


Fig. 1: The optimal partition P_1.

Proposition 7 *Let $G = \langle K, N, \mathbf{w}, v\rangle$ be a weighted game with n symmetric buyers. (a) If there is an almost-balanced partition of $\mathbf{w}$, then G admits a proportional WEQ. (b) symmetry is a necessary condition.*

The proof of part (a) is in Appendix A.4. Part (b) is demonstrated by Example 7 in Appendix B.2, which shows that a WEQ may not exist in asymmetric games, even when there is a fully balanced partition.

Experimental study of stability with three buyers The full details of our simulations appear in Appendix C. This is a short summary of our results.

- We generated over 6000 random instances of weighted games with three symmetric buyers and up to 14 items. Although less than a third of the instances admitted an almost balanced partition, in all but 4 instances a WEQ was found.
- In most generated instances, there is a proportional (in particular fair) WEQ that can be found using simple heuristics, even though a nearly balanced partition may not exist. This is by generalizing the notion of the "unit payoff" δ from the two agents case.
- By Proposition 7, this heuristic solution is always a WEQ when the gap $d = \max_i w(S_i) - \min_i w(S_i)$ is at most 1 (i.e. P^* is almost-balanced). The experimental results show that the heuristic solution induces an approximate WEQ, where the incentive to deviate grows with d.

[3] The integrality gap of this example is $\frac{LPR(G)}{ILP(G)} = \frac{23.5}{23} = \frac{47}{46}$.

[4] For example, if there are at least $n \cdot \max_j w_j$ items with weight 1, then an almost-balanced partition *must* exist.

– In more than half of the total tested instances, the maximal gain is 0, meaning that $(P^*, \mathbf{x}^*)$ is a WEQ. Further, only in 6% of the generated instances there was a buyer that could improve her profit by more than 5%.

We do not claim that our random sample is characteristic of every possible scenario. We do feel however that these experimental results strengthen the conclusion, that when faced with a given weighted game, then (a) it is very likely to have a WEQ; and (b) heuristic proportional payoffs will usually be quite stable.

Computation The problem of computing P^* in weighted games is NP-hard even for two firms (a trivial reduction from PARTITION). However when there are few firms and few worker types the following approach can be applied. First find P^* via a dynamic algorithm similar to the one used for KNAPSACK (A similar dynamic approach to solve combinatorial auctions is given in [10]). Once P^* is found, a simple linear program can be written where constraints explicitly rule out any possible deviation. While the size of this LP may be exponential in general, there is a limited number of constraints once the number of firms and worker types is fixed. Thus for the games with three firms described in Section 4.2 we can efficiently verify whether a WEQ exists.

Computing the heuristic payoff vector x^* does not even require us to solve any LP—it can be computed directly given P^* and the vector of weights.

5 Games over a social network

5.1 Network model

Consider a social network $H = \langle V, E_H \rangle$ (a directed graph), and a subset of "influencers" $K \subseteq V$. Given some diffusion scheme in the network, every set $S \subseteq K$ influences some portion of the nodes V, whose size is denoted by $I_H(S)$.

Given such a social network H and a set of buyers (firms) N, we define a symmetric game where buyers bid over a set of influential nodes K (the items), trying to advertise to as many people (all nodes of H) as possible. The value function of every buyer is thus $v_i(S) = v(S) = I_H(S)$.[5]

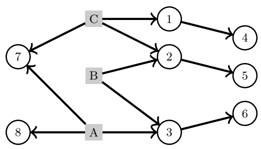
We apply one of the most widely known diffusion schemes, called the *independent cascade model*, which has been suggested by Goldenberg et al. [7] and further promoted in [12]. We briefly describe the diffusion process.

In the Independent Cascade model, every edge in the network H has an attached *probability* $p_{u,u'}$. Once a node u is activated, it tries to activate once each neighbor u', and succeeds w.p. of $p_{u,u'}$, independently of the state of any other node. Once a node is activated, it remains active. The influence of a set S, denoted by $I_H(S)$, is the expected number of nodes that end up as active if we activate the set S.

$$v_i(S) = v(S) = I_H(S) = \sum_{u \in V} pr(u \text{ is activated} | S \text{ is active}).$$

[5] We can define asymmetric games by using a different network H_i for each buyer.

(a) The network H



(b) The matrix M

	A	B	C
A	1	2	1
B	2	0	2
C	1	2	2
total	4	4	5

(c) The synergy graph M

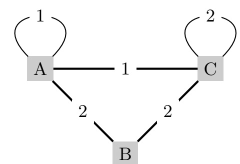


Fig. 2: An example of a sparse network / synergy graph with three items A, B and C. The maximal cut in M, which is also the optimal partition, is $P^* = (\{A, C\}, \{B\})$. Then $SW(P^*) = v(\{A, C\}) + v(\{B\}) = 8 + 4 = 12$.

This is equivalent to summing the probabilities of all percolations (subgraphs of H) in which there is a directed path from some node in S to u. We should note that $I_H(S)$ is a submodular function [12]. However, $I_H(S)$ does not necessarily hold the gross-substitute condition (see Appendix B.1)

While the independent cascade model seems to be more powerful than the weighted model studied in the previous section, it turns out that no model generalizes the other. Indeed, Example 8 in Appendix B.3 demonstrates a weighted value function over 3 homogeneous items, that cannot be represented as the influence in any graph H. Therefore, weighted value functions and influence value functions are two different classes of submodular valuations. A natural question is whether a WEQ always exists with two buyers in the independent cascade model. Unfortunately, the answer is negative in the general case, but we can prove existence is a special case of interest.

We say that a network $H = \langle V, E_H \rangle$ is t-sparse (w.r.t the set $K \subseteq V$), if every node $u \in V$ can be reached by at most t items from K.

Intuitively, t-sparsity means that the influence cones of different items hardly intersect. A 1-sparse network means that the cones of influence are pairwise mutually exclusive and thus that the influence is completely additive (a trivial case). A 2-sparse network means that two cones may intersect, but never three or more. A k-sparse network puts us back in the general case. In order to analyze sparse networks, it will be useful to formally define synergy graphs.

5.2 Synergy graphs and sparse social networks

A *synergy graph* is an undirected graph $M = \langle K, E_M \rangle$ with non-negative weights, where self-edges are allowed. It can thus be represented as a symmetric matrix, which is also denoted by M. Every synergy graph M induces a value function v_M, where the value of a subset $S \subseteq K$ is the sum of weights of edges between items in the subset (including self edges), and edges going outside the subset. That is,

$$v_M(S) = \sum_{j \in S} M(j, j) + \sum_{j, j' \in S, j < j'} M(j, j') + \sum_{j \in S} \sum_{j'' \notin S} M(j, j'').$$

Lemma 8 *A value function v over a set of items K can be described by a synergy graph if and only if it can be described as the influence in a 2-sparse network. I.e. there is M s.t. $v = v_M$ iff there is H s.t. $v = I_H$.*

As an intuition, the mapping is constructed s.t. $M(j, j')$ equals the expected number of nodes that are influenced by *both* item j and item j' (i.e. the intersection of their influence cones). Fig. 2 demonstrates a network H and its corresponding synergy graph M. Our main positive result in the network model is the following.

