Subsidies and Stability in Cooperative Games

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Abstract

A key issue in cooperative game theory is coalitional stability, usually captured by the notion of the *core*—the set of outcomes that are resistant to group deviations. However, some coalitional games have empty cores, and any outcome in such a game is unstable. We investigate the possibility of stabilizing a coalitional game by using subsidies. We consider scenarios where an external party that is interested in having the players work together offers a supplemental payment to the grand coalition, or, more generally, a particular coalition structure. This payment is conditional on players not deviating from this coalition structure, and may be divided among the players in any way they wish. We define the *cost of stability* (*CoS*) as the minimum external payment that stabilizes the game. We provide tight bounds on the cost of stability, both for games where the coalitional values are non-negative (profit-sharing games) and for games where the coalitional values are non-positive (expense-sharing games), under natural assumptions on the characteristic function, such as super- and subadditivity, anonymity, or both. We also investigate the relationship between the cost of stability and several variants of the least core.

Keywords: cooperative game theory, core, subsidy

1. Introduction

There are many settings where self-interested agents find it profitable to cooperate and form teams in order to achieve their individual goals. Such settings are modeled using the toolkit of *cooperative*, or *coalitional*, game theory, which studies what teams, or *coalitions*, are most likely to arise and how their members should distribute the gains from cooperation.

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Briefly, an outcome of a coalitional game is a payoff vector, which describes how the players share the joint profit (or cost). If multiple coalitions are allowed to form in parallel, then the outcome also specifies the partition into coalitions, and the payoff vector is required to distribute the value of each coalition among its members. An important consideration in identifying acceptable outcomes is *stability*: The agents should prefer the current outcome to the ones they can feasibly achieve by deviating. The most prominent solution concept that aims at formalizing the idea of stability in coalitional games is the *core*: An outcome is said to be in the core if it distributes the gains or costs so that no subset of agents has an incentive to abandon the existing arrangement and form a coalition of their own.

Arguably, the concept of the core captures the intuitive notion of stability in cooperative settings. However, it has an important drawback: The core of a game may be empty. In games with empty cores any outcome is unstable, and therefore there is always a group of agents that is tempted to deviate from the current plan. This observation has triggered the development of several alternative solution concepts, which are based on relaxing the core constraints (this includes the ε-core, the least core, and the (pre)nucleolus [1]) or employing alternative notions of stability (e.g., the bargaining set [2] and the kernel [3]).

In this paper, we approach this issue from a different perspective. Specifically, we examine the possibility of stabilizing an outcome of a game by means of subsidies. That is, we investigate settings where an external party, which can be seen as a central authority interested in a stable outcome of the interaction, attempts to incentivize the agents to cooperate. This party implements its agenda by offering the agents a supplemental payment that is conditional on the agents working together as a team (or a collection of teams). This payment is given to the coalition as a whole, and can be divided arbitrarily among the members of the coalition.

We will now present two simple examples where such a scenario is plausible.

Example 1.1 (Sharing the cost)**.** Three private hospitals in a large city plan to purchase an X-ray machine. The standard X-ray machine costs $5 million, and can fulfill the needs of up to two hospitals. There is also a more advanced machine, which is capable of serving all three hospitals, but costs $9 million. The hospital managers understand that the right thing to do is to buy the more expensive machine, which will serve all three hospitals and cost less than two standard machines, but cannot agree on how to allocate the cost of the more expensive machine among the hospitals: There will always be a pair of hospitals that (together) need to pay at least $6 million, and would then rather split off and buy the cheaper machine for themselves. The generous mayor decides to solve the problem by subsidizing the more advanced X-ray machine: She agrees to contribute $3 million, and asks each hospital to add $2 million. Pairs of hospitals now have no incentive to buy the less efficient machine, as each pair (together) pays only $4 million.

Example 1.2 (Sharing the profit)**.** A privately run infrastructure company is hiring contractors to work on a large project. The project can be completed by any set of three or more contractors. There are four contractors who are potentially interested in the project. The contractors can form teams to bid on the project: The company will accept a bid from any team of at least three companies, and pay a fixed amount to this team. Without external subsidies, no team of contractors will be stable: If the team consists of all contractors, one of them can be dropped, and if it consists of three contractors, the one who is not participating in the team can offer to replace the team member that receives the highest payoff share, at a lower cost.

However, a government agency that wants to see this project completed (and is also interested in reducing unemployment) may offer a subsidy to the company, conditioned on it hiring all four contractors.

When the supplemental payment is large enough, the resulting outcome is stable: The profit that the deviators can make on their own is dwarfed by the subsidy they could receive by sticking to the prescribed solution. For instance, in Example 1.1 above, an easy way to stabilize the game would be for the mayor to simply buy the expensive X-ray machine, for all three hospitals to share. However, normally the external party would want to minimize its expenditure: In Example 1.1, a subsidy of $1.5 million would suffice, so there is no reason to offer more than that.

In this paper we define and study the *cost of stability* (*CoS*), which is the minimum supplemental payment that is required to ensure stability in a coalitional game. We first consider the case where the central authority seeks to ensure that *all* agents cooperate, i.e., it offers a supplemental payment in order to stabilize the grand coalition. We use linear programming duality to characterize the cost of stability both for games where the coalitional values are non-negative (profit-sharing games) and for games where the coalitional values are non-positive (expense-sharing games). We use this characterization to prove tight bounds on the cost of stability in games where the characteristic function satisfies additional constraints, such as super-/subadditivity, anonymity, or both. We then explore the relationship between the cost of stability and several variants of the notion of the least core. In addition, we define a new natural variant of the least core, where negative payoffs are forbidden. Bounds on the cost of stability continue to hold under this stricter definition, which is interesting in its own right. Finally, we extend our analysis to the setting where the goal of the center is the stability of a *coalition structure*, i.e., a partition of all agents into disjoint coalitions. In this setting, the center does not expect the agents to work as a single team, but nevertheless wants each individual team to be immune to deviations.

The observation that an external party may be willing to provide subsidies to the grand coalition in the interest of stability has a long history. Indeed, in the context of expense-sharing games, there has been a number of papers that study the closely related notion of γ-core (sometimes also referred to as α-core), i.e., the set of all expense-sharing schemes that collect a γ fraction of the cost of the grand coalition without violating the stability constraints [4]. The smallest value of γ for which the γ-core is not empty is sometimes referred to as the *cost recovery ratio*, and is essentially equivalent to (the multiplicative version of) the cost of stability. We provide an overview of this stream of work in Section 7. However, the prior work focuses on analyzing the cost of stability (or its analogues) in specific combinatorial optimization games, whereas our goal is to establish general bounds for classes of games that are characterized by simple axiomatic conditions, such as super-/subadditivity and anonymity. Further, to the best of our knowledge, we are the first to explore the connection between the cost of stability and the least core, as well as to analyze the cost of stability for games with coalition structures.

Organization. The remainder of the paper is organized as follows. In Section 2, we provide the necessary background on coalitional games. In Section 3 we formally define the cost of stability and characterize it using linear programming techniques. Section 4 presents bounds on the cost of stability in various subclasses of profit-sharing and expense-sharing games. Section 5 compares the cost of stability with the closely related concept of the least core. Section 6 analyzes the cost of stability in games with coalition structures. The related work is surveyed in Section 7. We conclude and suggest directions for future research in Section 8. Some of the more technical proofs are deferred to the appendix.

2. Preliminaries

We now present the standard concepts of cooperative game theory that will be used in this paper; for a detailed background on coalitional games, see, e.g., the textbook by Peleg and Sudhölter [5].

2.1. Transferable Utility Games, Imputations, and the Core

We consider games with the set of players $N = \{1, \dots, n\}$, where $n \geq 2$. A *coalition* is simply a subset of N; N itself is referred to as the *grand coalition*. We denote by $\mathbb{R}_+$ the set of all non-negative real numbers. Given two vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$, we write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for all $i = 1, \dots, n$. Also, given a vector $\mathbf{x} = (x_1, \dots, x_n)$ and a set $S \subseteq N$, we write $x(S)$ to denote $\sum_{i \in S} x_i$.

Definition 2.1. *A* transferable utility (TU) game $G = \langle N, g \rangle$ *is given by a set of* players*, or* agents*,* $N = \{1, \dots, n\}$ *and a* characteristic function $g : 2^N \to \mathbb{R}$*, which for each subset of agents S outputs the payoff that the members of S can achieve by working together; we require $g(\emptyset) = 0$ and $g(N) \neq 0$.*

In most TU games studied in the literature, the payoffs of all coalitions have the same sign, i.e., it is assumed that agents get together either to share costs (as in Example 1.1) or to earn profits (as in Example 1.2). In the former case, we say that G is an *expense-sharing game* and write $G = \langle N, c \rangle$, where $c \equiv -g$, and in the latter case we say that G is a *profit-sharing game* and write $G = \langle N, v \rangle$, where $v \equiv g$; note that both v and c only take values in $\mathbb{R}_+$. We remark that expense-sharing games are usually referred to as *cost-sharing games*; we use the term "expense" to avoid confusion with the *cost* of stability. In this paper, we will only consider TU games that are either profit-sharing games or expense-sharing games.

An *outcome* of a TU game is a way of sharing the value (i.e., profit or expense) of the grand coalition among all players. Formally, a *payoff vector* for a TU game $G = \langle N, g \rangle$ is a vector $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$, where p_i is the profit received by (respectively, the cost imposed on) player i. A payoff vector $\mathbf{p}$ is said to be *budget-balanced* if $\sum_{i \in N} p_i = |g(N)|$. In general, we do not impose any constraints on the signs of the entries of $\mathbf{p}$. We denote by $\mathbb{I}(G)$ the set of all budget-balanced payoff vectors for the TU game G; the elements of $\mathbb{I}(G)$ are also referred to as *pre-imputations*. A pre-imputation $\mathbf{p} \in \mathbb{I}(G)$ is said to be an *imputation* if it is *individually rational*, i.e., $G = \langle N, v \rangle$ is a profit-sharing game and $p_i \geq v(\{i\})$ for all $i \in N$, or $G = \langle N, c \rangle$ is an expense-sharing game and $p_i \leq c(\{i\})$ for all $i \in N$.

Not all outcomes of coalitional games are equally attractive to the agents. In particular, an important consideration in the analysis of TU games is that of coalitional stability: For an outcome to be *stable*, it should be the case that no subset of players has an incentive to deviate. Formally, given a profit-sharing game $G = \langle N, v \rangle$ (respectively, an expense-sharing game $G = \langle N, c \rangle$), we say that a coalition S *blocks* a payoff vector $\mathbf{p}$ if it can improve its payoff by deviating, i.e., if $v(S) > p(S)$ (respectively, $c(S) < p(S)$). A payoff vector $\mathbf{p}$ for a TU game $G = \langle N, g \rangle$ is said to be *stable* if no coalition $S \subseteq N$ blocks it. We denote the set of all stable payoff vectors by $\mathbb{S}(G)$. The *core of* G, denoted by $\mathbb{C}(G)$, is the set of all payoff vectors that are both stable and budget-balanced. Thus, $\mathbb{C}(G) = \mathbb{I}(G) \cap \mathbb{S}(G)$.

While the core is a very appealing solution concept, there are games with empty cores; we will see an example of such a game at the end of this section.

2.2. Subclasses of TU Games

We will now define several important classes of TU games that are obtained by placing restrictions on the characteristic function, namely, monotone games, super- and subadditive games, anonymous games, and simple games.

Monotone games. A TU game $G = \langle N, g \rangle$ is called *monotone* if $|g(S)| \leq |g(T)|$ for all $S \subseteq T$. Monotonicity means that adding agents to a coalition can only increase its profit in a profit-sharing game, or its expenses in an expense-sharing game.

Super- and subadditive games. A profit-sharing game $\langle N, v \rangle$ is said to be *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$. An expense-sharing game $\langle N, c \rangle$ is said to be *subadditive* if $c(S \cup T) \leq c(S) + c(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$. Intuitively, in superadditive profit-sharing games and subadditive expense-sharing games it is never harmful to merge two non-overlapping coalitions. In what follows, we will refer to superadditive profit-sharing games and subadditive expense-sharing games as *s-additive* games.

Anonymous games. A TU game $G = \langle N, g \rangle$ is called *anonymous* if the payoff of a coalition depends on its size only, i.e., $g(S) = g(T)$ whenever $|S| = |T|$. Given an anonymous game $G = \langle N, g \rangle$, for every $k = 1, \dots, n$ we define $g_k = g(\{1, \dots, k\})$; we have $g_k = g(S)$ for every coalition $S \subseteq N$ of size k.

Simple games. A profit-sharing TU game $G = \langle N, v \rangle$ is called *simple* if it is monotone, $v(S) \in \{0, 1\}$ for all $S \subseteq N$, and $v(N) = 1$. In a simple game we say that a coalition $S \subseteq N$ *wins* if $v(S) = 1$, and *loses* if $v(S) = 0$. A player i in a simple game G is called a *veto* player if he is necessary to form a winning coalition, i.e., we have $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$. The notion of a veto player turns out to be useful for characterizing the core of a simple game.

Theorem 2.2 ([5]). *Let $G = \langle N, v \rangle$ be a simple game. If there are no veto players in G, then the core of G is empty. Otherwise, let $N' = \{i_1, \dots, i_m\}$ be the set of veto players in G. Then the core of G is the set of non-negative payoff vectors that distribute all the gains among the veto players only, i.e.,* $\mathbb{C}(G) = \{\mathbf{p} \in \mathbb{I}(G) \mid p(N') = 1, p_i \geq 0 \text{ for all } i \in N\}$.

An important subclass of simple games is that of *weighted voting games* (WVGs). In these games, each agent has a weight, and a coalition of agents wins the game if the sum of the weights of its members meets or exceeds a certain quota. Formally, a *weighted voting game* is given by a set of agents $N = \{1, \dots, n\}$, a vector of agents' *weights* $\mathbf{w} = (w_1, \dots, w_n) \in (\mathbb{R}_+)^n$, and a *quota* $q \in \mathbb{R}_+$; we write $G = [\mathbf{w}; q]$, or, dropping the vector parentheses,

$G = [w_1, \ldots, w_n; q]$. The *weight* of a coalition $S \subseteq N$ is $w(S) = \sum_{i \in S} w_i$; we require $0 < q \leq w(N)$. The characteristic function of a weighted voting game is given by

$$v(S) = \begin{cases} 1 \text{ if } w(S) \geq q \\ 0 \text{ if } w(S) < q. \end{cases}$$

It is easy to verify that weighted voting games are simple games; however, they are not necessarily superadditive.

Example 2.3. Four contractors, 1, 2, 3 and 4, compete for a construction project. The project requires at least 10 trucks. Contractor 1 has 2 trucks, contractor 2 has 5 trucks, contractor 3 has 8 trucks, and contractor 4 has 12 trucks. This situation corresponds to a weighted voting game $[2, 5, 8, 12; 10]$. In this game, $\{4\}$ and $\{1, 2, 3\}$ are winning coalitions (with value 1), whereas $\{1, 2\}$ is a losing coalition (with value 0). Note that there are no veto players in this game, and thus the core is empty, which means that if all four contractors cooperate, they have no stable way to divide the project's profits. For instance, the payoff vector $\mathbf{p} = (0, 1/4, 1/4, 1/2)$ is blocked by the coalition $S = \{2, 3\}$, since $p(\{2, 3\}) = 1/2 < 1 = v(\{2, 3\})$.