Theorem 9. *Let $G = \langle K, N, v_M \rangle$ be a symmetric game with 2 buyers over a synergy graph M (or, equivalently, a 2-sparse network). Then G has a WEQ.*

The outline of the proof is as follows. The maximal cut in M is the optimal partition in G, and we set the payoff of each item j to be the average of her total influence and her exclusive influence (i.e., $x_j = \frac{v(\{j\})+M(j,j)}{2}$), and show that $\mathbf{x}$ induce a WEQ.

Unfortunately, if we relax any of the conditions in Theorem 9 (symmetry, number of buyers, or sparsity) then the existence of a WEQ is no longer guaranteed. See Appendix A.5 for detailed proofs.

6 Discussion

We considered combinatorial auctions without gross substitutes, and showed that a Walrasian equilibrium (or, equivalently, a pure Nash equilibrium of the first price auction) is guaranteed to exist under certain restrictions, with a special focus on the case of few buyers.

In games based on capacities (weights) with subadditive production, we proved that a WEQ must exist if there are two buyers or if there is an almost balanced partition, and demonstrated empirically that at least for three symmetric buyers it often exists even when neither condition applies.

Finally, we showed that a WEQ always exists in a particular case of the network model, when the network is sparse and featuring two identical competing buyers Unfortunately, there may not be a WEQ if any of these conditions is violated. Our results are summarized in Table 1.

Related work and implications Following Kelso and Crawford's seminal paper, multiple authors studied the implications of the gross substitutes restriction. In particular, Lehmann et al. showed that only a tiny fraction of all submodular value functions are gross substitutes [13]. Further, Gul and Stacchetti proved that for *any* value function without this property (and with additional unit-demand buyers), it is possible to construct a market with no WEQ [8]. However, the construction by Gul and Stacchetti used an unbounded number of buyers/firms (one for every item/worker).

Some recent papers show that despite the Gul and Stacchetti negative result, the gross substitutes condition can be slightly relaxed. This is either by allowing very restricted complementarities [16], or by introducing a modified version of

<table>
<tr><th rowspan="2"># of buyers</th><th rowspan="2">homogeneous games</th><th colspan="2">weighted games</th><th colspan="2">network games</th></tr>
<tr><th>near-balanced</th><th>any</th><th>2-sparse (syn. graphs)</th><th>any</th></tr>
<tr><td>$n = 2$, symm.</td><td rowspan="4">V (Kelso & Crawford [11])</td><td rowspan="2">V ($\Leftarrow$)</td><td rowspan="2">V (T. 5)</td><td>V (T. 9)</td><td>X (P. 12)</td></tr>
<tr><td>$n = 2$, asym.</td><td>X (P. 11)</td><td rowspan="3">X ($\Rightarrow, \Downarrow$)</td></tr>
<tr><td>$n \geq 3$, symm.</td><td>V (P. 7a)</td><td>X (P. 6)</td><td>X (P. 10)</td></tr>
<tr><td>$n \geq 3$, asym.</td><td>X (P. 7b)</td><td>X ($\Downarrow$)</td><td>X ($\Downarrow$)</td></tr>
</table>

Table 1: Existence results for the cases where a WEQ is guaranteed to exist are marked with V. Cases marked with X mean that there are instances where *no WEQ* exists.

unit-demand [1]. Our results demonstrate that there are entire natural classes of value functions where items are neither substitutes not complements, yet existence of equilibrium can be guaranteed if the number of firms is low (and in particular for two firms).

Integrality gap Dobzinski and Schapira [5] study upper and lower bounds on the integrality gap of various submodular value functions. The integrality gap is an important factor in the construction of efficient approximation algorithms that find optimal allocations in combinatorial auctions. For general submodular functions, they show that the (maximal) integrality gap is between $\frac{8}{7}$ and $\frac{4}{3}$. Since our construction implements theirs, we get that the lower bound of $\frac{8}{7}$ still applies for value functions based on sparse networks. As for weighted functions, the integrality gap of Example 2 is $\frac{47}{46}$, which gives us a lower bound. An interesting challenge is to find the maximal integrality gap of instances that correspond to weighted or network games. In particular, it is an open question whether tighter bounds can be proved compared to general submodular functions.

Possible extensions While most real world networks are not 2-sparse, many of them demonstrate other forms of sparsity [6,14]. We would like to develop a heuristic solution similar to the one suggested for weighted functions, and test whether it provides us with an exact or approximate WEQ in a more general class of sparse networks.

Another direction concerns the context of the competition. In our network model the firms compete only for the services of the influencing nodes, in separation from other arenas in which they might affect one another. However, if companies are also competing for market share, then there are externalities: users in the network that are exposed to an ad of one company may become less likely to purchase the products of its competitor. Refining the model can contribute to the literature on competitive diffusion in networks (see e.g. [9]).

Other natural directions would be to study the relation between the integrality gap of various classes of valuation functions (such as those studied in [5,3]), and the number of buyers. In particular, it would be interesting to find new classes of valuations where the integrality gap with few buyers is trivial—and thus a Walrasian equilibrium exists.

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A Proofs

We provide proofs for propositions in the order they appear in the main text. Proofs for each Section X appear in Appendix A.X.

A.3 Properties

Lemma 1. *Let $(P, \mathbf{x})$ be a WEQ outcome in game G. Then for all $i \in N$ and $j \in S_i$,*

1. $x_j \leq m_i(j, S_i \setminus \{j\})$.
2. *For any* $i' \neq i$, $x_j \geq m_{i'}(j, S_{i'})$.

Proof. For the first part, note that if i pays for j more than its current marginal value, then i can gain by forgoing j. For the second part, if i pays for j less than its marginal value to i', then i' can gain by offering for j slightly more than its current price. All other payments remain unchanged, thus the new outcome will be identical to P, except item j will go to i' instead of i. □

Lemma 4. *If $(P, \mathbf{x})$ is a WEQ in game G, then there is a* fair *outcome $(P, \mathbf{x}^*)$ that is a WEQ in G, where the profit of each buyer remains the same.*

Proof. Recall that fairness means that items of the same type (i.e., they are complete substitutes) have the same price. Let $(P, \mathbf{x})$ be a WEQ, and suppose that items j, j' are of the same type, $j \in S_i$ and $j' \in S_{i'}$. If $i \neq i'$, then j, j' must have the same price. Otherwise, suppose that $x_j < x_{j'}$. buyer i' will forgo j' and purchase j instead,

If $i = i'$, then w.l.o.g. i can pay for j, j' the same by equalizing the payment. That is, there is another WEQ $(P, \mathbf{x}^*)$, where $x^*_j = x^*_{j'} = \frac{x_j + x_{j'}}{2}$. □

A.4 Weighted games

Lemma 1. *Let $u : \mathbb{N} \to \mathbb{R}^+$ be a subadditive function, and let $w : K \to \mathbb{N}$. Then $v(S) = u(w(S))$ is submodular.*