3. The Cost of Stability

In many coalitional games, the core is empty: For instance, the examples in Section 1 correspond to transferable utility games with empty cores. In such games, an external authority can provide a subsidy that increases the profit of the grand coalition in a profit-sharing game or lowers the cost of the grand coalition in an expense-sharing game. This subsidy is given to the grand coalition as a whole, and is conditional on agents forming the grand coalition. We will refer to the new game that arises as a result of this subsidy as the *adjusted coalitional game*. Technically, this game is derived from the original game by relaxing the budget-balance requirement.

Definition 3.1. *Given a TU game $G = \langle N, g \rangle$ and a real value $\Delta \geq 0$, the* adjusted coalitional game $G(\Delta) = \langle N, g' \rangle$ *is given by*

$$g'(S) = \begin{cases} g(S) & \text{if } S \neq N \\ g(S) + \Delta & \text{if } S = N. \end{cases}$$

We say that $g'(N) = g(N) + \Delta$ is the *adjusted payoff* of the grand coalition. We will refer to the quantity Δ as the *subsidy* for the game G. Note that if $\Delta \neq 0$, a pre-imputation $\mathbf{p}'$ for the adjusted game $G(\Delta)$ is *not* a pre-imputation for the original game G, since $p'(N) \neq |g(N)|$. We say that a subsidy Δ *stabilizes* the game G if the adjusted game $G(\Delta)$ has a non-empty core.

To illustrate the concepts introduced in the previous paragraph, consider Example 1.1. In this example, $c(N)$ was reduced from 9 (million) to $c'(N) = 6$ (i.e., by $\Delta = 3$), while for every other non-empty coalition S we have $c'(S) = c(S) = 5$. Thus, $\mathbf{p}' = (2, 2, 2)$ is a pre-imputation for the new game $G(3)$. Since $\mathbf{p}'$ satisfies all stability constraints, it is in the core of $G(3)$, demonstrating that a subsidy of $\Delta = 3$ stabilizes the game G.

We observe that any TU game can be stabilized by an appropriate choice of Δ. Indeed, if $G = \langle N, c\rangle$ is an expense-sharing game, it suffices to set $\Delta = c(N)$; then the payoff vector $(0, \dots, 0)$ is in the core of $G(\Delta)$. Similarly, if $G = \langle N, v\rangle$ is a profit-sharing game, we can set $\Delta = n \max_{S\subseteq N} v(S)$ and distribute the profits so that each player receives at least $\max_{S\subseteq N} v(S)$. However, the central authority typically wants to spend as little money as possible. Hence, we are interested in the *smallest* subsidy that stabilizes the grand coalition. We will refer to this quantity as the *cost of stability*. We will consider both the absolute value of this subsidy and the total payout obtained/distributed by the center relative to the value of the grand coalition, thus distinguishing between *additive* and *multiplicative* cost of stability.

Definition 3.2. *Given a TU game $G = \langle N, g\rangle$, its* additive cost of stability *is the quantity*

$$\mathrm{addCoS}(G) = \inf\{\Delta \in \mathbb{R}_+ \mid \mathbb{C}(G(\Delta)) \neq \emptyset\} = \inf\{|p(N) - |g(N)|| \mid \mathbf{p} \in \mathbb{S}(G)\}, \quad (1)$$

and its multiplicative cost of stability *is the quantity*

$$\mathrm{multCoS}(G) = \frac{|\mathrm{addCoS}(G) + g(N)|}{|g(N)|}. \quad (2)$$

(Recall that we assume that $g(N) \neq 0$ throughout the paper.)

In what follows, we will alternate between the additive and the multiplicative notation; obviously, all results formulated for the additive cost of stability can be restated for its multiplicative sibling, and vice versa. Note that we have $\mathrm{multCoS}(G) \geq 1$ if G is a profit-sharing game, and $0 \leq \mathrm{multCoS}(G) \leq 1$ if G is an expense-sharing game. For expense-sharing games, the multiplicative cost of stability is also known as the *cost recovery ratio* [6] and has a natural economic interpretation: This is the fraction of the cost of providing the service to the grand coalition that can be collected without giving the agents an incentive to deviate.

We will denote the game $G(\mathrm{addCoS}(G))$ by $\overline{G}$. As argued above, we can replace the constraint $\Delta \in \mathbb{R}_+$ in (1) with $0 \leq \Delta \leq n \max_{S\subseteq N} |g(S)|$. Since $[0, n \max_{S\subseteq N} |g(S)|]$ is a closed set, the infimum in (1) is in fact a minimum, and thus $\overline{G}$ has a non-empty core.

Remark 3.3. Note that in the definition of the cost of stability we require $\Delta \geq 0$, and hence $\mathrm{addCoS}(G) = 0$ if and only if G has a non-empty core. Nevertheless, the game $G(\Delta)$ would remain well-defined if we allowed Δ to take negative values. Under this definition, $G(\Delta)$ may have a non-empty core for some $\Delta < 0$ (of course, this could only happen if $\mathbb{C}(G) \neq \emptyset$). That is, the central authority may be able to extract some of the profit of the grand coalition in a profit-sharing game (or impose additional costs on the grand coalition in a cost-sharing game) without destabilizing the game. However, we do not study this type of scenarios in this paper.

The notion of cost of stability presupposes that the subsidy is provided *before* the agents decide how to share profits/expenses. Alternatively, one can ask if a given pre-imputation can be transformed into a stable payoff vector by providing a subsidy of Δ for some $\Delta \geq 0$ [7]. Formally, given a profit-sharing game $G = \langle N, v\rangle$ and a vector $\mathbf{p} \in \mathbb{I}(G)$, we define the *cost of stability* of $\mathbf{p}$ as the smallest payment required to extend $\mathbf{p}$ to a stable payoff vector, i.e.,

$$\mathrm{addCoS}(\mathbf{p}, G) = \inf\{\Delta \geq 0 \mid \text{ there exists a } \mathbf{p}' \geq \mathbf{p} \text{ s.t. } \mathbf{p}' \in \mathbb{C}(G(\Delta))\ \}; \quad (3)$$

for expense-sharing games, the definition has to be modified by replacing the inequality $\mathbf{p}' \geq \mathbf{p}$ in (3) with $\mathbf{p}' \leq \mathbf{p}$. In general, we have $\mathrm{addCoS}(\mathbf{p}, G) \geq \mathrm{addCoS}(G)$. The *extended core* of a TU game G consists of all pre-imputations for which this inequality holds with equality: we define

$$\mathbb{EC}(G) = \{\mathbf{p} \in \mathbb{I}(G) \mid \mathrm{addCoS}(\mathbf{p}, G) = \mathrm{addCoS}(G)\}.$$

If $\mathbf{p} \in \mathbb{EC}(G)$ and $\mathbf{p}'$ satisfies $\mathbf{p}' \geq \mathbf{p}$, $\mathbf{p}' \in \mathbb{C}(G(\Delta))$, where $\Delta = \mathrm{addCoS}(G)$, we say that $\mathbf{p}'$ is a *stable extension of* $\mathbf{p}$.

To illustrate the notions introduced in this section, we will now derive bounds on the cost of stability for two classes of games. Our first example is weighted voting games where all players have the same weight; for such games, we can compute the cost of stability exactly.

Example 3.4. Consider a weighted voting game $G = [w, w, \ldots, w; q]$. We will show that $\mathrm{multCoS}(G) = \max\left\{\frac{n}{\lceil q/w \rceil}, 1\right\}$, or, equivalently, $\mathrm{addCoS}(G) = \max\left\{\frac{n}{\lceil q/w \rceil} - 1, 0\right\}$.

By scaling w and q we can assume that $w = 1$. Set $\Delta = n/\lceil q \rceil - 1$. Suppose first that $\Delta \leq 0$, i.e., $n \leq \lceil q \rceil$. As we stipulate that $n \geq q$, we have $n - 1 < q \leq n$, and hence all players are veto players and the core is non-empty, i.e., $\mathrm{addCoS}(G) = 0$.

On the other hand, suppose that $\Delta > 0$. Consider the payoff vector $\mathbf{p} = (p_1, \ldots, p_n)$ given by $p_i = 1/\lceil q \rceil$ for $i = 1, \ldots, n$. Clearly, we have $p(N) = n/\lceil q \rceil$, so $\mathbf{p} \in \mathbb{I}(G(\Delta))$. Moreover, for any winning coalition S, we have $|S| \geq \lceil q \rceil$, so $p(S) \geq \lceil q \rceil \cdot 1/\lceil q \rceil = 1$. Therefore, $\mathbf{p}$ is in the core of $G(\Delta)$, and hence $\mathrm{addCoS}(G) \leq \Delta$.

Conversely, let $\mathbf{p}$ be in the core of $\overline{G}$. Set $s = \lceil q \rceil$. Consider a collection of coalitions $S^1, \ldots, S^n$, where $S^i = \{(i \bmod n) + 1, (i+1 \bmod n) + 1, \ldots, (i+s-1 \bmod n) + 1\}$; for example, we have $S^{n-1} = \{n, 1, \ldots, s-2, s-1\}$. We have $|S^i| = s$ and hence $p(S^i) \geq 1$ for all $i = 1, \ldots, n$, so $p(S^1) + \cdots + p(S^n) \geq n$. On the other hand, each player i occurs in exactly s of these coalitions, so we have $p(N) \cdot s = p(S^1) + \cdots + p(S^n)$. Hence, $p(N) \geq n/s = n/\lceil q \rceil$, and therefore $\mathrm{addCoS}(G) \geq \Delta$. As $\mathrm{addCoS}(G) \geq 0$ by definition, our claim follows.

For example, if $w = 1$, $n = 3k$, and $q = 2k$ for some integer $k > 0$, i.e., $q = 2n/3$, we have $\mathrm{multCoS}(G) = 3/2$.

Our second example is given by simple games defined by finite projective planes; subsequently, we will use this example to prove lower bounds on the cost of stability.

Example 3.5. Let q be a prime number. Consider the finite projective plane of order q. It has $q^2 + q + 1$ points and the same number of lines, every line contains $q + 1$ points, every two lines intersect at a single point, and every point belongs to exactly $q + 1$ lines. We construct a simple game $G_q = \langle N, v \rangle$ as follows. We let N be the set of points in P, and for every $S \subseteq N$, we let $v(S) = 1$ if S contains a line, and $v(S) = 0$ otherwise. Observe that this game is superadditive: Since any two lines intersect, there do not exist two disjoint winning coalitions.

Now, consider a stable payoff vector $\mathbf{p}$. For each line R we have $p(R) \geq 1$. Summing over all $q^2 + q + 1$ lines, and using the fact that each point belongs to $q + 1$ lines, we obtain $(q+1) \sum_{i \in N} p_i \geq q^2 + q + 1$, i.e., $p(N) \geq \frac{q^2+q+1}{q+1} = q + \frac{1}{q+1}$. Since $n = |N| = q^2 + q + 1$, we have $q + 1 > \sqrt{n}$ and hence

$$\mathrm{multCoS}(G_q) \geq p(N) > q > \sqrt{n} - 1.$$

3.1. The Cost of Stability: A Linear Programming Formulation

We will now give two alternative definitions of the cost of stability, which can be expressed in terms of linear programming and will prove to be useful later in the paper. We formulate our results for profit-sharing games only; similar results can be derived for expense-sharing games.

Fix a profit-sharing game $G = \langle N, v\rangle$ and consider the following linear program $\mathcal{LP}^*$ with variables $p_1, \dots, p_n, \Delta$:

$$
\begin{aligned}
\min \Delta & \quad \text{subject to:} & \\
\Delta &\geq 0 & (4)\\
\sum_{i\in N} p_i &= v(N) + \Delta & (5)\\
\sum_{i\in S} p_i &\geq v(S) \text{ for all } S \subseteq N. & (6)
\end{aligned}
$$

It is not hard to see that the optimal value of this linear program is exactly $\mathrm{addCoS}(G)$. Moreover, any optimal solution of $\mathcal{LP}^*$ corresponds to a payoff vector in the core of $\overline{G}$. It will be convenient to modify this linear program by removing constraints (4) and (5) and replacing the objective function with $p_1 + \cdots + p_n$: If the core of G is empty, the optimal value of the resulting linear program, which we will denote by $\mathcal{LP}'$, is exactly $\mathrm{addCoS}(G) + v(N)$.

The cost of stability can also be written in a closed form, using the Bondareva–Shapley characterization of the core. To state the Bondareva–Shapley theorem, we first need to introduce the notion of a (minimal) balanced collection of subsets.

Definition 3.6. *A collection $\mathcal{D}$ of subsets of a finite set N is said to be* balanced *if there exists a vector $\{\delta_S\}_{S\in\mathcal{D}}$ such that $\delta_S \in \mathbb{R}_+$ for every $S \in \mathcal{D}$, and for every agent $i \in N$ it holds that $\sum_{S\in\mathcal{D}:i\in S} \delta_S = 1$; the vector $\{\delta_S\}_{S\in\mathcal{D}}$ is called the* balancing weight vector *for $\mathcal{D}$. We denote the set of all balanced collections of subsets of N by $\mathcal{BC}(N)$; the collection of all balancing weight vectors for a balanced collection $\mathcal{D}$ is denoted by $\mathcal{B}(\mathcal{D})$.*

A balanced collection of subsets $\mathcal{D}$ is called minimal *if there exists no $\mathcal{D}' \subsetneq \mathcal{D}$ such that $\mathcal{D}'$ is balanced.*

Theorem 3.7 (Bondareva–Shapley Theorem)**.** *A profit-sharing game $G = \langle N, v\rangle$ has a non-empty core if and only if for every [minimal][1] balanced collection $\mathcal{D}$ and every balancing weight vector $\{\delta_S\}_{S\in\mathcal{D}}$ for $\mathcal{D}$ it holds that $\sum_{S\in\mathcal{D}} \delta_S v(S) \leq v(N)$.*

Theorem 3.7 can be obtained by considering the dual program to $\mathcal{LP}'$. This linear program, which we will denote by $\mathcal{LP}'_{\text{dual}}$, can be written as follows:

[1]There are versions of the theorem with and without the minimality requirement.

$$\max \sum_{S \subseteq N} \delta_S v(S) \qquad \text{subject to:}$$

$$\begin{aligned} \sum_{i \in S} \delta_S &\leq 1 \text{ for each } i = 1, \ldots, n \\ \delta_S &\geq 0 \text{ for all } S \subseteq N. \end{aligned} \tag{7}$$

For a full proof and a more detailed discussion of this result, see, e.g., the textbook by Maschler, Solan and Zamir [8].

Now, fix a profit-sharing game $G = \langle N, v \rangle$ with an empty core and consider the game $\overline{G}$. It is not hard to see that for $\overline{G}$ there exists a collection $\mathcal{D}$ with a balancing weight vector $\{\delta_S\}_{S \in \mathcal{D}}$ for which the inequality in the statement of the Bondareva–Shapley theorem holds with equality. Thus, we can write the multiplicative cost of stability of the game G as follows:

$$\text{multCoS}(G) = \frac{1}{v(N)} \max \left\{ \sum_{S \in \mathcal{D}} \delta_S v(S) \;\middle|\; \mathcal{D} \in \mathcal{BC}(N), \{\delta_S\}_{S \in \mathcal{D}} \in \mathcal{B}(\mathcal{D}) \right\}. \tag{8}$$

Furthermore, let $\mathcal{D}$ be a collection of sets with a balancing weight vector $\{\delta_S\}_{S \in \mathcal{D}}$ such that

$$\text{multCoS}(G) = \frac{1}{v(N)} \sum_{S \in \mathcal{D}} \delta_S v(S). \tag{9}$$

We can assume without loss of generality that $\mathcal{D} = 2^N$ by setting $\delta_S = 0$ for $S \notin \mathcal{D}$ [9]. Any such balancing weight vector (with an entry for each subset of N) satisfies the constraints of $\mathcal{LP}'_{\text{dual}}$. We will therefore call such weight vectors *solutions to* $\overline{G}$.

Remark 3.8. Note that any partition of N into pairwise disjoint coalitions $S_1, \ldots, S_k$ (such partitions are known as *coalition structures* and are discussed in detail in Section 6) is a minimal balanced collection of sets, with a balancing weight vector $\{\delta_{S_i}\}_{i=1,\ldots,k}$ given by $\delta_{S_i} = 1$ for $i = 1, \ldots, k$. Thus, we have $\text{multCoS}(G) \geq \frac{1}{v(N)} \sum_{i=1}^{k} v(S_i)$, or, equivalently, $\text{addCoS}(G) \geq \sum_{i=1}^{k} v(S_i) - v(N)$. In words, if the agents can collectively benefit from forming multiple coalitions rather than staying in the grand coalition, the additive cost of stability is at least the loss they incur by not doing so. Equation (8) can be seen as a generalization of this observation to "fractional" coalition structures, for which it becomes tight.

Example 3.4 with $w = 1$, $n = 3k$, and $q = 2k$ illustrates that considering such fractional coalition structures in (8) is necessary: In this game, any partition of players into coalitions contains at most one coalition with value 1, yet the multiplicative cost of stability is $3/2$. On the other hand, $\{S_1, S_2, S_3\}$, where $S_1 = \{1, \ldots, 2k\}$, $S_2 = \{k+1, \ldots, 3k\}$, $S_3 = \{2k+1, \ldots, 3k, 1, \ldots, k\}$, is a balanced collection of subsets with $\delta_{S_1} = \delta_{S_2} = \delta_{S_3} = 1/2$, $v(S_1) = v(S_2) = v(S_3) = 1$, and we have $\text{multCoS}(G) = \sum_{i=1}^{3} \delta_{S_i} v(S_i)$.

Unfortunately, in general neither the linear program $\mathcal{LP}^*$ nor equation (8) provide an efficient way to compute the cost of stability. This remains true even if the value of each coalition can be easily computed [10].

4. Bounds on the Cost of Stability

In this section, we will provide bounds on the cost of stability for several classes of coalitional games. It will be convenient to consider profit-sharing games and expense-sharing games separately. However, at the end of this section, we will also discuss how to relate the bounds on the cost of stability for these two types of games.

4.1. The Cost of Stability in Profit-Sharing Games

Consider an arbitrary profit-sharing game $G = \langle N, v \rangle$ with an empty core. We have observed that G can be stabilized by paying the maximum possible coalitional value to each agent, i.e.,

$$\mathrm{addCoS}(G) \le n \cdot \max_{S \subseteq N} v(S) - v(N).$$

For monotone games we have $\max_{S \subseteq N} v(S) = v(N)$, so this bound can be simplified to $\mathrm{addCoS}(G) \le (n-1)v(N)$, or, equivalently, $\mathrm{multCoS}(G) \le n$. In fact, this bound is tight, as illustrated by the (simple, anonymous) game G' given by $v'(S) = 1$ for all $S \neq \emptyset$: Clearly, in this game any payoff vector that offers some agent less than 1 will not be stable, whereas setting $p_i = 1$ for all $i \in N$ ensures stability. Thus, the multiplicative cost of stability can be as large as the number of agents. We summarize these observations as follows.

Observation 4.1. *Let $G = \langle N, v \rangle$ be a monotone profit-sharing game. Then* $\mathrm{multCoS}(G) \le$ *n and this bound is tight, even if G is simple and anonymous.*

We will now show how to refine this upper bound for specific subclasses of profit-sharing games.

4.1.1. Superadditive Profit-Sharing Games

For superadditive profit-sharing games, the upper bound of n can be strengthened considerably. Note that in such games the grand coalition maximizes the social welfare, so its stability is particularly desirable. Yet, we will see that ensuring stability may turn out to be quite costly even in this restricted setting.

We start by stating a technical lemma that will be used throughout the paper; we relegate its proof to Appendix A.

Lemma 4.2. *Let $G = \langle N, v \rangle$ be a superadditive profit-sharing game. Then there exists a solution $\{\delta_S\}_{S \subseteq N}$ to $\overline{G}$ such that for every $R, T \subseteq N$ with $\delta_R \neq 0$, $\delta_T \neq 0$ we have $R \cap T \neq \emptyset$.*

Lemma 4.2 enables us to prove a tight upper bound on the cost of stability in superadditive games.

Theorem 4.3. *Let $G = \langle N, v \rangle$ be a superadditive profit-sharing game. Then* $\mathrm{multCoS}(G) \le$ *$\sqrt{n}$, and this bound is asymptotically tight.*

Proof. Consider a solution $\{\delta_S\}_{S \subseteq N}$ to $\overline{G}$ such that $R \cap T \neq \emptyset$ for every pair of sets $R, T \subseteq N$ with $\delta_R, \delta_T > 0$; its existence is ensured by Lemma 4.2. Since $\{\delta_S\}_{S \subseteq N}$ is a solution to $\overline{G}$ and G is monotone, we obtain

$$\mathrm{multCoS}(G) = \frac{1}{v(N)} \left(\sum_{S \subseteq N} \delta_S v(S) \right) \le \sum_{S \subseteq N} \delta_S.$$

To complete the proof, we will argue that $\sum_{S\subseteq N}\delta_S \le \sqrt{n}$.

Suppose first that there exists a set $T \subseteq N$ with $|T| \le \sqrt{n}$, $\delta_T > 0$. Every set $S \subseteq N$ with $\delta_S > 0$ intersects T. Thus, we have

$$\sum_{S\subseteq N}\delta_S \le \sum_{i\in T}\sum_{S\subseteq N: i\in S}\delta_S = \sum_{i\in T}1 = |T| \le \sqrt{n}.$$

Now, suppose that $|S| > \sqrt{n}$ for every $S \subseteq N$ with $\delta_S > 0$. Then we have

$$\sqrt{n}\sum_{S\subseteq N}\delta_S < \sum_{S\subseteq N}|S|\delta_S = \sum_{S\subseteq N}\sum_{i\in S}\delta_S = \sum_{i\in N}\sum_{S\subseteq N: i\in S}\delta_S = \sum_{i\in N}1 = n,$$

which implies $\sum_{S\subseteq N}\delta_S \le \sqrt{n}$.

To see that this bound is asymptotically tight, consider the game G_q described in Example 3.5. Indeed, we have argued that G_q is superadditive and $\mathrm{multCoS}(G_q) > \sqrt{n} - 1$. ■

In many scenarios it is not realistic to assume that an arbitrary group of players can deviate from the grand coalition. In particular, deviations by large groups of players may be infeasible due to communication, coordination, or trust issues. In such cases, it is meaningful to ask whether the game in question has an outcome that is resistant to deviations by coalitions of size at most k (where k is a given parameter), or whether it can be made resistant to such deviations by a subsidy of at most Δ.

Formally, given a profit-sharing game $G = \langle N, v\rangle$ and an integer k, $1 \le k \le n$, we define a game $G|_k = \langle N, v|_k\rangle$ by setting $v|_k(S) = v(S)$ for every S such that $|S| \le k$ or $S = N$ and $v|_k(S) = 0$ otherwise. For such games, we have the following result, whose proof is similar to that of Theorem 4.3 (see Appendix A).

Theorem 4.4. *Let $G = \langle N, v\rangle$ be a superadditive profit-sharing game. Then for every positive integer $k < |N|$ we have* $\mathrm{multCoS}(G|_k) \le k$.

Combining Theorem 4.3 and Theorem 4.4, we conclude that for any superadditive profit-sharing game G it holds that $\mathrm{multCoS}(G|_k) \le \min\{\sqrt{n}, k\}$.

4.1.2. Anonymous Profit-Sharing Games

Recall that an anonymous profit-sharing game $G = \langle N, v\rangle$ can be specified by a list of n numbers $v_1, \ldots, v_n$, where $v_k = v(\{1, \ldots, k\})$: We have $v_k = v(S)$ for every coalition $S \subseteq N$ of size k. Using this notation, we can simplify equation (8) for anonymous games.

Theorem 4.5. *Let $G = \langle N, v\rangle$ be an anonymous profit-sharing game. Then*

$$\mathrm{multCoS}(G) = \frac{n}{v_n}\cdot\max_{k\le n}\frac{v_k}{k}.$$

Proof. Pick $k^* \in \mathrm{argmax}_{k\le n} v_k/k$, and let $\mathbf{p}$ be the payoff vector given by $p_i = v_{k^*}/k^*$ for all $i \in N$. Clearly, $\mathbf{p}$ is stable: For every $S \subseteq N$, we have $p(S) = |S|v_{k^*}/k^* \ge v(S)$ by our choice of k^*.

Now, suppose that there is a stable payoff vector $\mathbf{q}$ with $q(N) < p(N)$. Renumber the players so that $q_1 \leq \cdots \leq q_n$ and set $S^* = \{1, \ldots, k^*\}$. Clearly, we have $q(S^*)/k^* \leq q(N)/n$, and hence

$$q(S^*) \leq \frac{k^*}{n} q(N) < \frac{k^*}{n} p(N) = v_{k^*},$$

which means that $\mathbf{q}$ is not stable. Hence,

$$\text{multCoS}(G) = \frac{p(N)}{v(N)} = \frac{n}{v_n} \cdot \frac{v_{k^*}}{k^*},$$

which completes the proof. ∎

If we assume both superadditivity and anonymity, we can strengthen Theorem 4.3 considerably.

Theorem 4.6. *Let $G = \langle N, v \rangle$ be an anonymous superadditive profit-sharing game. Then* $\text{multCoS}(G) \leq 2 - \frac{2}{n+1}$*, and this bound is tight.*

Proof. Fix an anonymous superadditive profit-sharing game $G = \langle N, v \rangle$ with $|N| = n$.

For every $k \geq (n+1)/2$, we have

$$\frac{n}{v_n} \cdot \frac{v_k}{k} \leq \frac{2nv_k}{(n+1)v_n} \leq \frac{2n}{n+1}.$$

Now, consider a $k < (n+1)/2$ and let $q = \lfloor n/k \rfloor$. Since k is integer, we have $q \geq 2$. By superadditivity we have $v_n \geq qv_k$. Let $\alpha = n/k - q < 1$. Then if $n \geq 3$, we have

$$\frac{n}{v_n} \cdot \frac{v_k}{k} = (q + \alpha)\frac{v_k}{v_n} \leq (q + \alpha)\frac{1}{q} = 1 + \frac{\alpha}{q} < \frac{3}{2} \leq \frac{2n}{n+1},$$

and if $n = 2$, we obtain $k = 1$ and $(n/v_n) \cdot (v_k/k) = 2v_1/v_2 \leq 1 < 4/3$.

Thus, by Theorem 4.5 we obtain

$$\text{multCoS}(G) = \max_{k \leq n} \frac{n}{v_n} \cdot \frac{v_k}{k} \leq \frac{2n}{n+1} = 2 - \frac{2}{n+1}.$$

To show that this bound is tight, for every odd $n > 1$ we define a game $G_n = \langle N, v \rangle$ with $|N| = n$ by setting $v(S) = 1$ if $|S| > n/2$, and $v(S) = 0$ otherwise. In G_n every two winning coalitions intersect, and thus this game is superadditive. Let $k^* = \lceil n/2 \rceil$. Then by Theorem 4.5 we have

$$\text{multCoS}(G_n) \geq \frac{n}{v_n} \cdot \frac{v_{k^*}}{k^*} = \frac{n}{\lceil n/2 \rceil} = \frac{2n}{n+1},$$

which completes the proof. ∎

4.2. The Cost of Stability in Expense-Sharing Games

Having analyzed profit-sharing games in detail, we now turn to study expense-sharing games. The cost of stability in these games can be characterized via the Bondareva–Shapley theorem, similarly to equation (8): For a game $G = \langle N, c\rangle$ with an empty core, we have

$$\mathrm{multCoS}(G) = \frac{1}{c(N)} \min\left\{ \sum_{S\in\mathcal{D}} \delta_S c(S) \;\middle|\; \mathcal{D} \in \mathcal{BC}(N), \{\delta_S\}_{S\in\mathcal{D}} \in \mathcal{B}(\mathcal{D}) \right\}. \qquad (10)$$

We will use equation (10) to prove bounds on the cost of stability for several classes of expense-sharing games.

4.2.1. Subadditive Expense-Sharing Games

A well-studied class of expense-sharing games is that of *set cover games* [11]. Briefly, a set cover game is described by an instance of the set cover problem: The agents are elements of the ground set, and the cost of a coalition S is the cost of the cheapest collection of subsets that covers all elements of S. More formally, a set cover game is an expense-sharing game given by a tuple $\langle N, \mathcal{F}, w\rangle$, where $N = \{1, \dots, n\}$ is a set of agents, $\mathcal{F}$ is a collection of subsets of N that satisfies $\bigcup_{F\in\mathcal{F}} F = N$, and $w : \mathcal{F} \to \mathbb{R}_+$ is a mapping that assigns a non-negative weight to each set in $\mathcal{F}$. The cost of a coalition $S \subseteq N$ is given by

$$c(S) = \min\left\{ \sum_{F\in\mathcal{F}'} w(F) \;\middle|\; \mathcal{F}' \subseteq \mathcal{F}, S \subseteq \bigcup_{F\in\mathcal{F}'} F \right\}.$$

We will write $\mathcal{F}^*(S)$ to denote a cheapest cover of the set S.

It is easy to see that the hospital game described in Example 1.1 is a set cover game with three agents (the hospitals). This observation can be extended as follows.

Proposition 4.7. *Set cover games are monotone and subadditive. Furthermore, every monotone and subadditive expense-sharing game can be described as a set cover game.*

Proof. Fix a set cover game G given by a tuple $\langle N, \mathcal{F}, w\rangle$. Clearly, G is monotone. Further, for every pair of subsets $S, T \subseteq N$ we have $S \cup T \subseteq \mathcal{F}^*(S) \cup \mathcal{F}^*(T)$. Therefore, $c(S \cup T) \le c(S) + c(T)$, i.e., G is subadditive.