Proof. Let $S, T \subseteq K$, and suppose that $|S| \leq |T|$. Note first that $u(a+b) - u(a) \leq u(a'+b) - u(a')$ for $a > a'$. Thus

$$u(w(S) + w(T \setminus S)) - u(w(S)) \leq u(w(T \cap S) + w(T \setminus S)) - u(w(T \cap S)).$$

Then,

$$\begin{aligned} v(S \cap T) + v(S \cup T) &= u(w(S \cap T)) + u(w(S) + w(T \setminus S)) \\ &= u(w(S \cap T)) + u(w(S)) + u(w(S) + w(T \setminus S)) - u(w(S)) \\ &\leq u(w(S \cap T)) + u(w(S)) + u(w(S \cap T) + w(T \setminus S)) - u(w(S \cap T)) \\ &= u(w(S)) + u(w(S \cap T) + w(T \setminus S)) = u(w(S)) + u(w(T)) = v(S) + v(T). \end{aligned}$$

□

Theorem 5. *Let $G = \langle K, N, \mathbf{w}, v_1, v_2 \rangle$ be a weighted game with two buyers, then G admits a proportional WEQ.*

Proof. For any partition $P = (S_1, S_2)$ we denote the social welfare by $SW(P) = v_1(S_1) + v_2(S_2)$. Note that the input $w(S_1)$ contains all the required information to compute SW, thus we can write

$$SW(w(S_1)) \stackrel{\text{def.}}{=} v_1(w(S_1)) + v_2(W - w(S_1)).$$

Let $P^* = (S_1^*, S_2^*)$ be a partition maximizing the social welfare, and $q_i = w(S_i^*)$. Let $j_i' = \text{argmin}_{j \in S_i^*} w_j$ and $\check{w}_i = w_{j_i'}$.

We first handle the case where $q_i = 0$ for some i, that is where all items go to the same buyer. In such case let $w^* = \min_{j \in K} w_j$. By setting the unit payment to $\delta = v_i(w^*)/w^*$, and $\mathbf{x} = \delta \cdot \mathbf{w}$, we get a WEQ: the marginal value of every set S with weight $w = w(S) \geq w^*$ is at most $x(S) = \delta \cdot w$ to buyer i, and at least $x(S)$ to buyer $-i$.

Assume therefore that both S_1^*, S_2^* are non-empty. Since value functions are different, the marginal contribution of each unit of weight to each buyer may also be different. We therefore replace the "slope" δ with four different quantities y_1, z_1, y_2, z_2. We denote by y_i, z_i the *normalized* marginal value of the "lightest" item below and above the threshold q_i, respectively. Formally, $y_i = \frac{1}{\check{w}_i}(v_i(q_i) - v_i(q_i - \check{w}_i))$, and $z_i = \frac{1}{\check{w}_{-i}}(v_i(q_i + \check{w}_{-i}) - v_i(q_i))$. Now, by subadditivity, $y_i \geq z_i$ for both buyers. Also, $y_i \geq z_{-i}$, as otherwise there is a more efficient partition where $-i$ has item j_i'.

Next, set

$$d_i^* = \min\left\{d \geq 1 : \frac{v_i(q_i + d) - v_i(q_i)}{d} \leq y_{-i}\right\},$$

and

$$z_i^* = \frac{v_i(q_i + d_i^*) - v_i(q_i)}{d_i^*}.$$

Clearly, $y_i, y_{-i} \geq z_i^* \geq z_i$ and $d_i^* \leq w_{-i}$. For a graphical illustration, see Figure 3.

We define a unit payment $\delta = \max\{z_1^*, z_2^*\}$. In particular, $y_i \geq \delta \geq z_i^* \geq z_i$ for both buyers. We set the (proportional) prices $x_j = \delta w_j$.

The profit of buyer i is

$$r_i = v_i(S_i^*) - \sum_{j \in S_i} x_j = v_i(q_i) - \delta w(q_i).$$

Assume, toward a contradiction, that some buyer can deviate (w.l.o.g. buyer 1), and keep a total weight of $q' = w(S_1')$.

case 1. $q' < q_1$. Denote $d' = q_1 - q'$, and the marginal value (per unit) of the range $[q', q_1]$ by $t' = \frac{v_1(q_1) - v_1(q')}{d'}$ (see Figure 4).

Since buyer 1 increased the profit, t' must be strictly smaller than δ, as otherwise $r_1' = r_1 - d' \cdot t' + d' \cdot \delta < r_1$. Thus it is not possible that $d' \geq \check{w}_1$, as this would entail $t' \geq y_1 \geq \delta$.

On the other hand, $t' \geq z_1^*$ by subadditivity of v_1. As $\delta = \max\{z_1^*, z_2^*\}$, and $\delta > t' \geq z_1^*$, we get that $t' < \delta = z_2^*$. Therefore,

$$\begin{aligned}
SW(q') &= v_1(q') + v_2(W - q') \\
&= (v_1(q_1) - t' \cdot d') + v_2(q_2 + d') \\
&= (v_1(q_1) - t' \cdot d') + v_2(q_2) + d' \frac{v_2(q_2 + d') - v_2(q_2)}{d'} \\
&\geq (v_1(q_1) - t' \cdot d') + v_2(q_2) + d' \frac{v_2(q_2 + \check{w}_1) - v_2(q_2)}{\check{w}_1} && \text{(since } d' < \check{w}_1 \text{ and } v_2 \text{ is subadditive)} \\
&= (v_1(q_1) - t' \cdot d') + v_2(q_2) + z_2^* \cdot d' \\
&= (v_1(q_1) - t' \cdot d') + v_2(q_2) + \delta \cdot d' \\
&> v_1(q_1) + v_2(q_2) = SW(q_1),
\end{aligned}$$

in contradiction to the optimality of P.

case 2. $q' > q_1$. Similarly, we denote $d' = q' - q_1$, and it holds that d' is smaller than $\check{w}_2$. Also, $t = \frac{v_1(q')-v_1(q_1)}{d'}$ must hold $\delta < t' < y_1$.

Now, if $d' \geq d_1^*$, then we have $t' \leq z_1^*$. However this is impossible as $z_1^* \leq \delta < t'$. Thus $d' < d_1^*$, and by definition, $t' > y_2$. Then, similarly to case 1,

$$\begin{aligned}
SW(q') &= (v_1(q_1) + t' \cdot d') + v_2(q_2 - d') \\
&= SW(q_1) + t' \cdot d' - d' \frac{v_2(q_2) - v_2(q_2 - d')}{d'} \\
&\geq SW(q_1) + t' \cdot d' - d' \frac{v_2(q_2) - v_2(q_2 - \check{w}_2)}{\check{w}_2} && \text{(since } d' < \check{w}_2 \text{ and } v_2 \text{ is subadditive)} \\
&= SW(q_1) + d' \cdot t' - d' \cdot y_2 \\
&= SW(q_1) + d'(t' - y_2) > SW(q_1),
\end{aligned}$$

which again contradicts the optimality of P. □

Proposition 7. *Let $G = \langle K, N, \mathbf{w}, v \rangle$ be a weighted game with n symmetric buyers. (a) If there is an almost-balanced partition of $\mathbf{w}$, then G admits a proportional WEQ. (b) symmetry is a necessary condition.*

Proof (of Proposition 7a). Let $W = \sum_{j \in K} w_j$, and $q = \lfloor W/n \rfloor$. Set δ to be the marginal gain of a unit weight in an almost-balanced partition, that is, $\delta = v(q+1) - v(q)$. We set prices as $x_j = \delta \cdot w_j$.

Let P^* be an almost balanced partition, i.e. $w(S_i) = q$ or $q+1$ for all $i \in N$. We argue that $(P^*, \mathbf{x})$ is a WEQ.

The profit of each buyer is

$$r_i = v(S_i) - \delta \cdot w(S_i) = v(q) - \delta q = v(q+1) - \delta(q+1).$$

Suppose some buyer i deviates, and ends up with S_i' of weight q'. This means that the items in S_i' cost at least $\delta \cdot w(S_i') = \delta q'$ in total (as otherwise there is an item j with price less than x_j). Thus the profit of i becomes

$$r_i' \leq v(q') - \delta q' \leq r_i,$$

where the last inequality is since $\{q, q+1\} \subseteq \text{argmax}_{q' \in \mathbb{N}} v(q') - \delta q'$. □

Part (b) of the proposition follows from Example 7 in Appendix B.

A.5 Networks games

Lemma 8. *A value function v over a set of items K can be described by a synergy graph if and only if it can be described as the influence in a 2-sparse network. Formally, for any 2-sparse influence network H there is a synergy graph M s.t. $v_M = I_H$; and for any synergy graph M (with integer weights) there is a deterministic 2-sparse influence network H s.t. $I_H = v_M$.*

Note that since we can always multiply v by a positive constant, we can represent any synergy graph with *rational* weights as an influence network.

Proof. For simplicity, we will start by considering deterministic networks (where an edge means influence w.p. 1), and synergy graphs with integer weights.

"⇒" Given a network $H = \langle V, E_H \rangle$ and $K \subseteq V$, we define the following synergy matrix/graph of size $k \times k$. For every $j \in K, u \in V$, we say that j influences u if there is a directed path from j to u in H (recall that influence is deterministic).

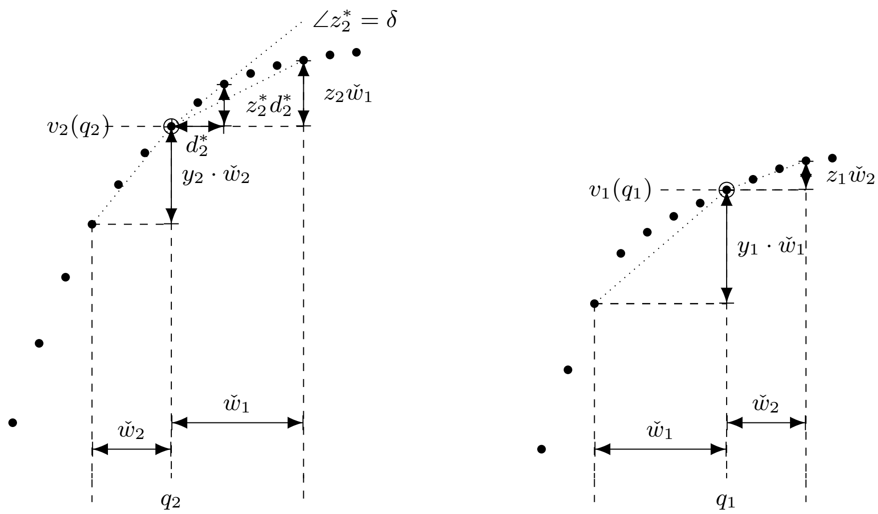

Fig. 3: The value functions of both buyers in the partition P^*. Here $\delta = \max\{z_1^*, z_2^*\} = z_2^*$.

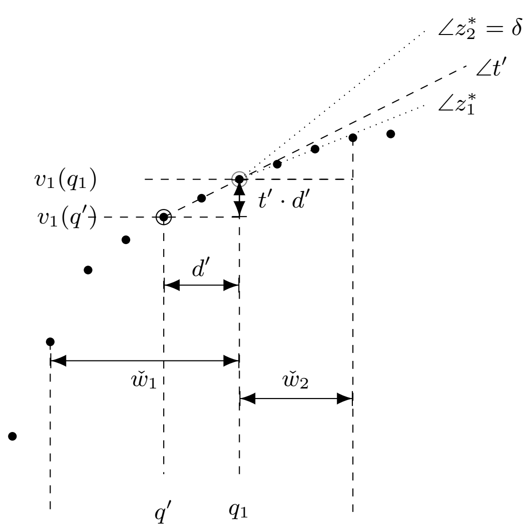

Fig. 4: A deviation of buyer 1 from the outcome $(P^*, \mathbf{x})$ to a bundle S'_1 where $q' = w(S'_1) = q_1 - 2$.

We set $M(j,j')$ as the number of nodes influenced by *both* j and j'. In particular, the weight of the self-edge $M(j,j)$ is the number of nodes that only j influences. Since H is 2-sparse, every node is influenced by exactly 1 or 2 items, and thus $I_H(j) = M(j,j) + \sum_{j'\neq j} M(j,j') = \sum_{j'\in K} M(j,j')$. I.e., the sum of row (or column) j in the matrix M. For a set S, we need to sum the rows of $j \in S$, and then remove all nodes $u \in V$ that have been counted twice, i.e. cells $M(j,j')$ s.t. $j,j' \in S$ and $j \neq j'$. Formally,

$$I_H(S) = \sum_{j\in S}\left(M(j,j) + \frac{1}{2}\sum_{j'\in S\setminus\{j\}} M(j,j') + \sum_{j''\in K\setminus\{S\}} M(j,j'')\right) = v_M(S).$$

Next, we extend the proof to the case of general probabilities on edges. For $j \neq j'$, suppose that a certain node $u \in V$ is influenced by j w.p. p_j, and by j' w.p. $p_{j'}$. Then we add $p_j p_{j'}$ to $M(j,j')$ for each such node. Finally, we set $M(j,j) = I_H(j) - \sum_{j'\neq j} M(j,j')$. Note that $M(j,j)$ is exactly the marginal contribution of j, i.e. the fraction of nodes that only j influences. Thus $I_H(S) = v_M(S)$ as in the equation above.

"$\Leftarrow$" Given a synergy graph/matrix M, we define a set of nodes $V_{j,j'}$ for every pair $j,j' \in K$, whose size is $M(j,j')$. Let $V = K \cup \bigcup_{j,j'} V_{j,j'}$. We connect an edge in H from every $j \in K$, to every $u \in \bigcup_{j'\in K} V_{j,j'}$. Thus we have once again $I_H(j) = \sum_{j'\in K} M(j,j') = v_M(j)$, and similarly for sets as in the equation above. □

By applying both directions of the lemma, we get as a corollary that any 2-sparse network with rational probabilities can be replaced by a deterministic one.