Conversely, given a monotone subadditive expense-sharing game $G = \langle N, c\rangle$, we construct a set cover game by setting $\mathcal{F} = 2^N$, $w(F) = c(F)$ for every $F \in \mathcal{F}$. We will now argue that the resulting game $G' = \langle N, c'\rangle$ is equivalent to G. Indeed, consider a set S and its cheapest cover $\mathcal{F}^*(S)$. We have $c'(S) = \sum_{F\in\mathcal{F}^*(S)} c(F)$. Since G is monotone, we can assume that the sets in $\mathcal{F}^*(S)$ are pairwise disjoint: If we have $F_1 \cap F_2 \neq \emptyset$ for some $F_1, F_2 \in \mathcal{F}^*(S)$, we can replace F_2 with $F_2 \setminus F_1$ without increasing the overall cost. Now, set $F' = \cup_{F\in\mathcal{F}^*(S)} F$. The subadditivity of G implies that $c(F') \le \sum_{F\in\mathcal{F}^*(S)} c(F) = c'(S)$. Further, since S is a subset of F', we have $c(S) \le c(F')$ and hence $c(S) \le c'(S)$. On the other hand, $\{S\}$ is a cover of S, so we have $c'(S) \le c(S)$. Thus, $c'(S) = c(S)$. Since this holds for every set $S \subseteq N$, the games G and G' are equivalent. ■

We remark that the construction in the proof of Proposition 4.7 produces a set cover game with exponentially many sets. However, sometimes the number of sets can be reduced. In particular, if $c(S) = c(S_1) + \cdots + c(S_k)$ for some partition $\{S_1, \ldots, S_k\}$ of S, then the set S can be removed from $\mathcal{F}$. This observation is inspired by the *synergy coalition groups* representation of superadditive profit-sharing games due to Conitzer and Sandholm [12].

The cost of stability in set cover games is closely related to the integrality gap of the standard linear program for the set cover problem. Specifically, consider an instance $\langle N, \mathcal{F}, w\rangle$ of the set cover problem. The cost of the grand coalition in the corresponding game G can be written as the following integer linear program (ILP) over the variables $\{y_j\}_{F_j \in \mathcal{F}}$:

$$\min \sum_{F_j \in \mathcal{F}} w(F_j) y_j \qquad \text{subject to:}$$

$$\sum_{j : i \in F_j} y_j \geq 1 \text{ for each } i \in N, \tag{11}$$

$$y_j \in \{0, 1\} \text{ for each } F_j \in \mathcal{F}. \tag{12}$$

In this ILP, setting $y_j = 1$ corresponds to picking the set F_j for the cover. The linear relaxation of this program is obtained by replacing condition (12) with the condition $y_j \geq 0$ for each $F_j \in \mathcal{F}$ (clearly, in the optimal solution we will have $y_j \leq 1$ for each $F_j \in \mathcal{F}$); we will denote the resulting linear program by $\mathcal{LP}(G)$.

For a given instance of the set cover problem, the ratio between the value of its optimal integer solution and that of its optimal fractional solution is known as the *integrality gap*. Formally, let $\mathrm{ILP}(G)$ and $\mathrm{LP}(G)$ denote, respectively, the values of optimal integer and fractional solutions of the linear program $\mathcal{LP}(G)$ corresponding to the set cover game G. The integrality gap of G is defined as $\mathrm{IG}(G) = \frac{\mathrm{ILP}(G)}{\mathrm{LP}(G)}$; note that $\mathrm{ILP}(G) = c(N)$. The following theorem relates the integrality gap of a set cover game to its multiplicative cost of stability.

Theorem 4.8. *Let G be a set cover game. Then* $\mathrm{multCoS}(G) = \frac{1}{\mathrm{IG}(G)}$.

The proof of Theorem 4.8 can be obtained by modifying the proof of Theorem 1 in Deng et al. [13]; an alternative proof using the Bondareva–Shapley condition is given in [4] (Corollary 15.9). For completeness, we provide a direct proof in Appendix A, and also demonstrate how an optimal stable payoff vector can be computed efficiently.

The integrality gap of the set cover problem is well-studied in the literature: It is known to be bounded from above by $H_n = \sum_{i=1}^{n} 1/i < \ln n + 1$, and n can be replaced with k when all set sizes are bounded by k (see [14]). Moreover, these bounds are essentially tight, even when the sets are non-weighted [15]. Thus, we obtain the following corollary.

Corollary 4.9. *Let $G = \langle N, c\rangle$ be a subadditive expense-sharing game with $|N| = n$. Then for every positive integer $k \leq n$, we have* $\mathrm{multCoS}(G|_k) \geq \frac{1}{\ln k + 1}$*, and this bound is asymptotically tight. In particular,* $\mathrm{multCoS}(G) \geq \frac{1}{\ln n + 1}$.

It is interesting to note that the worst-case bound on the multiplicative cost of stability for subadditive expense-sharing games (Corollary 4.9) is much stronger than the one for superadditive profit-sharing games (Theorem 4.3).

4.2.2. Anonymous Expense-Sharing Games

An anonymous expense-sharing game $G = \langle N, c\rangle$ can be described by a list of n numbers $c_1, \dots, c_n$, where $c_k = c(\{1, \dots, k\})$ for $k = 1, \dots, n$. As with profit-sharing games, anonymity allows us to simplify equation (10) (the proof is similar to that of Theorem 4.5 and is relegated to Appendix A).

Theorem 4.10. *Let $G = \langle N, c\rangle$ be an anonymous expense-sharing game. Then*

$$\mathrm{multCoS}(G) = \frac{n}{c_n} \cdot \min_{k \le n} \frac{c_k}{k}.$$

Without further assumptions, the multiplicative cost of stability of an anonymous expense-sharing game can still be as low as 0: Consider, for instance, the game $G = \langle N, c\rangle$ given by $c(N) = 1$ and $c(S) = 0$ for every $S \subsetneq N$. However, if we assume both anonymity and subadditivity, we can bound the multiplicative cost of stability by $\frac{1}{2} + \frac{1}{2n-2}$. Although the bound is similar to the one for profit-sharing games (slightly above $\frac{1}{2}$ rather than slightly below 2), the proof is somewhat more complicated than that of Theorem 4.6 (see Appendix A).

Theorem 4.11. *Let $G = \langle N, c\rangle$ be an anonymous subadditive expense-sharing game. Then $\mathrm{multCoS}(G) \ge \frac{1}{2} + \frac{1}{2n-2}$, and this bound is tight.*

4.3. Expense–Profit Duality

We have obtained bounds on the cost of stability for several classes of profit-sharing and expense-sharing games. The reader may wonder if the bounds for expense-sharing games can be derived directly from the bounds for profit-sharing games or vice versa. However, it seems that this is not the case.

There exists a natural mapping between profit-sharing games and expense-sharing games that works as follows. The *dual* (not to be confused with LP duality) of a given profit-sharing game $G_v = \langle N, v\rangle$ is the expense-sharing game $G_c = \langle N, c\rangle$ defined by

$$c(S) = \begin{cases} v(S) & \text{if } S = N \\ v(N) - v(N \setminus S) & \text{if } S \neq N. \end{cases}$$

Similarly, the *dual* of an expense-sharing game $G_c = \langle N, c\rangle$ is the profit-sharing game $G_v = \langle N, v\rangle$ defined by

$$v(S) = \begin{cases} c(S) & \text{if } S = N \\ c(N) - c(N \setminus S) & \text{if } S \neq N. \end{cases}$$

It can be shown that the core of G_v is empty if and only if the core of G_c is empty (see [16, p. 4]), i.e., $\mathrm{multCoS}(G_c) = 1$ if and only if $\mathrm{multCoS}(G_v) = 1$. Also, G_c is monotone if and only if G_v is. One might therefore expect that it is possible to compute the cost of stability of an expense-sharing game by analyzing its dual, i.e., that there exists a function f such that $\mathrm{multCoS}(G_c) = f(\mathrm{multCoS}(G_v))$. Unfortunately, the following example shows that the cost of stability of a game does not reveal much about the cost of stability of its dual (as long as the core is known to be empty).

Example 4.12. Consider an expense-sharing game $G_c = \langle N, c\rangle$ with $c(N) = 1$ and $c(\{i\}) = 0$ for all $i \in N$. We have $\mathrm{multCoS}(G_c) = 0$: Since each agent can get the service for free by forming a singleton coalition, the center will not recover any of the costs. The dual game $G_v = \langle N, v\rangle$ satisfies $v(N) = 1$ and $v(S) = 1$ for each $S \subseteq N$ with $|S| = n - 1$. However, the values of all other coalitions can be arbitrary, and hence $\mathrm{multCoS}(G_v)$ can be as low as $\frac{n}{n-1}$ (if $v(S) = 0$ for every coalition $S \subseteq N$ such that $|S| < n - 1$) or as high as n (if $v(S) = 1$ for every $S \subseteq N$).

5. The Cost of Stability and the Least Core

In this section we explore the relationship between the cost of stability, the extended core, and several variants of the least core. We focus on profit-sharing games.

We start by formally defining the *strong* and the *weak* least core. Consider a profit-sharing game $G = \langle N, v\rangle$ and some $\varepsilon \geq 0$. Following Shapley and Shubik [17], we define the *strong ε-core of G* as the set $\mathrm{SC}_\varepsilon(G)$ of all pre-imputations for G such that no coalition can gain more than ε by deviating:

$$\mathrm{SC}_\varepsilon(G) = \{\mathbf{p} \in \mathbb{I}(G) \mid p(S) \geq v(S) - \varepsilon \text{ for all } S \subseteq N\}.$$

Clearly, if ε is large enough, we have $\mathrm{SC}_\varepsilon(G) \neq \emptyset$. The quantity $\varepsilon_{\mathbf{S}}(G) = \inf\{\varepsilon \geq 0 \mid \mathrm{SC}_\varepsilon(G) \neq \emptyset\}$ is called the *value of the strong least core of G*. We remark that there are variants of this definition with and without the constraint $\varepsilon \geq 0$; we impose this constraint since we also require the cost of stability to be non-negative. The strong $\varepsilon_{\mathbf{S}}$-core of G is referred to as the *strong least core of G*, and is denoted by $\mathbb{SLC}(G)$; a simple continuity argument shows that for every profit-sharing game G the set $\mathbb{SLC}(G)$ is non-empty.

A particular payoff vector that is known to be contained in the strong least core is the *pre-nucleolus* $\mathrm{PN}(G)$. This is the pre-imputation that (1) minimizes the *excess* $v(S) - p(S)$ of the least satisfied coalition; (2) minimizes the excess of the second-least satisfied coalition subject to (1), etc. (see [1, 18] for a formal definition).

In contrast, the *weak ε-core of G* (see, e.g., [7]) consists of pre-imputations such that no coalition can deviate in a way that profits each deviator by more than ε:

$$\mathrm{WC}_\varepsilon(G) = \{\mathbf{p} \in \mathbb{I}(G) \mid p(S) \geq v(S) - \varepsilon|S| \text{ for all } S \subseteq N\}.$$

Just as for the strong least core, we define the *value of the weak least core of G* as $\varepsilon_{\mathbf{W}}(G) = \inf\{\varepsilon \geq 0 \mid \mathrm{WC}_\varepsilon(G) \neq \emptyset\}$; the *weak least core of G* (denoted by $\mathbb{WLC}(G)$) is its $\varepsilon_{\mathbf{W}}$-core. Again, a continuity argument shows that every profit-sharing game has a non-empty weak least core.

Note that $\mathrm{SC}_\varepsilon(G) \subseteq \mathrm{WC}_\varepsilon(G)$ for any $\varepsilon > 0$ and, consequently, $\varepsilon_{\mathbf{W}}(G) \leq \varepsilon_{\mathbf{S}}(G)$.

Positive payoffs. Note that both weak and strong ε-core may contain payoff vectors with negative coordinates. However, in many cases negative payoffs are not acceptable. To capture such settings, we define the *positive strong ε-core of G* as

$$\mathrm{PSC}_\varepsilon(G) = \{\mathbf{p} \in \mathbb{I}(G) \mid p_i \geq 0 \text{ for all } i \in N, p(S) \geq v(S) - \varepsilon \text{ for all } S \subseteq N\}.$$

Similarly, the notions of the *positive weak ε-core* (denoted by $\mathrm{PWC}_\varepsilon(G)$), the *value of the positive strong/weak least core* (denoted by $\varepsilon_{\mathbf{PS}}$ and $\varepsilon_{\mathbf{PW}}$, respectively) and the *positive strong/weak least core* (denoted by $\mathbb{PSLC}(G)$ and $\mathbb{PWLC}(G)$, respectively), are defined by adding the requirement that $p_i \geq 0$ for all $i \in N$ to the corresponding definitions above. Clearly, we have $\mathrm{PSC}_\varepsilon(G) \subseteq \mathrm{SC}_\varepsilon(G)$, $\mathrm{PWC}_\varepsilon(G) \subseteq \mathrm{WC}_\varepsilon(G)$ and $\varepsilon_{\mathbf{PS}} \geq \varepsilon_{\mathbf{S}}$, $\varepsilon_{\mathbf{PW}} \geq \varepsilon_{\mathbf{W}}$.

Similarly to the additive cost of stability, the values of the (positive) strong least core and the (positive) weak least core can be obtained as optimal values of certain linear programs. We can think of all these notions as different *measures* of (in)stability. For instance, it is clear that conditions $\varepsilon_{\mathbf{S}}(G) > 0$, $\varepsilon_{\mathbf{W}}(G) > 0$, $\varepsilon_{\mathbf{PS}}(G) > 0$, $\varepsilon_{\mathbf{PW}}(G) > 0$, and $\mathrm{addCoS}(G) > 0$ are all equivalent, as each of them holds if and only if the core of G is empty. We will now discuss the relationship between the (positive) weak least core, the (positive) strong least core, and the cost of stability in more detail.

5.1. The Weak Least Core

We start by observing that the value of the weak least core is closely related to the additive cost of stability (a similar observation was made by Bejan and Gómez [7]).

Proposition 5.1. *Let $G = \langle N, v\rangle$ be a profit-sharing game. Then*

1. $\mathrm{addCoS}(G) = n\varepsilon_{\mathbf{W}}(G)$.
2. $\mathbb{WLC}(G) \subseteq \mathbb{EC}(G)$.

Proof. If G has a non-empty core, we have $\mathrm{addCoS}(G) = \varepsilon_{\mathbf{W}}(G) = 0$ and $\mathbb{WLC}(G) = \mathbb{EC}(G) = \mathbb{C}(G)$, so we are done.

Now, suppose that the core of G is empty, and let $\Delta = \mathrm{addCoS}(G)$ and $\varepsilon = \varepsilon_{\mathbf{W}}(G)$. If a pre-imputation $\mathbf{q}$ is in the weak least core of G, then the payoff vector $\mathbf{q}'$ given by $q'_i = q_i + \varepsilon$ for $i = 1, \ldots, n$ satisfies $q'(S) \geq v(S)$ for every $S \subseteq N$. Thus, this payoff vector belongs to the core of $G(n\varepsilon)$. This implies $\Delta \leq n\varepsilon$.

Conversely, consider a payoff vector $\mathbf{p}$ in the core of $G(\Delta)$, and let $\mathbf{p}'$ be a payoff vector given by $p'_i = p_i - \Delta/n$ for $i = 1, \ldots, n$. We have $p'(N) = v(N)$, so $\mathbf{p}'$ is a pre-imputation for G. Moreover, for every $S \subseteq N$ we have $p'(S) = p(S) - (\Delta/n)|S| \geq v(S) - (\Delta/n)|S|$, so $\mathbf{p}'$ is in the weak $\frac{\Delta}{n}$-core of G, and therefore $\varepsilon \leq \Delta/n$.

We conclude that $\Delta = n\varepsilon$. As we have argued that every pre-imputation in the weak least core can be stabilized with a subsidy of $n\varepsilon$, it follows that $\mathbb{WLC}(G) \subseteq \mathbb{EC}(G)$. ■

We remark, however, that the proof of Proposition 5.1 does not go through for the positive weak least core: It may happen that a payoff vector $\mathbf{p}$ in the core of $G(\Delta)$ is such that $p_i < \Delta/n$ for some $i \in N$. Our next proposition relates the value of the positive weak least core and the cost of stability for monotone profit-sharing games.

Proposition 5.2. *Let $G = \langle N, v\rangle$ be a monotone profit-sharing game. Then*

$$2\varepsilon_{\mathbf{PW}}(G) \leq \mathrm{addCoS}(G) \leq n\varepsilon_{\mathbf{PW}}(G),$$

and these bounds are tight.

Proof. The upper bound is immediate, since $\mathrm{addCoS}(G) = n\varepsilon_{\mathbf{W}}(G) \leq n\varepsilon_{\mathbf{PW}}(G)$. To see that it is tight, consider the game $G = \langle N, v\rangle$ such that $v(S) = 1$ for every non-empty coalition S. We have $\mathrm{addCoS}(G) = n - 1$, $\varepsilon_{\mathbf{PW}}(G) = (n-1)/n$.

For the lower bound, let $\Delta = \mathrm{addCoS}(G)$ and consider a payoff vector $\mathbf{q} \in \mathbb{C}(G(\Delta))$; note that $q_i \geq v(\{i\}) \geq 0$ for all $i \in N$. Suppose first that there exist two players i and j, $i \neq j$, such that $q_i \geq \Delta/2$ and $q_j \geq \Delta/2$. Consider the payoff vector $\mathbf{q}'$ defined by

$$q'_\ell = \begin{cases} q_\ell & \text{if } \ell \neq i, j, \\ q_\ell - \Delta/2 & \text{if } \ell = i \text{ or } \ell = j. \end{cases}$$

Clearly, $\mathbf{q}'$ is a pre-imputation for G, all of its entries are non-negative, and for every coalition $S \subseteq N$ we have $q'(S) \geq q(S) - (\Delta/2)|S| \geq v(S) - (\Delta/2)|S|$, so $\varepsilon_{\mathbf{PW}}(G) \leq \Delta/2$.

Now, suppose that there is at most one player i such that $q_i \geq \Delta/2$; assume without loss of generality that $q_i < \Delta/2$ for $i = 2, \ldots, n$ and set $S = \{2, \ldots, n\}$.

Observe that $q_1 \leq v(N)$. Indeed, if $q_1 > v(N)$, the payoff vector $\mathbf{q}'$ given by $q'_1 = v(N)$, $q'_i = q_i$ for $i \in S$ satisfies $q'(N) < q(N)$. Further, $\mathbf{q}'$ is stable: We have $q'(T) = q(T) \geq v(T)$ for every $T \subseteq S$ and, since G is monotone, $q'(T) \geq v(N) \geq v(T)$ for every $T \subseteq N$ such that $1 \in T$. This is a contradiction with our choice of $\mathbf{q}$.

Thus, we have $q(S) = q(N) - q_1 \geq \Delta$. This means that there exists a pre-imputation $\mathbf{q}''$ for G such that $q''_1 = q_1$ and $0 \leq q''_i \leq q_i$ for $i \in S$. Since $q_i < \Delta/2$ for all $i \in S$, we have $q_i - q''_i \leq \Delta/2$ for all $i \in N$. Consequently, for every set $T \subseteq N$ we have $q''(T) \geq q(T) - (\Delta/2)|T|$, which means that $\mathbf{q}'' \in \mathrm{PWC}_{\Delta/2}(G)$.

To see that this bound is tight, consider the weighted voting game $G = [1, 1, 0, 0, \ldots, 0; 1]$: We have $\mathrm{addCoS}(G) = 1$ and $\varepsilon_{\mathbf{PW}}(G) = 1/2$. ■

5.2. The Strong Least Core

In this section we derive upper and lower bounds on the ratio $\mathrm{addCoS}(G)/\varepsilon_{\mathbf{S}}(G)$. Since $\varepsilon_{\mathbf{W}}(G) \leq \varepsilon_{\mathbf{S}}(G)$, Proposition 5.1 immediately implies that $\mathrm{addCoS}(G)/\varepsilon_{\mathbf{S}}(G) \leq n$. For general profit-sharing games this bound is tight. To see this, consider the game $G = \langle N, v\rangle$ with $v(S) = 1$ for all $S \neq \emptyset$: We have $\varepsilon_{\mathbf{S}}(G) = (n-1)/n$, $\mathrm{addCoS}(G) = n - 1$. We will now explore whether this bound can be improved if we place additional restrictions on the characteristic function.

In what follows, we use the following construction. Given a game G with an empty core, we set $\varepsilon = \varepsilon_{\mathbf{S}}(G)$ and define a new game $G_\varepsilon = \langle N, v_\varepsilon\rangle$, where $v_\varepsilon(S) = v(S) - \varepsilon$ for all $S \subsetneq N$, and $v_\varepsilon(N) = v(N)$. Intuitively, G_ε is obtained by imposing the minimum penalty on deviating coalitions that ensures stability, just as $\overline{G}$ is obtained by providing the minimum subsidy that ensures stability. Clearly, $\mathbb{C}(G_\varepsilon) = \mathbb{SLC}(G)$.

We will now show that for superadditive games we can strengthen the upper bound on the ratio $\mathrm{addCoS}(G)/\varepsilon_{\mathbf{S}}(G)$ to $\sqrt{n}$.

Theorem 5.3. *Let $G = \langle N, v\rangle$ be a superadditive profit-sharing game, Then $\mathrm{addCoS}(G) \leq \sqrt{n} \cdot \varepsilon_{\mathbf{S}}(G)$, and this bound is tight up to a small additive constant.*

Proof. By Lemma 4.2 there exists a solution $(\delta_S)_{S \in \mathcal{D}}$ to $\overline{G}$ such that every two sets S and T with $\delta_S \neq 0$ and $\delta_T \neq 0$ have a non-empty intersection.

Since $(\delta_S)_{S\in\mathcal{D}}$ is a balancing weight vector for $\mathcal{D} = 2^N$, applying the Bondareva–Shapley theorem to the game G_ε (which has a non-empty core), we obtain

$$\sum_{S\subseteq N} \delta_S(v(S) - \varepsilon) = \sum_{S\subseteq N} \delta_S v_\varepsilon(S) \le v_\varepsilon(N) = v(N).$$

Together with the fact that $\sum_{S\subseteq N} \delta_S \le \sqrt{n}$ (cf. the proof of Theorem 4.3), this implies

$$\begin{aligned} \text{addCoS}(G) &= \sum_{S\subseteq N} \delta_S v(S) - v(N) \le \sum_{S\subseteq N} \delta_S v(S) - \sum_{S\subseteq N} \delta_S(v(S) - \varepsilon) \\ &= \varepsilon \sum_{S\subseteq N} \delta_S \le \sqrt{n}\varepsilon = \sqrt{n}\varepsilon_{\mathbf{S}}(G), \end{aligned}$$

which completes the proof of the upper bound.

To see that this bound is tight, consider the game G_q (see Example 3.5). Since G_q is a simple game, we have $\varepsilon_{\mathbf{S}}(G_q) \le 1$. Moreover, consider the payoff vector $\mathbf{p}$ given by $p_i = 1/n$ for all $i \in N$. If S is a winning coalition in G_q, then $|S| \ge q + 1$ and thus $p(S) \ge (q+1)/n \ge 1/\sqrt{n}$. Therefore $\varepsilon_{\mathbf{S}}(G_q) \le 1 - 1/\sqrt{n}$, and hence $\sqrt{n}\cdot\varepsilon_{\mathbf{S}}(G) \le \sqrt{n} - 1$. On the other hand, we have seen that $\text{multCoS}(G_q) > \sqrt{n} - 1$. Since G_q is a simple game, this implies

$$\text{addCoS}(G_q) = \text{multCoS}(G_q) - 1 > (\sqrt{n} - 1) - 1 \ge \sqrt{n}\cdot\varepsilon_{\mathbf{S}}(G) - 1,$$

which completes the proof. ∎

It is instructive to compare Theorem 4.3 with Theorem 5.3: For any superadditive profit-sharing game $G = \langle N, v\rangle$, the former shows that $\text{multCoS}(G) \le \sqrt{n}$, while the latter can be rewritten as

$$\text{multCoS}(G) = 1 + \frac{\text{addCoS}(G)}{v(N)} \le 1 + \sqrt{n}\,\frac{\varepsilon_{\mathbf{S}}(G)}{v(N)},$$

We have $\varepsilon_{\mathbf{S}}(G) \le v(N)$ for every profit-sharing game $G = \langle N, v\rangle$, and for many games $\varepsilon_{\mathbf{S}}(G)$ is significantly smaller than $v(N)$; for such games Theorem 5.3 is substantially stronger than Theorem 4.3.

Our next proposition establishes a lower bound on the ratio $\text{addCoS}(G)/\varepsilon_{\mathbf{S}}(G)$.

Proposition 5.4. *Let $G = \langle N, v\rangle$ be a profit-sharing game. Then $\text{addCoS}(G) \ge \frac{n}{n-1}\varepsilon_{\mathbf{S}}(G)$, and this bound is tight.*[2]

Note that for monotone games the proposition follows directly from Proposition 5.6, as $\varepsilon_{\mathbf{S}}(G) \le \varepsilon_{\mathbf{PS}}(G)$.

[2] While we later noticed that the proposition in fact follows from a very simple argument, the original proof used in an early version of this paper may provide some more intuition on how the least core and the extended core spatially relate. See [19, 20] for details.

Proof. Observe that $\mathrm{WC}_\varepsilon(G) \subseteq \mathrm{SC}_{(n-1)\varepsilon}(G)$ for any $\varepsilon > 0$. Indeed, let $\varepsilon > 0$, and consider a pre-imputation $\mathbf{p} \in \mathrm{WC}_\varepsilon(G)$. For every coalition $S \subsetneq N$ we have $p(S) \geq v(S) - |S|\varepsilon \geq v(S) - (n-1)\varepsilon$, and hence $\mathbf{p} \in \mathrm{SC}_{(n-1)\varepsilon}(G)$. This means that $\varepsilon_{\mathbf{S}} \leq (n-1)\varepsilon_{\mathbf{W}}$. Now, by Proposition 5.1 we obtain $\mathrm{addCoS}(G) = n\varepsilon_{\mathbf{W}}(G) \geq \frac{n}{n-1}\varepsilon_{\mathbf{S}}(G)$.

To see that this bound is tight, consider the game $G = \langle N, v\rangle$, where $v(S) = 1$ if $|S| \geq n-1$, and $v(S) = 0$ otherwise: We have $\mathrm{addCoS}(G) = 1/(n-1)$ and $\varepsilon_{\mathbf{S}}(G) = 1/n$. ■

For superadditive games, we can use Proposition 5.1, Theorem 5.3, and Proposition 5.4 to relate the value of the strong least core and the value of the weak least core.

Corollary 5.5. *Let $G = \langle N, v\rangle$ be a superadditive profit-sharing game. Then*

$$\sqrt{n}\varepsilon_{\mathbf{W}}(G) \leq \varepsilon_{\mathbf{S}}(G) \leq (n-1)\varepsilon_{\mathbf{W}}(G),$$

(superadditivity is not required for the second inequality).

In fact, we can strengthen Proposition 5.4, by showing that it extends to the positive strong least core, as long as we assume that the game is monotone.

Proposition 5.6. *Let $G = \langle N, v\rangle$ be a monotone profit-sharing game. Then $\mathrm{addCoS}(G) \geq \frac{n}{n-1}\varepsilon_{\mathbf{PS}}(G)$, and this bound is tight.*

The full proof requires some more notation, but we provide here some intuition. For a given minimal subsidy Δ, we can construct a pre-imputation in $\mathrm{PSC}_{\frac{n-1}{n}\Delta}(G)$. This is by picking the (non-negative) pre-imputation $\mathbf{q}$ that can be stabilized with the most "balanced" subsidies. That is, where the subsidy is distributed as equally as possible among the n agents. We then show that for every $i \in N$, either $q_i = 0$, or the subsidy for i in the stable extension of $\mathbf{q}$ is at least Δ/n. Finally, consider a coalition S that does not include some i of the second type, then the total subsidy for S is at most $\frac{n-1}{n}\Delta$ (and thus it is sufficient to relax $v(S)$ by $\frac{n-1}{n}\Delta$). If S contains all type 2 agents, then $q(S) = q(N) = v(N) \geq v(S)$, and no subsidy/relaxation is required to stabilize S. Tightness follows immediately from Proposition 5.4. We emphasize that monotonicity is a necessary condition both in Proposition 5.2 and in Proposition 5.6.

We have established a quantitative relationship between the additive cost of stability and the value of the strong least core. We conjecture that the strong least core is also closely related to the extended core.

Conjecture 5.7. *For every monotone profit-sharing game G with an empty core, we have $\mathbb{SLC}(G) \subseteq \mathbb{EC}(G)$.*

In other words, we conjecture that payoff vectors in the strong least core are among the ones that are the easiest to stabilize: For every $\mathbf{p} \in \mathbb{SLC}(G)$, it holds that $\mathrm{addCoS}(\mathbf{p}, G) = \mathrm{addCoS}(G)$. It is possible to verify that Conjecture 5.7 holds for $n = 2$ and $n = 3$ (see Appendix B); proving it for $n \geq 4$ is an interesting open problem. Note that this conjecture implies that the pre-nucleolus $\mathrm{PN}(G)$ (which is contained in the strong least core) is one of the easiest vectors to stabilize. A yet stronger conjecture would be that $\mathrm{PN}(G) \leq \mathrm{PN}(\overline{G})$, i.e., the pre-nucleolus of $\overline{G}$ is itself a stable extension of $\mathrm{PN}(G)$.

Another implication of Conjecture 5.7 is as follows. Suppose that we are given a game G with an empty core, and decide to divide the profits according to some vector $\mathbf{p}$ in the strong least core (e.g., according to the pre-nucleolus). Assume that at a later time an additional subsidy becomes available, so that the core of G becomes non-empty. Then Conjecture 5.7 implies that we can complete $\mathbf{p}$ to a vector in the core of this new game using positive transfers only. We note that a similar conjecture can be formulated for the positive strong least core, i.e., one can ask whether $\mathbb{PSLC}(G) \subseteq \mathbb{EC}(G)$.

6. The Cost of Stability in Games with Coalition Structures

So far, we have tacitly assumed that the only possible outcome of a coalitional game is the formation of the grand coalition. This makes sense when the grand coalition is optimal (as happens, for example, in s-additive games), or when the context dictates that only one coalition can be formed (as with companies competing for a contract). However, in other cases the agents may be better off forming several disjoint coalitions, each of which can focus on its own task. For instance, suppose that agent i has w_i units of some resource (e.g., time or money), and there is a number of identical tasks each of which requires q units of this resource to be completed: While this setting can be modeled as a weighted voting game, the formation of the grand coalition means that only one task will be completed, even if there are enough resources for several tasks.

Settings where agents may split into several teams, with each team working on its own project, can be modeled by *TU games with coalition structures*; such games were introduced by Aumann and Dréze [21]. In this section, we consider the problem of stabilizing TU games with coalition structures. We start by introducing the basic framework of these games, followed by the exposition of our results.