Theorem 9. *Let $G = \langle K, N, v_M\rangle$ be a symmetric game with 2 buyers over a synergy graph M. Then G has a WEQ.*

Proof. Every partition of K to the two buyers, is a *cut* in M. Let S_1, S_2 be such a cut. By definition (after rearranging terms), $v(S) = v_M(S)$ equals to

$$\sum_{j\in S}\left(M(j,j) + \frac{1}{2}\sum_{j'\in S\setminus\{j\}} M(j,j') + \sum_{j''\in K\setminus\{S\}} M(j,j'')\right).$$

The sum over the rightmost term is the weight of the cut $(S, K \setminus \{S\})$.

It is therefore easy to see that social welfare is maximized by taking the maximal cut in M. We still need to set the prices properly to induce stability.

Let $P^* = (S_1^*, S_2^*)$ be some maximal cut in M, and let $j \in S_i^*$. We set $x_j = \frac{1}{2}(v(j) + M(j,j)) = M(j,j) + \frac{1}{2}\sum_{j'\in K\setminus\{j\}} M(j,j')$. We claim that the price vector $\mathbf{x}$, with S_1^*, S_2^* is a WEQ.

Indeed, the profit of either buyer is

$$\begin{aligned}
r_i = v(S_i^*) - \sum_{j\in S_i^*} x_j &= v(S_i^*) - \sum_{j\in S_i^*}\left(M(j,j) + \frac{1}{2}\sum_{j'\in K\setminus\{j\}} M(j,j')\right)\\
&= \sum_{j\in S_i^*}\left(M(j,j) + \frac{1}{2}\sum_{j'\in S_i^*\setminus\{j\}} M(j,j') + \sum_{j''\in S_{-i}^*} M(j,j'')\right) - \\
&\quad \sum_{j\in S_i^*}\left(M(j,j) + \frac{1}{2}\sum_{j'\in S_i^*\setminus\{j\}} M(j,j') + \frac{1}{2}\sum_{j''\in S_{-i}^*} M(j,j'')\right)\\
&= \frac{1}{2}\sum_{j\in S_i^*}\sum_{j'\in S_{-i}^*} M(j,j') = \frac{1}{2}\sum_{j\in S_1^*}\sum_{j'\in S_2^*} M(j,j'),
\end{aligned}$$

i.e. half the weight of the cut P^* (in particular $r_1 = r_2$).

To see that S_1^* is the optimal package at prices $\mathbf{x}$, observe that for any set of items S_1':

$$\begin{aligned}
r_1' &= v(S_1') - \sum_{j \in S_1'} x_j' \le v(S_1') - \sum_{j \in S_1'} x_j \\
&= \sum_{j \in S_1'} \left(M(j,j) + \frac{1}{2} \sum_{j' \in S_1' \setminus \{j\}} M(j,j') + \sum_{j'' \in S_2} M(j,j'') \right) - \\
&\qquad \sum_{j \in S_1} \left(M(j,j) + \frac{1}{2} \sum_{j' \in S_1 \setminus \{j\}} M(j,j') + \frac{1}{2} \sum_{j'' \in S_2} M(j,j'') \right) \\
&= \frac{1}{2} \sum_{j \in S_1'} \sum_{j' \in S_2'} M(j,j') = w(S_1', S_2') \le w(S_1^*, S_2^*) = r_1.
\end{aligned}$$

The same holds for S_2^*, and thus $(P^*, \mathbf{x})$ is a WEQ. □

Lemma 13. *Let $G = \langle K, \{1,2\}, v_1, v_2 \rangle$ be a submodular game with two buyers. Let $Z = \max\{v_1(K), v_2(K)\}$, $Z' > Z$, and $K' = K \cup \{x, y\}$. Define a new value function v s.t. for every $\emptyset \ne S \subseteq K$,*[6]

$$\begin{aligned}
v(S) &= v_1(S) + v_2(S) \\
v(S_x) &= v_1(S) + Z + Z' \\
v(S_y) &= v_2(S) + Z + Z' \\
v(S_{x,y}) &= 2Z + Z',
\end{aligned}$$

where $S_x = S \cup \{x\}$, and likewise for y and $\{x, y\}$. Then the symmetric game $G' = \langle K', \{1,2\}, v \rangle$ is submodular. Further, G' has a WEQ iff G has a WEQ.

Proof. Set $Z'' = Z + Z'$. We first show that v is submodular. Let $S, T \subseteq K$. Clearly $v(S \cup T) \le v(S) + v(T) - v(S \cap T)$.

$$\begin{aligned}
v(S_x \cup T) &= v_1(S \cup T) + Z'' \\
&\le v_1(S) + v_1(T) - v_1(S \cap T) + Z'' \qquad (1) \\
&= v(S_x) + (v_1(T) - v_1(S \cap T)) \\
&= v(S_x) + (v(T_x) - v(S_x \cap T_x)). \\
v(S_{x,y} \cup T) &= 2Z + Z' = v(S_{x,y}) \\
&\le v(S_{x,y}) + v(T) - v(S_{x,y} \cap T) \qquad (2) \\
v(S_x) + v(T_y) &= v_1(S) + v_2(T) + 2Z'' \\
&\ge v_1(S \cap T) + v_2(S \cap T) + 2Z'' = v(S \cap T) + 2Z'',
\end{aligned}$$

Thus,

$$v(S_x \cup T_y) = 2Z + Z' < 2Z'' \le v(S_x) + v(T_y) - v(S \cap T).$$

All other cases follow directly from these cases.

Since any WEQ is maximizing the social welfare, x and y must go to distinct buyers. This is since for every $S, T \subseteq K$,

$$v(S_{x,y}) + v(T) = 2Z + Z' + v(T) = Z + v(T_y) < v(S_x) + v(T_y).$$

[6] The notation x is overloaded, but should be clear from the context.

Finally, note that $v(S_x), v(T_y)$ are just $v_1(S), v_2(T)$ shifted by a constant Z''. Thus under any price vector for K, S maximizes $v_1(S) - x(S)$ if and only if S maximizes $v(S_x) - x(S)$, and similarly for v_2. If there is a WEQ $((S_1, S_2), \mathbf{x})$ in G, then $((S_1 \cup \{x\}, S_2 \cup \{y\}), (\mathbf{x}, Z, Z))$ is a WEQ in G'. In the other direction, if there is a WEQ in G', its projection on K is a WEQ in G. □

Negative results for networks

Proposition 10 *There is a symmetric game with 3 buyers over a synergy graph, that has no WEQ.*

Proof. Intuitively, it seems that a generalization of Theorem 9 to $k > 2$ buyers can be easily constructed by considering a multicut (rather than a cut). However, this intuition is misleading. The following example shows where this intuition fails. Consider a network H, containing three influencers and three other nodes with weights $1, 2, 3$. Each node is influenced by exactly two influencers (w.p. 1). The resulting 3×3 matrix induces a graph M that is a triangle, whose edge weights are $1, 2$ and 3 (i.e. all three edges have different weights). If payments are set according to the scheme above then the profit of every firm will be half the weight of the cut *between himself and the others*. Then the firm that recruited the lightest influencer (and hence has the lightest part of the cut) is envious in the other firms. However, this game does have the following WEQ: $P = (\{a\}, \{b\}, \{c\})$, $\mathbf{x} = (2, 3, 4)$, where the profit of every firm is 1. □

Proposition 11 *There is an asymmetric game with two buyers, each over a different 2-sparse network, that has no WEQ.*

Proof. Consider the following example by Dobzinski and Schapira [5] of a submodular combinatorial auction with two buyers and 4 items.