6.1. TU Games with Coalition Structures: Basic Definitions

A *coalition structure* for a TU game $G = \langle N, g \rangle$ is a partition of N, i.e., a list $CS = (S^1, \ldots, S^m)$ of non-empty subsets of N that satisfies

- $\bigcup_{j=1}^{m} S^j = N$;
- $S^i \cap S^j = \emptyset$ for all $i, j \in N$ such that $i \neq j$.

Given a TU game $G = \langle N, g \rangle$, we denote by $g(CS)$ the total payoff/expenses that the players obtain/incur by forming the coalition structure CS: $g(CS) = \sum_{j=1}^{m} g(S^j)$. Also, we denote by $\mathcal{CS}(N)$ the set of all coalition structures over N, and extend this notation to subsets of N: given a set $S \subseteq N$, we denote by $\mathcal{CS}(S)$ the set of all partitions of S.

A *pre-imputation for a coalition structure* $CS = (S^1, \ldots, S^m)$ in a coalitional game $G = \langle N, g \rangle$ is a vector $\mathbf{p} = (p_1, \ldots, p_n)$ that satisfies $\sum_{i \in S^j} p_i = |g(S^j)|$ for every $j = 1, \ldots, m$. That is, just as in the setting without coalitions structures, p_i is the profit received (respectively, the expenses incurred) by player i. We emphasize that the profit (respectively, expenses) of each coalition in a coalition structure is distributed among the coalition members. We denote the set of all pre-imputations for a coalition structure CS by $\mathbb{I}(CS)$.

An *outcome* of a TU game $G = \langle N, g \rangle$ with coalition structures is a pair $(CS, \mathbf{p})$, where $CS \in \mathcal{CS}(N)$ and $\mathbf{p} \in \mathbb{I}(CS)$. Just as for games without coalition structures, we are interested in outcomes that are stable. Such outcomes are said to form the *CS-core of* G. More formally, an outcome $(CS, \mathbf{p})$ is said to be in the *CS-core* of G if $\mathbf{p}$ is not blocked by any coalition, i.e., $\mathbf{p} \in \mathbb{S}(G)$. We will denote the CS-core of a game G by $\mathbb{CSC}(G)$. Also, for every coalition structure CS we set

$$\mathbb{C}(G, CS) = \{\mathbf{p} \in \mathbb{I}(CS) \mid (CS, \mathbf{p}) \in \mathbb{CSC}(G)\} = \mathbb{I}(CS) \cap \mathbb{S}(G).$$

Note that $\mathbf{p}$ is in the core of G if and only if $(\{N\}, \mathbf{p})$ is in the CS-core of G.

6.2. Subsidizing Socially Optimal Coalition Structures

Given a TU game $G = \langle N, g\rangle$, let $\widehat{CS} \in \mathcal{CS}(N)$ be a coalition structure that maximizes the social welfare, i.e., $g(\widehat{CS}) \geq g(CS)$ for every $CS \in \mathcal{CS}(N)$. Stabilizing $\widehat{CS}$ makes sense if the central authority wants the agents to cooperate in a stable manner, possibly by forming a coalition structure, but does not care about the specific cooperation pattern and simply wants to minimize its own expenses.

We define the quantity $\mathrm{addCoS}_{\mathrm{cs}}(G)$, which we will call the *additive coalitional cost of stability of* G, as the minimum subsidy needed to stabilize G if agents are allowed to form coalition structures. Since the only difference from $\mathrm{addCoS}(G)$ is the amount that agents can generate without the subsidy, and the stability constraints remain the same, we have $\mathrm{addCoS}_{\mathrm{cs}}(G) = \mathrm{addCoS}(G) - (v(\widehat{CS}) - v(N))$. We also define the *multiplicative coalitional cost of stability* of G as

$$\mathrm{multCoS}_{\mathrm{cs}}(G) = \frac{|g(\widehat{CS}) + \mathrm{addCoS}_{\mathrm{cs}}(G)|}{g(\widehat{CS})}.$$

We will now show that bounds on the cost of stability of superadditive games translate into bounds on $\mathrm{addCoS}_{\mathrm{cs}}(G)$ and $\mathrm{multCoS}_{\mathrm{cs}}(G)$.

Following Aumann and Dréze [21], we define the *superadditive cover* of a profit-sharing game $G = \langle N, v\rangle$ as a game $G^* = \langle N, v^*\rangle$ given by $v^*(S) = \max_{CS \in \mathcal{CS}(S)} v(CS)$. Note that for $S = N$ we get $v^*(N) = v(\widehat{CS})$. It is easy to see that G^* is a superadditive game: For every pair of non-overlapping coalitions, S^1 and S^2, we have $(S^1, S^2) \in \mathcal{CS}(S^1 \cup S^2)$ and hence $v^*(S^1 \cup S^2) \geq v^*(S^1) + v^*(S^2)$. The *subadditive cover* $G^* = \langle N, c^*\rangle$ of an expense-sharing game $G = \langle N, c\rangle$ is defined similarly: we set $c^*(S) = \min_{CS \in \mathcal{CS}(S)} c(CS)$. Clearly, $G^* = \langle N, c^*\rangle$ is subadditive. We will refer to both the superadditive covers of profit-sharing games and the subadditive covers of expense-sharing games as *s-additive covers* of the respective games.

It turns out that the coalitional cost of stability of a given game equals the cost of stability of its s-additive cover. The following proposition follows almost immediately from Theorem 3.5 in [22]; for completeness, we provide a proof in Appendix C.

Proposition 6.1. *Let* $G = \langle N, g\rangle$ *be a coalitional game, and let* $G^* = \langle N, g^*\rangle$ *be its s-additive cover. Then* $\mathrm{addCoS}_{\mathrm{cs}}(G) = \mathrm{addCoS}(G^*)$ *and* $\mathrm{multCoS}_{\mathrm{cs}}(G) = \mathrm{multCoS}(G^*)$.

Combining Proposition 6.1 with Theorem 4.3 and Corollary 4.9, we obtain the following corollary.

Corollary 6.2. *For every n-player profit-sharing game G we have* $\mathrm{multCoS}_{\mathrm{cs}}(G) \leq \sqrt{n}$*, and for every n-player expense-sharing game G we have* $\mathrm{multCoS}_{\mathrm{cs}}(G) \geq \frac{1}{\ln n+1}$.

In fact, the s-additivity condition in Theorem 4.3 and Corollary 4.9 can be weakened significantly. A coalitional game $G = \langle N, g \rangle$ is called *cohesive* if $g(CS) \leq g(N)$ for every $CS \in \mathcal{CS}(N)$ (see, e.g., [22]); every s-additive game is cohesive, but the converse is not necessarily true. Since $\{N\}$ is a coalition structure over N, we obtain the following result.

Corollary 6.3. *If G is a cohesive n-player profit-sharing game then* $\mathrm{multCoS}(G) \leq \sqrt{n}$*, and if G is a cohesive n-player expense-sharing game then* $\mathrm{multCoS}(G) \geq \frac{1}{\ln n+1}$.

We remark that the results of this section can be obtained using the notion of *cohesive* cover rather than s-additive cover [22], i.e., by considering the game $G^\star = \langle N, g^\star \rangle$ defined as $g^\star(N) = g(\widehat{CS})$ and $g^\star(S) = g(S)$ for $S \neq N$.

7. Related Work

The term "cost of stability" was introduced in a preliminary version of this work by Bachrach et al. [23], who defined this concept formally, proved several bounds on the additive cost of stability, and presented computational complexity results for the cost of stability in weighted voting games. This short paper was later extended by Bachrach et al. [10], who studied more general families of games, including games with coalition structures. Two other papers, by subsets of the current set of authors [24, 19], studied the cost of stability in expense-sharing games and games with restricted cooperation, as well as the relationship between the cost of stability and the least core. The current paper includes most of the previous results (excluding results on computational complexity), as well as improved bounds and a number of results that do not appear in the earlier papers.

7.1. Recent Research on Subsidies and the Cost of Stability

Since the first of these papers has been published, several groups of researchers studied the cost of stability, focusing mainly on computational questions. Resnick et al. [25] examined the cost of stability in threshold network flow games, a family of simple games played on flow networks where a coalition of edges wins if it can guarantee a sufficient flow from the source to the sink. Aziz et al. [26] studied the complexity of computing the cost of stability and the least core in a variety of coalitional games, comparing games with thresholds (such as threshold network flow games and weighted voting games) to their variants without a threshold. Aadithya et al. [27] showed that for coalitional games represented by algebraic decision diagrams the cost of stability can be computed in polynomial time. Greco et al. [22] proved bounds on the complexity of computing the cost of stability, for games with and without coalition structures. Very recently, Meir et al. [28] studied bounds on the cost of stability when cooperation is restricted by a network of social connections, rather than by the size of the coalition.

A model for subsidies in coalitional games was independently suggested by Bejan and Gómez [7], who focused (as we do in Section 5) on the relationship between subsidies and other solution concepts. We adopted some of their notation, which is useful in our case as well.

However, in their work the additional payment required to stabilize a game is collected from the participating agents by means of a specific *taxation* system, rather than injected into the game by an external authority, whereas we do not assume any form of taxation. The taxation approach was extended by Zick et al. [29], who also studied the connections between taxes and the CoS. The relation between the cost of stability and another property of TU games (which aims to measure how far a game is from being a weighted voting game) was studied by Freixas and Kurz [30].

7.2. Approximate Core

Several other researchers studied subsidies and other incentive issues in expense-sharing games using different terminology. Specifically, Deng et al. [13] show that a coalitional game whose characteristic function is given by an integer program of a certain form has a non-empty core if and only if the linear relaxation of this problem has an integer solution. Their argument can be used to relate the multiplicative cost of stability in such games and the integrality gap of the respective program. The connection between the integrality gap and the multiplicative cost of stability is made explicit in the work of Goemans and Skutella [31] in the context of facility location games. The observation that the cost of stability can be characterized using the Bondareva–Shapley theorem can be found in [4].

A number of other authors have studied the cost of stability in a variety of combinatorial optimization games, sometimes adding other requirements on top of minimization of subsidies. A common assumption is that players gain some *private utility* from participating in the game; in contrast, our model assumes that participation is mandatory, or, equivalently, that the utility derived from participation is sufficiently high to guarantee participation at any cost. Specifically, Devanur et al. [11] suggested a mechanism that recovers at least a fraction of $\frac{1}{\ln n+1}$ of the total cost in set cover games, and a constant fraction (namely, 0.462) in metric facility location games, with the additional requirement of strategyproofness. Our results imply that for set cover games this bound is tight, even if the strategyproofness requirement is dropped.

An application that has drawn much attention is routing in networks, which was initially formulated as a minimum spanning tree game [32]. In the minimum spanning tree game the agents are nodes of a graph, and each edge is a connection that has a fixed price. The cost of a coalition is the price of the cheapest tree that connects all participating nodes to the source node. The additive cost of stability in this particular game is always 0, as its core is never empty [33]. However, there is a more realistic variation of this game known as the Steiner tree game, where nodes are allowed to route through nodes that are not part of their coalition. Megiddo [34] showed that the core of the Steiner tree game may be empty, and therefore its cost of stability is nontrivial. Jain and Vazirani [35] proposed a mechanism for the Steiner tree game with multiplicative cost of stability of $1/2$, under the stronger requirements of group strategyproofness.[3] Könemann et al. [36, 37] put forward mechanisms for the more general Steiner forest game that have the same cost of stability, and suggest that this bound is tight.

[3]More precisely, Jain and Vazirani [35] demanded full cost recovery and relaxed stability constraints. The bound on the cost of stability is achieved if we divide their proposed payments by 2.

It is interesting to note that the value of the optimal Steiner tree (i.e., the value of the grand coalition) can be written as an integer linear program whose integrality gap is lower-bounded by $1/2$ as well [38, Example 22.10, p. 206]. However, in contrast to set cover games, for Steiner tree games we do not know if the integrality gap is always equal to the inverse of the multiplicative cost of stability, or if better cost-sharing mechanisms are possible when the strategyproofness constraint is relaxed. In fact, a different line of research [39] suggested a cost-sharing mechanism for Steiner trees that does not guarantee strategyproofness, and showed *empirically* that it allocates at least 92% of the cost on all tested instances.

Other cost-sharing mechanisms have been suggested for many different games. For example, Moulin and Shenker [40] studied the tradeoff between efficiency and the cost of stability in subadditive games; see Pal and Tardos [41], Jain and Mahdian [4], and Immorlica et al. [42] for more results and an overview. Some of the proofs in this paper use similar techniques, especially in Section 4.2. Some of the proposed mechanisms impose strong requirements such as group strategyproofness, in addition to stability. Therefore, it is an interesting question whether tighter bounds on the cost of stability for specific families of games can be derived once these requirements are relaxed.

7.3. Subsidies in Normal-Form Games

The idea of providing subsidies to ensure stability has also been explored in the context of normal-form games. Monderer and Tennenholtz [43] investigate the setting where an interested party wishes to influence the behavior of agents in a game not under its control. In spirit, their approach is close to the one we take here: The interested party may commit to making non-negative payments to the agents if certain strategy profiles are selected. Payments are given to agents individually, but they are dependent on the strategies selected by all agents. As in our work, it is assumed that the interested party wishes to minimize its expenses. Determining the optimal monetary offers to be made in order to implement a desired outcome is shown to be NP-hard in general, but becomes tractable under certain constraints. Also, it is sometimes possible for the external party to stabilize a particular outcome without paying anything, which is clearly impossible in our setting.

Another closely related paper is that by Buchbinder et al. [44], who study subsidies in a normal-form version of the set cover game. However, the focus of this paper is on efficiency rather than stability, and the subsidy is financed by taxes collected from the users.

8. Conclusions and Future Work

We have examined the possibility of stabilizing a coalitional game by offering the agents additional payments in order to discourage them from deviating. We defined the additive cost of stability, which is the minimum subsidy that allows a stable division of the gains or costs, and the multiplicative cost of stability, which is the ratio between the total payment given to/obtained from the agents and the profit/cost of the grand coalition. We provided bounds on the cost of stability both for general games and under various restrictions on the characteristic function, such as super- and subadditivity and anonymity. Our results are summarized in Table 1. We extended our results to the case where the goal is to stabilize an optimal coalition structure rather than the grand coalition. We have also explored the relationship between the

	unrestricted	s-additive		
		all	k-size	anonymous
Profit (upper bound)	n (Ob. 4.1)	$\sqrt{n}$ (Thm. 4.3)	k (Thm. 4.4)	$\frac{2n}{n+1}$ (Thm. 4.6)
Expense (lower bound)	0 (Ex. 4.12)	$\frac{1}{\ln n+1}$ (Cor. 4.9)	$\frac{1}{\ln k+1}$ (Cor. 4.9)	$\frac{n}{2(n-1)}$ (Thm. 4.11)

Table 1: The multiplicative cost of stability for different classes of TU games. All bounds are either exactly or asymptotically tight.

cost of stability, the (strong and weak) least core, and its variations such as the positive least core.

Future Work

There are several lines of possible future research. First, it would be interesting to study the cost of stability in other restricted classes of games. Another direction is to explore how the cost of stability is affected by additional constraints on the expense/profit-sharing mechanisms, such as truthfulness, fairness, and efficiency. For example, we could require the core of the adjusted game to contain a "fair" imputation (e.g., the Shapley imputation), rather than just be non-empty. Another challenge is to extend the notion of the cost of stability to games with nontransferable utility and to partition function games.

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A. Proofs for Section 4

Lemma 4.2. *Let $G = \langle N, v\rangle$ be a superadditive profit-sharing game. Then there exists a solution $\{\delta_S\}_{S\subseteq N}$ to $\overline{G}$ such that for every $R, T \subseteq N$ with $\delta_R \neq 0$, $\delta_T \neq 0$ we have $R \cap T \neq \emptyset$.*

Proof. For every solution $\{\delta_S\}_{S\subseteq N}$ to $\overline{G}$, let $\sigma(\{\delta_S\}_{S\subseteq N})$ be a vector of length n whose i-th coordinate $\sigma_i(\{\delta_S\}_{S\subseteq N})$ is the number of sets of size i with non-zero coefficients:

$$\sigma_i(\{\delta_S\}_{S\subseteq N}) = |\{S \subseteq N \mid \delta_S \neq 0, |S| = i\}|;$$

we will refer to this vector as the *signature of* $\{\delta_S\}_{S\subseteq N}$. Given two signatures $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$, we set $\sigma \prec \tau$ if there exists an $i \in N$ such that $\sigma_j = \tau_j$ for all $j < i$ and $\sigma_i < \tau_i$. We write $\sigma \preceq \tau$ if $\sigma \prec \tau$ or $\sigma = \tau$. Let $\{\delta^*_S\}_{S\subseteq N}$ be a solution to $\overline{G}$ with the minimal signature, i.e., one that satisfies $\sigma(\{\delta^*_S\}_{S\subseteq N}) \preceq \sigma(\{\delta_S\}_{S\subseteq N})$ for every $\{\delta_S\}_{S\subseteq N}$ that is a solution to $\overline{G}$.

We will now argue that $\{\delta^*_S\}_{S\subseteq N}$ satisfies the condition in the statement of the lemma. Suppose for the sake of contradiction that this is not the case. Then there exists a pair of non-empty sets (R, T) such that $\delta^*_R > 0$ and $\delta^*_T > 0$, but $R \cap T = \emptyset$. Let $\varepsilon = \min\{\delta^*_R, \delta^*_T\}$. Consider the vector $\{\delta^{**}_S\}_{S\subseteq N}$ given by

$$\delta^{**}_S = \begin{cases} \delta^*_S & \text{for } S \neq R, T, R\cup T, \\ \delta^*_S - \varepsilon & \text{for } S = R, T, \\ \delta^*_S + \varepsilon & \text{for } S = R\cup T. \end{cases}$$

First, observe that since R and T are disjoint, $\{\delta^{**}_S\}_{S\subseteq N}$ is also a feasible solution to the dual program. Furthermore, by superadditivity we have

$$\sum_{S\subseteq N} v(S)\delta^{**}_S = \sum_{S\subseteq N} v(S)\delta^*_S - v(R)\varepsilon - v(T)\varepsilon + v(R\cup T)\varepsilon \geq \sum_{S\subseteq N} v(S)\delta^*_S,$$

so $\{\delta^{**}_S\}_{S\subseteq N}$ is an optimal solution to the dual program, too, and hence a solution to $\overline{G}$. Further, assume without loss of generality that $\varepsilon = \delta^*_R$. Then, if S is distinct from R, T, and $R \cup T$, we have $\delta^*_S = \delta^{**}_S$, and, moreover, $\delta^*_R \neq 0$, $\delta^{**}_R = 0$, and $|R\cup T| > |R|$. In other words, if we set $i = |R|$, we get

$$\sigma_i(\{\delta^{**}_S\}_{S\subseteq N}) < \sigma_i(\{\delta^*_S\}_{S\subseteq N}), \quad \sigma_j(\{\delta^{**}_S\}_{S\subseteq N}) \leq \sigma_j(\{\delta^*_S\}_{S\subseteq N}) \text{ for all } j < i.$$

It follows that $\sigma(\{\delta^{**}_S\}_{S\subseteq N}) \prec \sigma(\{\delta^*_S\}_{S\subseteq N})$, which is a contradiction with our choice of $\{\delta^*_S\}_{S\subseteq N}$. Thus, there exists a solution $\{\delta_S\}_{S\subseteq N}$ in which any two sets R and T with $\delta_R \neq 0$, $\delta_T \neq 0$ intersect. ■

Theorem 4.4. *Let $G = \langle N, v\rangle$ be a superadditive profit-sharing game. Then for every positive integer $k \leq |N|$ we have $\mathrm{multCoS}(G|_k) \leq k$.*

Proof. If $k \geq \sqrt{n}$ then by Theorem 4.3 we are done. Thus, we can assume that $k < \sqrt{n}$. Let $\{\delta_S\}_{S \subseteq N}$ be a solution to $\overline{G|_k}$ that satisfies the conditions of Lemma 4.2. If $\delta_N = 1$ then we have $\delta_S = 0$ for every $S \neq \emptyset, N$, and hence equation (9) implies $\mathrm{multCoS}(G|_k) = 1$. Now, suppose that $\delta_N < 1$. We claim that in this case $\delta_T > 0$ for some coalition T with $|T| \leq k$. Indeed, otherwise all terms in equation (9) except possibly for $\delta_N v|_k(N)$ would be equal to 0 and we would get $\mathrm{multCoS}(G|_k) = \frac{1}{v|_k(N)} \delta_N v|_k(N) = \delta_N < 1$, a contradiction.

Now, consider a coalition T with $|T| \leq k$ and $\delta_T > 0$. We have

$$\begin{aligned} \mathrm{multCoS}(G|_k) &= \frac{1}{v|_k(N)} \sum_{S \subseteq N} \delta_S v|_k(S) \leq \frac{1}{v|_k(N)} \sum_{i \in T} \sum_{S \subseteq N : i \in S} \delta_S v|_k(S) \\ &\leq \sum_{i \in T} \sum_{S \subseteq N : i \in S} \delta_S = \sum_{i \in T} 1 = |T| \leq k, \end{aligned}$$

which completes the proof. ∎

Theorem 4.8. *Let G be a set cover game. Then* $\mathrm{multCoS}(G) = \frac{1}{\mathrm{IG}(G)}$.

Proof. Consider a set cover game $G = \langle N, \mathcal{F}, w \rangle$. The dual to the linear program $\mathcal{LP}(G)$ is the following linear program over the set of variables $\{p_i\}_{i \in N}$:

$$\begin{aligned} \max \sum_{i \in N} p_i & \qquad \text{subject to:} \\ \sum_{i \in F_j} p_i &\leq w(F_j) \text{ for each } F_j \in \mathcal{F} \\ p_i &\geq 0 \text{ for each } i \in N. \end{aligned} \tag{A.1}$$

Let $\mathbf{p}$ be a feasible solution of this linear program. Consider a coalition S with cost $c(S)$. By definition of the cost function, the set S can be covered by a collection of subsets $\mathcal{F}^*(S) = \{F_1, \ldots, F_k\}$ of cost $c(S) = \sum_{\ell=1}^{k} w(F_\ell)$. Note that

$$\sum_{i \in S} p_i \leq \sum_{\ell=1}^{k} \sum_{i \in F_\ell} p_i \leq \sum_{\ell=1}^{k} w(F_\ell) = c(S).$$

That is, the payoff vector $\mathbf{p}$ is stable.

Now, let $\mathbf{p}^*$ be an optimal solution to the LP (A.1). By strong LP duality, we have $\sum_{i \in N} p_i^* = \mathrm{LP}(G)$ and hence $\mathrm{multCoS}(G) \geq \sum_{i \in N} p_i^* / c(N) = \mathrm{LP}(G)/\mathrm{ILP}(G)$.

Conversely, let $\mathbf{p}$ be a stable payoff vector for G. Since $\mathbf{p}$ is not blocked by any coalition, it satisfies all constraints of the linear program (A.1), and therefore $\sum_{i \in N} p_i \leq \mathrm{LP}(G)$. Hence $\mathrm{multCoS}(G) \leq \mathrm{LP}(G)/\mathrm{ILP}(G)$, and the proof is complete. ∎

Theorem 4.10. *Let $G = \langle N, c \rangle$ be an anonymous expense-sharing game. Then*

$$\mathrm{multCoS}(G) = \frac{n}{c_n} \cdot \min_{k \leq n} \frac{c_k}{k}.$$

Proof. Pick $k^* \in \operatorname{argmin}_{k \leq n} c_k/k$, and let $\mathbf{p}$ be a payoff vector given by $p_i = c_{k^*}/k^*$ for all $i \in N$. Clearly, $\mathbf{p}$ is stable: For every $S \subseteq N$, we have $p(S) = |S|c_{k^*}/k^* \leq c(S)$ by our choice of k^*.

Now, suppose that there is a stable payoff vector $\mathbf{q}$ with $q(N) > p(N)$. Renumber the players so that $q_1 \geq \cdots \geq q_n$ and set $S^* = \{1, \ldots, k^*\}$. Clearly, we have $q(S^*)/k^* \geq q(N)/n$, and hence

$$q(S^*) \geq \frac{k^*}{n} q(N) > \frac{k^*}{n} p(N) = c_{k^*},$$

which means that $\mathbf{q}$ is not stable. Hence,

$$\operatorname{multCoS}(G) = \frac{p(N)}{c(N)} = \frac{n}{c_n} \cdot \frac{c_{k^*}}{k^*},$$

which completes the proof. ∎

Theorem 4.11. *Let $G = \langle N, c \rangle$ be an anonymous subadditive expense-sharing game. Then* $\operatorname{multCoS}(G) \geq \frac{1}{2} + \frac{1}{2n-2}$*, and this bound is tight.*

Proof. For $n = 2$ the theorem is trivially true, so assume $n \geq 3$. Fix some $k^* \in \operatorname{argmin} c_k/k$. If $k^* = n$, the theorem follows immediately from Theorem 4.10, so assume $k^* \leq n - 1$. Let $q = n/k^*$. We have $q \geq n/(n-1) > 1$, and thus $\lceil q \rceil \geq 2$.

Suppose first that $n \geq 4$ and $q > 2$. Then

$$\frac{q}{\lceil q \rceil} \geq \frac{q}{q+1} > \frac{2}{3} \geq \frac{n}{2n-2}.$$

Further, if $n = 3$ and $q > 2$, it has to be the case that $k^* = 1$, and hence we obtain

$$\frac{q}{\lceil q \rceil} = \frac{3}{3} = 1 > \frac{n}{2n-2}.$$

Finally, suppose that $q \leq 2$ and hence $\lceil q \rceil = 2$. We have $q \geq n/(n-1)$, so

$$\frac{q}{\lceil q \rceil} \geq \frac{n/(n-1)}{2} = \frac{n}{2n-2}.$$

We showed that in all cases $q/\lceil q \rceil \geq n/(2n-2)$. Further, since G is subadditive, we have $c_n \leq \lceil n/k^* \rceil c_{k^*}$. Combining this with Theorem 4.10, we obtain

$$\begin{aligned} \operatorname{multCoS}(G) &= \frac{n}{c_n} \cdot \frac{c_{k^*}}{k^*} \geq \frac{n}{k^*} \cdot \frac{1}{\lceil \frac{n}{k^*} \rceil} = \frac{q}{\lceil q \rceil} \\ &\geq \frac{n}{2n-2} = \frac{1}{2} + \frac{1}{2n-2}. \end{aligned}$$

To see that this bound is tight, for each $n \geq 2$ we define a game $G_n = \langle N, c \rangle$ with $|N| = n$ by setting $c(N) = 2$ and $c(S) = 1$ for every $S \subsetneq N$. In this game $k^* = n - 1$, so by Theorem 4.10 we obtain

$$\operatorname{multCoS}(G_n) = \frac{n}{c_n} \cdot \frac{c_{k^*}}{k^*} = \frac{n}{2(n-1)} = \frac{1}{2} + \frac{1}{2n-2},$$

which completes the proof. ∎

B. The Cos and the Least-core

Proposition 5.6. *Let $G = \langle N, v\rangle$ be a monotone profit-sharing game. Then* $\mathrm{addCoS}(G) \geq \frac{n}{n-1}\varepsilon_{\mathbf{PS}}(G)$*, and this bound is tight.*

Proof. Let $\Delta = \mathrm{addCoS}(G)$. We will construct a pre-imputation $\mathbf{q}^* \in \mathrm{PSC}_{\frac{n-1}{n}\Delta}(G)$.

Set $\mathbb{EC}^+(G) = \mathbb{EC}(G) \cap \mathbb{R}^n_+$. First, we will argue that $\mathbb{EC}^+(G) \neq \emptyset$. To see this, pick an arbitrary pre-imputation $\mathbf{p} \in \mathbb{EC}(G)$, and let $\overline{\mathbf{p}}$ be some stable extension of $\mathbf{p}$. Note that $\overline{p}_i \geq \max\{p_i, 0\}$ for all $i \in N$. Assume without loss of generality that $p_1 \leq \cdots \leq p_n$, and let j be the smallest index such that $\sum_{i=1}^{j} p_i \geq 0$; note that $p(N) = v(N) > 0$, so j is well-defined. If $j = 1$, we are done, since $\mathbf{p} \in \mathbb{R}^n_+$. Otherwise, consider the pre-imputation $\mathbf{p}'$ given by $p'_i = 0$ for $i = 1, \ldots, j-1$, $p'_j = \sum_{i=1}^{j} p_i$, $p'_i = p_i$ for $i = j+1, \ldots, n$. Note that $p'(N) = p(N)$. We have $0 \leq p'_j \leq p_j \leq \overline{p}_j$ by our choice of j. Further, for $i < j$ we have $p'_i = 0 \leq \overline{p}_i$ and for $i > j$ we obtain $0 \leq p'_i = p_i \leq \overline{p}_i$. Hence, $\mathbf{p}' \in \mathbb{R}^n_+$ and $\overline{\mathbf{p}}$ is a stable extension of $\mathbf{p}'$, which means that $\mathbf{p}' \in \mathbb{EC}^+(G)$.

Now, for each pre-imputation $\mathbf{p} \in \mathbb{EC}^+(G)$, we let $\overline{\mathbf{p}}$ be its most balanced stable extension. Formally, given a pre-imputation $\mathbf{p} \in \mathbb{EC}^+(G)$, let $E_{\mathbf{p}} = \{\mathbf{q} \mid \mathbf{q} \geq \mathbf{p}, \mathbf{q} \in \mathbb{C}(G(\Delta))\}$ be the set of its stable extensions. Consider a function $f_{\mathbf{p}} : E_{\mathbf{p}} \to \mathbb{R}$ given by $f_{\mathbf{p}}(\mathbf{q}) = \sum_{i \in N: q_i - p_i \leq \Delta/n}(q_i - p_i)$. The set $E_{\mathbf{p}}$ is compact, and $f_{\mathbf{p}}$ is a continuous function, so $f_{\mathbf{p}}$ reaches its maximum on $E_{\mathbf{p}}$. Let $\overline{\mathbf{p}}$ be some payoff vector in $\arg\max_{\mathbf{q} \in E_{\mathbf{p}}} f_{\mathbf{p}}(\mathbf{q})$. Further, consider a function $g : \mathbb{EC}^+(G) \to \mathbb{R}$ given by $g(\mathbf{p}) = f_{\mathbf{p}}(\overline{\mathbf{p}})$; note that the value of g at $\mathbf{p}$ does not depend on the choice of $\overline{\mathbf{p}}$ in $\arg\max_{\mathbf{q} \in E_{\mathbf{p}}} f_{\mathbf{p}}(\mathbf{q})$. The set $\mathbb{EC}^+(G)$ is compact, and g is a continuous function, so g reaches its maximum on $\mathbb{EC}^+(G)$. Pick a pre-imputation $\mathbf{q}^*$ in $\arg\max_{\mathbf{p} \in \mathbb{EC}^+(G)} g(\mathbf{p})$. We will argue that $\mathbf{q}^* \in \mathrm{PSC}_{\frac{n-1}{n}\Delta}(G)$.