Example 3 ([5]). The first valuation function v_1 is defined as follows:

$$v_1(S) = \begin{cases} 2, \ |S| = 1 \\ 3, \ |S| = 2, S \neq \{1,3\} \text{ and } S \neq \{2,4\} \\ 4, \ S = \{1,3\} \text{ or } S = \{2,4\} \\ 4, \ |S| \geq 3 \end{cases}$$

The value function v_2 is defined similarly, replacing $\{1, 3\}$ and $\{2, 4\}$ with $\{1, 2\}$ and $\{3, 4\}$. Dobzinski and Schapira show that this game has no equilibrium. ◊

We next use Example 3 as a starting point, and further develop it do derive negative results for games over asymmetric and dense influence networks.

We show how to construct a particular network where the influence equals the value function v. We first describe a single network that equals v_1. The network H_1 contains our four items $W = \{1, 2, 3, 4\}$ and four other nodes $A = \{a_{1,2}, a_{2,3}, a_{3,4}, a_{4,1}\}$. To every $a_{i,j}$, there are edges with influence probability 1 both from i and from j. It is not hard to see that for every $S \subseteq A$, $I_{H_1}(S) = v_1(S)$. Note that v_2 can be constructed in a similar way by a network H_2. □

Proposition 12 *There is a game over a 3-sparse network H with two identical buyers and no WEQ.*

The following lemma describes a "symmetrization" that can be applied to any submodular game with two buyers.

Lemma 13 *Let $G = \langle K, \{1, 2\}, v_1, v_2 \rangle$ be a submodular game with two buyers. Let $Z = \max\{v_1(K), v_2(K)\}$, $Z' > Z$, and $K' = K \cup \{x, y\}$. Define a new value function v s.t. for every $S \subseteq K$,*

$$\begin{aligned} v(S) &= v_1(S) + v_2(S) \qquad & v(S_x) &= v_1(S) + Z + Z' \\ v(S_y) &= v_2(S) + Z + Z' \qquad & v(S_{x,y}) &= 2Z + Z', \end{aligned}$$

where $S_x = S \cup \{x\}$, and likewise for y and $\{x, y\}$. Then the symmetric game $G' = \langle K', \{1,2\}, v\rangle$ is submodular. Further, G' has a WEQ iff G has a WEQ.

Proof (of Proposition 12). Next, we apply the symmetrization described in Lemma 13 to construct the full network H. H has the set of items $\{1, 2, 3, 4, x, y\}$. We have three sets of nodes: A is defined as above, $B = \{b_{1,3}, b_{3,2}, b_{2,4}, b_{4,1}\}$, and another set C of size $Z' > Z = 4$. We add edges from W to A as in H_1 and to B as in H_2. In addition, x is connected to all of $B \cup C$, and y is connected to all of $A \cup C$. It is easy to see that for all $S \subseteq \{1, 2, 3, 4\}$, $v(S_x) = v_1(S)+|C|+|B| = v_1(S)+Z'+Z = v(S_x)$, and similarly for y. Also, $v(S_{x,y}) = |A| + |B| + |C| = 2Z + Z'$, and $v(S) = v_1(S) + v_2(S)$, as required in Lemma 13.

Note that every node in H_1, H_2 is affected by exactly two items, and every node in H is affected by three. Thus the proof is completed. □

Together with Proposition 11 above it shows that the conditions in Theorem 9 are minimal.

B Examples

B.1 The Gross-substitutes condition

Informally, the gross-substitutes (GS) condition states that items can "substitute" one another—if the price of some items rises and the price of other remains the same, then a buyer would never want to forgo or replace items whose price remains the same [11].

Definition 1 (Gross substitutes [11]). *A value function $v : 2^K \to \mathbb{R}_+$ holds the* gross substitutes *condition if the following holds.*

Suppose that under prices $\mathbf{x}$ the set $S \subseteq K$ maximizes the profit $r(S, \mathbf{x}) = v(S) - x(S)$, and let $T \subseteq S$, $\mathbf{x}' \geq \mathbf{x}$ where $x'_j = x_j$ for all $j \in T$. Then there is a set $S' \subseteq K$ maximizing $r(S', \mathbf{x}')$ s.t. $T \subseteq S'$.

The following example shows that a weight-based valuation function may not hold the GS condition.

Example 4. Consider the valuation function $v(w) = \min\{w, 6\}$. We have five items, two of weight 3 and three of weight 2, i.e. $\mathbf{w} = (3, 3, 2, 2, 2)$. With the price vector $\mathbf{x} = \mathbf{w}/2$, there are two optimal subsets: $S_1 = \{3, 3\}$ and $S_2 = \{2, 2, 2\}$, each yielding a profit of $v(6) - 6/2 = 3$. Suppose the buyer has S_1, and we now raise the price of one of the 3-items from $x_1 = 1.5$ to $x'_1 = 2$. The unique optimal selection under $\mathbf{x}' = (x'_1, x_{-1})$ is now S_2. This already violates the GS condition, since x_2 did not change, yet item 2 is not part of any optimal solution.

Similarly, if the firm has S_2 and we raise the price of one of the 2-items, then the unique optimal solution S_1 does not contain the 2-items whose price remains the same. ◊

We can similarly show that valuation functions defined by synergy graphs/sparse influence networks may not hold the GS condition. Indeed, consider the valuation function v_1 from Example 3. Under the price vector $\mathbf{x} = (1, 1, 1, 1)$ the set $S_1 = \{1, 3\}$ is optimal. However if the price of item 1 increases, then the unique optimal set is $S_2 = \{2, 4\}$. S_2 does not include item 3 even though $x'_3 = x_3$, and thus violates the GS condition.

B.2 Weighted games with no Walrasian equilibrium

A non-subadditive game

Example 5. consider a homogeneous and symmetric game with a weighted (non-subadditive) value function $v = (0, 3, 4, 6)$, two buyers, and three items. Assume towards a contradiction that there is a WEQ $((S_1, S_2), \mathbf{x})$. W.l.o.g. $\mathbf{x}$ is fair (by Lemma 4). By the FWT, (S_1, S_2) must be an optimal partition, thus one buyer (w.l.o.g. buyer 1) has two items, $v(S_1) = v(2) = 4, v(S_2) = v(1) = 3$. The marginal value of each item to buyer 1 is 1, thus their price is *at most* 1 each. The marginal value of an additional item to this buyer is 2. Thus buyer 2 must pay for this item *at least* 2. However, by fairness all items must have the same price. A contradiction. $\Diamond$

An experimental analysis shows that in non-subadditive weighted games generated at random, no WEQ usually exists (see Appendix C.1).