Let

$$
\begin{aligned}
N^+ &= \{i \in N \mid \overline{q^*}_i - q^*_i > \Delta/n\},\\
N^0 &= \{i \in N \mid \overline{q^*}_i - q^*_i = \Delta/n\},\\
N^- &= \{i \in N \mid \overline{q^*}_i - q^*_i < \Delta/n\}.
\end{aligned}
$$

We claim that for every $i \in N^-$ it holds that $q^*_i = 0$. Indeed, if $N^- = \emptyset$ this claim is trivially true, so suppose that $N^- \neq \emptyset$ and hence $N^+ \neq \emptyset$. Now suppose that there is an agent $i \in N^-$ with $q^*_i > 0$. Pick an agent $j \in N^+$ and set

$$\delta = \min\{q^*_i, \Delta/n - (\overline{q^*}_i - q_i), \overline{q^*}_j - q_j - \Delta/n\};$$

note that $\delta > 0$. Construct a pre-imputation $\mathbf{r}$ by setting

$$
r_k = \begin{cases} q^*_k - \delta/2 & \text{if } k = i,\\ q^*_k + \delta/2 & \text{if } k = j,\\ q^*_k & \text{if } k \in N \setminus \{i, j\}. \end{cases}
$$

We have

$$0 < \overline{q^*}_i - r_i = \overline{q^*}_i - q^*_i + \delta/2 < \Delta/n, \quad \overline{q^*}_j - r_j = \overline{q^*}_j - q^*_j - \delta/2 > \Delta/n.$$

Thus, $\overline{\mathbf{q}^*} \in E_{\mathbf{r}}$ and $f_{\mathbf{r}}(\overline{\mathbf{q}^*}) = f_{\mathbf{q}^*}(\overline{\mathbf{q}^*}) + \delta/2$, so

$$g(\mathbf{r}) = f_{\mathbf{r}}(\overline{\mathbf{r}}) \geq f_{\mathbf{r}}(\overline{\mathbf{q}^*}) > f_{\mathbf{q}^*}(\overline{\mathbf{q}^*}) = g(\mathbf{q}^*),$$

a contradiction with our choice of $\mathbf{q}^*$. Hence, $q^*(N^-) = 0$.

Now, consider an arbitrary coalition S. If $N^+ \cup N^0 \subseteq S$, we have

$$q^*(S) = q^*(N) = v(N) \geq v(S) > v(S) - \frac{n-1}{n}\Delta.$$

On the other hand, suppose that $N^+ \cup N^0 \not\subseteq S$. Consider some agent $i \in (N^+ \cup N^0) \setminus S$ and let $\Delta_i = \overline{q^*}_i - q^*_i$; note that $\Delta_i \geq \Delta/n$. We have $\overline{q^*}(S) \leq q^*(S) + \Delta - \Delta_i \leq q^*(S) + \frac{n-1}{n}\Delta$, so

$$q^*(S) \geq \overline{q^*}(S) - \frac{n-1}{n}\Delta \geq v(S) - \frac{n-1}{n}\Delta.$$

Thus $\mathbf{q}^* \in \mathrm{SC}_{\frac{n-1}{n}\Delta}(G)$, and since $\mathbf{q}^*$ is also in $\mathbb{R}^n_+$, we obtain $\mathbf{q}^* \in \mathrm{PSC}_{\frac{n-1}{n}\Delta}(G)$. Consequently, $\varepsilon_{\mathbf{PS}} \leq \frac{n-1}{n}\Delta$, or, equivalently, $\Delta = \mathrm{addCoS}(G) \geq \frac{n}{n-1}\varepsilon_{\mathbf{PS}}$.

Tightness follows immediately from Proposition 5.4. ■

Conjecture 5.7. *For every profit-sharing game G with an empty core, we have $\mathbb{SLC}(G) \subseteq \mathbb{EC}(G)$.*

Proof for $n = 2, 3$. Suppose first that $n = 2$. We have argued that in this case $\mathbb{SLC}(G) = \mathbb{WLC}(G)$, and Proposition 5.1 implies that $\mathbb{WLC}(G) \subseteq \mathbb{EC}(G)$ for every TU game G. Combining these observations, we conclude that for $n = 2$ our conjecture is true.

Now suppose that $n = 3$, i.e., $N = \{1, 2, 3\}$. Consider a payoff vector $\mathbf{p}$ in the strong least core of G, and let $\varepsilon = \varepsilon_{\mathbf{S}}(G)$. Clearly, $\mathbf{p}$ belongs to the core of G_ε. Consider a collection of subsets that corresponds to the tight constraints of the respective linear program, i.e., set $\mathcal{D} = \{S \subseteq N \mid p(S) = v(S) - \varepsilon\}$. It follows from the Bondareva-Shapley theorem that $\mathcal{D}$ is a balanced collection of subsets, and we can assume without loss of generality that it is minimal (if this is not the case, we can consider a minimal balanced subcollection of $\mathcal{D}$).

For $n = 3$, there are only five minimal balanced collections other than $\{N\}$, namely,

(a) $\{\{1\}, \{2\}, \{3\}\}$;

(b) $\{\{1\}, \{2, 3\}\}$;

(c) $\{\{2\}, \{1, 3\}\}$;

(d) $\{\{3\}, \{1, 2\}\}$; and

(e) $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$.

We will consider all these possibilities for $\mathcal{D}$. In each case, we will show how to construct a payoff vector $\mathbf{p}'$ such that

(1) $p'_i \geq p_i$ for all $i \in N$,

(2) $\mathbf{p}'$ is stable, i.e., $\mathbf{p}' \in \mathbb{S}(G)$, and

(3) $p'(N) \le q(N)$ for every $\mathbf{q} \in \mathbb{S}(G)$.

Conditions (1)–(3) imply that $\mathbf{p}$ is in the extended core of G. To simplify notation, in the rest of the proof we write $v(i)$ in place of $v(\{i\})$ and $v(ij)$ in place of $v(\{i,j\})$ for all $i,j \in N$.

Case (a): $\mathcal{D} = \{\{1\},\{2\},\{3\}\}$.

In this case, we can simply set $p'_i = p_i + \varepsilon$ for each $i \in N$. Clearly, the resulting payoff vector is stable. Further, for any $\mathbf{q} \in \mathbb{S}(G)$ we have $q_i \ge v(i) = p_i + \varepsilon = p'_i$ for each $i \in N$, and therefore $q(N) \ge p'(N)$.

Cases (b)–(d): $\mathcal{D} = \{\{i\}, N \setminus \{i\}\}$ for some $i \in N$.

Assume without loss of generality that $i = 1$, i.e., $\mathcal{D} = \{\{1\},\{2,3\}\}$. We set

$$p'_1 = v(1), \qquad p'_2 = \max\{p_2, v(2)\}, \qquad p'_3 = \max\{v(3), v(23) - p'_2)\}.$$

Note that $p'_3 \ge p_3$, since $p'_3 \ge v(23) - p'_2 = p_2 + p_3 + \varepsilon - p'_2$ and $p'_2 \le p_2 + \varepsilon$ by the choice of ε. We will now argue that $\mathbf{p}'$ is stable. Indeed, by construction, we have $p'_i \ge v(i)$ for all $i \in N$. Further, since $p'_1 = v(1) = p_1 + \varepsilon$, we have $p'_1 + p'_2 \ge p_1 + p_2 + \varepsilon \ge v(12)$ and $p'_1 + p'_3 \ge p_1 + p_3 + \varepsilon \ge v(13)$. Finally, $p'_2 + p'_3 \ge p'_2 + v(23) - p'_2 = v(23)$.

Now, consider an arbitrary stable payoff vector $\mathbf{q} \in \mathbb{S}(G)$. We have $q_1 \ge v(1) = p'_1$. Suppose first that $p_2 > v(2)$, and hence $p'_2 = p_2$. Then $p'_3 = \max\{v(3), p_3 + \varepsilon\} = p_3 + \varepsilon$, and hence $p'_2 + p'_3 = p_2 + p_3 + \varepsilon = v(23) \le q_2 + q_3$. On the other hand, if $p_2 \le v(2)$, we have $p'_2 = v(2)$. Now, if $p'_3 = v(3)$, we have $q_2 \ge v(2) = p'_2$, $q_3 \ge v(3) = p'_3$, and if $p'_3 = v(23) - p'_2$, we have $q_2 + q_3 \ge v(23) = p'_2 + p'_3$. Thus, in all cases we have $q(N) \ge p'(N)$.

Case (e): $\mathcal{D} = \{\{1,2\},\{1,3\},\{2,3\}\}$.

Let $\delta_i = v(i) - p_i$ for all $i \in N$.

If $\delta_i \le \varepsilon/2$ for all $i \in N$, we set $p'_i = p_i + \varepsilon/2$ for all $i \in N$. Clearly, the resulting vector $\mathbf{p}'$ satisfies all stability constraints. Moreover, for every $\mathbf{q} \in \mathbb{S}(G)$ we have

$$\begin{aligned} q_1 + q_2 &\ge v(12) = p_1 + p_2 + \varepsilon, \\ q_1 + q_3 &\ge v(13) = p_1 + p_3 + \varepsilon, \\ q_2 + q_3 &\ge v(23) = p_2 + p_3 + \varepsilon, \end{aligned}$$

so $q(N) \ge p(N) + 3\varepsilon/2 = p'(N)$.

Now, suppose that $\delta_i > \varepsilon/2$ for exactly one agent $i \in N$; assume without loss of generality that $\delta_1 > \varepsilon/2$ and $\delta_2, \delta_3 \le \varepsilon/2$. We can then set

$$p'_1 = v(1), \qquad p'_2 = p_2 + \varepsilon/2, \qquad p'_3 = p_3 + \varepsilon/2.$$

Note that $p'_1 - p_1 = \delta_1 > \varepsilon/2$, and hence $\mathbf{p}'$ satisfies all stability constraints. Further, for every $\mathbf{q} \in \mathbb{S}(G)$ we have $q_1 \ge v(1) = p'_1$, $q_2 + q_3 \ge v(23) = p_2 + p_3 + \varepsilon = p'_2 + p'_3$, so $q(N) \ge p'(N)$.

Next, suppose that $\delta_i \leq \varepsilon/2$ for exactly one agent $i \in N$; assume without loss of generality that $\delta_1 \leq \varepsilon/2$ and $\delta_2, \delta_3 > \varepsilon/2$. We set

$$p'_2 = v(2), \qquad p'_3 = v(3), \qquad p'_1 = \max\{v(1), v(12) - p'_2, v(13) - p'_3\}.$$

Clearly, $p'_2 = v(2) = p_2 + \delta_2 > p_2$ and $p'_3 = v(3) = p_3 + \delta_3 > p_3$. Also note that $p'_1 \geq p_1$, since $p'_1 \geq v(12) - p'_2 = p_1 + p_2 + \varepsilon - v(2)$ and $p_2 \geq v(2) - \varepsilon$ by the choice of ε. Observe that all stability constraints are satisfied: By construction we have $p'_i \geq v(i)$ for all $i \in N$, $p'_1 + p'_2 \geq v(12)$, $p'_1 + p'_3 \geq v(13)$, and finally $p'_2 + p'_3 \geq p_2 + \varepsilon/2 + p_3 + \varepsilon/2 = p_2 + p_3 + \varepsilon \geq v(23)$. Now, consider an arbitrary stable payoff vector $\mathbf{q} \in \mathbb{S}(G)$. If $p'_1 = v(1)$, we have $q_i \geq v(i) = p'_i$ for all $i \in N$, so $q(N) \geq p'(N)$. If $p'_1 = v(12) - p'_2$, we have $q_1 + q_2 \geq v(12) = p'_1 + p'_2$, $q_3 \geq v(\{3\}) = p'_3$, so again we obtain $q(N) \geq p'(N)$. Similarly, if $p'_1 = v(13) - p'_3$, we have $q_1 + q_3 \geq v(13) = p'_1 + p'_3$, $q_2 \geq v(2) = p'_2$, and $q(N) \geq p'(N)$ in this case as well.

Finally, suppose that $\delta_i > \varepsilon/2$ for all $i \in N$. Then we can set $p'_i = v(i)$ for all $i \in N$. The resulting vector $\mathbf{p}'$ satisfies all stability constraints, and for every $\mathbf{q} \in \mathbb{S}(G)$ we have $q_i \geq v(i) = p'_i$ for all $i \in N$, so $q(N) \geq p'(N)$.

This completes the proof. ■

C. Proof of Proposition 6.1

Proposition 6.1. *Let $G = \langle N, g \rangle$ be a coalitional game, and let $G^* = \langle N, g^* \rangle$ be its s-additive cover. Then* $\mathrm{addCoS}_{\mathrm{cs}}(G) = \mathrm{addCoS}(G^*)$ *and* $\mathrm{multCoS}_{\mathrm{cs}}(G) = \mathrm{multCoS}(G^*)$.

Proof. We will prove this claim for the case when $G = \langle N, v \rangle$ is a profit-sharing game; for expense-sharing games the argument is similar. Let $\Delta = \mathrm{addCoS}(G^*)$.

Observe that $\mathbb{S}(G) = \mathbb{S}(G^*)$ and every pre-imputation for $G(\Delta)$ with coalition structures is a pre-imputation for $G^*(\Delta)$. This immediately implies that $\mathrm{addCoS}(G^*) \leq \mathrm{addCoS}_{\mathrm{cs}}(G)$.

We will now argue that $\mathrm{addCoS}_{\mathrm{cs}}(G) \leq \mathrm{addCoS}(G^*)$. Pick an arbitrary payoff vector $\mathbf{q} \in \mathbb{C}(G^*(\Delta)) = \mathbb{S}(G^*) \cap \mathbb{I}(G^*(\Delta))$. For $j = 1, \ldots, m$, set $\Delta^j = q(S^j) - v(S^j)$. Note that $\Delta^j \geq 0$, since otherwise coalition S^j would have an incentive to deviate under $\mathbf{q}$. We have

$$\sum_{j=1}^{m} \Delta^j = q(N) - v(\widehat{CS}) = \Delta.$$

Further, under the payoff vector $\mathbf{q}$ no coalition has an incentive to deviate. Thus, the vector $(\Delta^1, \ldots, \Delta^m)$ stabilizes $\widehat{CS}$, and hence $\mathrm{addCoS}_{\mathrm{cs}}(G) \leq \Delta = \mathrm{addCoS}(G^*)$.

We conclude that $\mathrm{addCoS}_{\mathrm{cs}}(G) = \mathrm{addCoS}(G^*)$. Further, since $v(\widehat{CS}) = v^*(N)$, we also have $\mathrm{multCoS}_{\mathrm{cs}}(G) = \mathrm{multCoS}(G^*)$. ■