A symmetric game with three buyers

Example 6. We consider the following game with 3 buyers, which has no WEQ. The game has 11 items, with weights $\mathbf{w} = (8, 8, 8, 3, 3, 3, 3, 3, 2, 2, 2)$. The value function is $v(w) = \sqrt{w}$. It is easy to verify that there are exactly four optimal partitions (up to permutations of agents of the same type), in all of which the total weights are $w(S_1) = 14$, $w(S_2) = 15$, and $w(S_3) = 16$. The partitions are as follows.

$$P_1 = (\{8, 2, 2, 2\}, \{3, 3, 3, 3, 3\}, \{8, 8\})$$
$$P_2 = (\{3, 3, 3, 3, 2\}, \{8, 3, 2, 2\}, \{8, 8\})$$
$$P_3 = (\{8, 3, 3\}, \{3, 3, 3, 2, 2, 2\}, \{8, 8\})$$
$$P_4 = (\{8, 3, 3\}, \{8, 3, 2, 2\}, \{8, 3, 3, 2\})$$

For each of these partitions, Matlab's `linprog` function returns a set of conflicting constraints in the corresponding linear program. Therefore, none of these optimal partitions can be stabilized, and by FWT there is no WEQ.

For higher values of n, we can use the same example with additional buyers. For each extra buyer $i > 3$, we add an item with weight 100. Thus each additional buyer will buy one heavy item without affecting the competition between the original buyers. $\Diamond$

An asymmetric, balanced game

Example 7. We define a game with three buyers and three items of weight 11, and six items of weight 2. The value functions are defined as $v_1(w) = \min\{w, 20\}; v_2(w) = \min\{w, 19\}; v_3(w) = \min\{w, 6\}$.

First note that a balanced partition exists, where each buyer has one 11-item and two 2-items. The optimal partition is $P = (\{11, 11\}; \{11, 2, 2, 2, 2\}; \{2, 2\})$, and has a welfare of

$$SW(P) = v_1(22) + v_2(19) + v_3(4) = 20 + 19 + 4 = 33.$$

(if all three buyers have an 11-item, then $SW \leq 22 + 12 + 6 = 40$). Assume towards a contradiction that there is a WEQ $(P, (x_2, x_{11}))$.

By Lemma 1, $x_2 = 2$. Now, suppose buyer 2 buys an 11-item instead of all four 2-items. Then

$$r_2 = 19 - x_{11} - 4x_2 \geq r'_2 = 19 - 2x_{11} \Rightarrow x_{11} \geq 4x_2 = 8.$$

On the other hand, if buyer 1 buys three 2-items instead of an 11-item,

$$r_1 = 20 - 2x_{11} \geq r'_1 = 19 - x_{11} - 3x_3 \Rightarrow x_{11} \leq 1 + 3x_2 = 7.$$

Thus we have a contradiction. $\Diamond$

B.3 Network games

There is a weighted value function over 3 homogeneous items, that cannot be represented as the influence in any graph H.

Example 8. Denote by $f(S)$ the expected fraction of nodes in V that is influenced by *all* items in S. For a deterministic network this simply means the number of nodes $u \in V$ s.t. for every $j \in S$ there is a path from j to u. More generally, $f(S) = \sum_{u \in V} f(u, S)$ where $f(u, S)$ is the the total probability of all percolations in which there are paths to u from every $j \in S$ and from no $j' \in K \setminus S$. It is straightforward to see that $I_H(S) = \sum_{T \cap S \neq \emptyset} f(T)$ for all $S \subseteq K$.

Now, consider the weighted value function $v = (0, 3, 6, 8)$. Since $m_v(1, 2) = 8 - 6 = 2$, we have that $f(j) = m(j, \{j', j''\}) = m_v(1, 2) = 2$ for all $j \in \{1, 2, 3\}$. Next,

$$\begin{aligned} m(j', j) &= I(\{j, j'\}) - I(\{j\}) \\ &= f(j) + f(j') + f(j, j') + f(j, j'') + f(j', j'') + f(j, j', j'') \\ &\quad - (f(j) + f(j, j') + f(j, j'') + f(j, j', j'')) \\ &= f(j') + f(j', j'') = 2 + f(j', j''), \end{aligned}$$

thus

$$2 + f(j', j'') = m(j', j) = m_v(1, 1) = 6 - 3 = 3,$$

which entails that $f(j', j'') = 1$ for every pair of items. Finally,

$$\begin{aligned} 3 = v(1) = I(\{1\}) &= f(1) + f(1, 2) + f(1, 3) + f(1, 2, 3) \\ &= 2 + 1 + 1 + f(1, 2, 3) = f(1, 2, 3) + 4, \end{aligned}$$

i.e. $f(1, 2, 3) = -1$. This is a contradiction since every $f(T)$ is a sum of probabilities and thus cannot be negative. $\diamondsuit$

A symmetric network game with three buyers and no WEQ

Example 9. Consider a game with three buyers, and a set of items $K = \{a_1, a_2, b_1, b_2, b_3\}$, where in the graph M every pair of items in connected (with weight 1), except the pair (a_1, a_2). In the optimal partition (which maximizes the weight of the multicut $P = (S_1, S_2, S_3)$), a_1 and a_2 must share a buyer. Thus w.l.o.g. $S_1 = \{a_1, a_2\},\ S_2 = \{b_1, b_2\}, S_3 = \{b_3\}$, and $SW(P) = 6 + 7 + 4 = 17$.

Assume, w.l.o.g., that a WEQ $(P, \mathbf{x})$ exists. Then by fairness all of the a_i items have the same price x_a, and similarly for the b_i items. Thus we only need to find the values x_a and x_b. By envy-freeness, buyer 2 and 3 make the same profit, thus $r_2 = v(S_2) - 2x_b = v(S_3) - x_b$, i.e. $x_b = 7 - 4 = 3$, and $r_2 = r_3 = 1$. buyer 1 must have the same profit as well, thus $1 = r_1 = v(S_1) - 2x_a = 6 - 2x_a$, and $x_a = 2.5$. However, buyer 1 has a deviation, by forgoing item a_2, and buying a b item instead. Then

$$r'_1 = v(\{a_1, b_1\}) - x_a - x_b = 7 - 5.5 > 1 = r_1.$$

$\diamondsuit$

C Experimental results

A priory computing WEQ in weighted games (and also in network games) is a computationally hard problem.

See the full version of this paper for a more detailed discussion on complexity, and for the algorithms we used to efficiently solve the problems at hand ([15], Sections 3.3, A.3).

We implemented a program that solves any given symmetric weighted game G, using Matlab 7. We used a variation of the algorithm described in [15] to first find an arbitrary optimal partition

P^*, then applied the Matlab `linprog` function to solve the induced linear program, i.e. to find a price vector $\mathbf{x}$ s.t. $(P^*, \mathbf{x})$ is a WEQ of G (thus we did not need to use any ILP solver).

An instance with $n = 3$ and $k \leq 15$ is typically solved in less than a second, so that it is possible to collect statistics over many instances. In the few cases where a stable price vector $\mathbf{x}$ was not found, we labeled this instance as "no solution".

C.1 Experimental setting

We generated three datasets with the following characteristics.

- Dataset D_1 had 3000 instances. Each instance had between 5 and 14 items, divided to 2-4 types. The weight of each type was in the range 2-15, and the total weight was limited to under 80. All of the parameters have been sampled uniformly at random from their range, and we re-sampled if the constraint on the total weight was violated. 1053 instances (32%) had a balanced or nearly balanced partition (in which case we know a WEQ exists by Proposition 7). In 4 instances the optimal partition could not be stabilized with prices, and thus a WEQ does not exist. One of these instances is shown in Example 6.
- Dataset D_2 had 3000 instances. Each instance had between 4 and 11 items, with weights in the range 2-15, and no restriction on the number of types. The total weight limit was 80. 1548 instances (52%) had a balanced or nearly balanced partition. In all instances, a WEQ was found.
- Dataset D_3 includes the first 200 instances of D_1.

In datasets D_1 and D_2 we used a random subadditive value function for each instance. This is by sampling the marginal value of a unit uniformly from $[0, 1]$, and then sort them in decreasing order. For D_3 we used the value function $f(w) = w^\alpha$, for various values of $0 < \alpha < 1$.

Since we only got a handful of instances with no WEQ, we did not make a statistical analysis of these samples. To test the importance of subadditivity, we also generated random value functions without enforcing subadditivity. In this case only 910 from the instances in D_1 (27%) had a WEQ, and slightly more (33%) in D_2.

C.2 Experimental validation of the heuristic prices

For every instance in each of the datasets D_1 and D_2, we computed the heuristic payment vector x^* as follows.

Let $P^* = (S_1, S_2, S_3)$ be the optimal partition, $i_+ = \text{argmax}_{i \in N}\, w(S_i)$, $i_- = \text{argmin}_{i \in N}\, w(S_i)$, then $z_0 = \frac{v(w(S_{i_+})) - v(w(S_{i_-}))}{w(S_{i_+}) - w(S_{i_-})}$. In the induced proportional payoff vector, $x^*(j) = z_0 \cdot w_j$ for all $j \in K$. For every instance G, we measured the maximal amount a buyer can gain by deviating from the profile $(P^*, \mathbf{x}^*)$. Formally, $h(G, x^*) = \max_{i \in N, S' \subseteq K}(v(S') - x^*(S')) - (v(S_i) - x^*(S_i))$. The profits are normalized to 1, so that $h(G, x^*) = 0.2$ for example, means that there is a buyer in G that can increase its profit by 20% by deviating from x^*.

We sorted the instances according to $h(G, x^*)$, and plotted a survival graph showing the percentage of instances for which $h(G, x^*) < h$ (Figures 5 and 6). The following trends are apparent from the graphs:

1. For most instances, $h(G, x^*) = 0$. That is, (P^*, x^*) is WEQ.
2. For roughly 95% of the instances in D_1 (97% in D_2), $h(G, x^*) \leq 0.05$.
3. As the gap $d = w(S_{i_+}) - w(S_{i_-})$ increases, $h(G, x^*)$ also increases.
4. In D_2, where the types are more diverse, $h(G, x^*)$ is lower, i.e. the heuristic solution is more stable. However if we condition on the gap d (see dashed lines), then there is no significant difference.

In addition, there was no substantial difference between D_1 and D_3. It seems therefore that the diversity of types is responsible mainly for the reduction in the average gap d, which in turn explains the improvement of the heuristic solution. Other parameters such as the function $v : [k] \to \mathbb{R}$ (as long as it is subadditive) have little effect on stability.

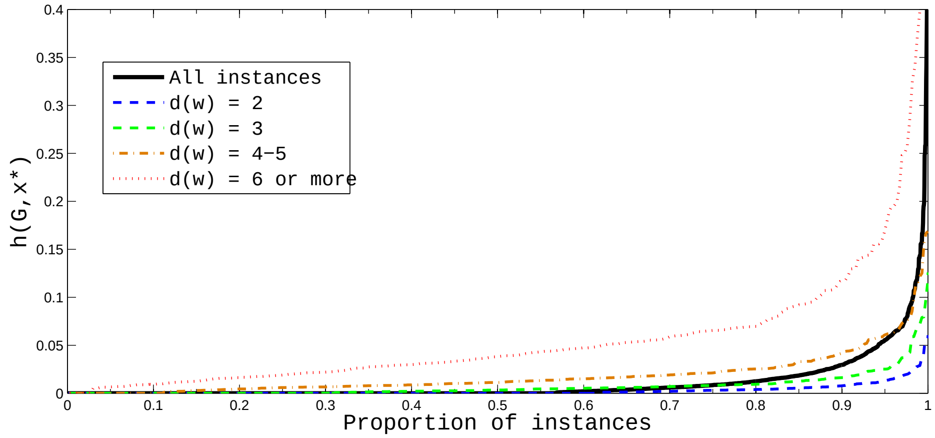

Fig. 5: A survival curve of the percentage of instances from the dataset D_1, for which $h(G, x^*) \leq h$. The solid line is showing statistics for all instances. The dashed lines are the curves of instances with a particular gap $d = q_{i_+} - q_{i_-}$. We can see that as the gap increases, the stability of the heuristic solution x^* deteriorates. We can also see that for almost 60% of the instances, $h(G, x^*) = 0$, i.e. the heuristic solution is stable.

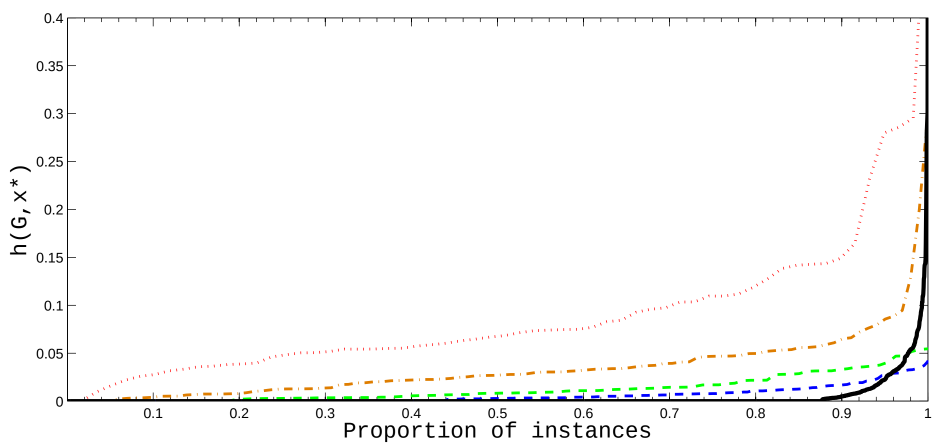

Fig. 6: The survival curve of the dataset D_2. Here the heuristic solution is stable ($h(G, x^*) = 0$) for over 85% of the instances.