Mathematical Foundations for the Precautionary Principle

Nassim Nicholas Taleb

This is a companion piece (in progress) to the Precautionary Principle by Taleb, Read, Norman, Douady, and Bar-Yam (2014, 2016); the ideas will be integrated in a future version.

CONTENTS

I. PROBABILISTIC SUSTAINABILITY

Remark 1. *If you take the risk –any risk – **repeatedly**, the way to count is in exposure per lifespan, or in the way it shortens the remaining lifespan.*

> **Commentary 1.** *Ruin probabilities are in the time domain for a single agent and do not correspond to state-space (or ensemble) tail probabilities. Nor are expectations fungible between the two domains. Statements on the "overestimation" of tail events (entailing ruin) by agents that are derived from state-space estimations are accordingly flawed. Many theories of "rationality" of agents from are based on wrong estimation operators and/or probability measures.*
>
> *This is the main reason behind the barbell strategy.*

This is a special case of the conflation between a random variable and the payoff of a time-dependent path-dependent derivative function.

> **Commentary 2.** *(Less technical translation)*
>
> *Never cross a river if it is on average only 4 feet deep.*
>
> *(Debate of author with P. Jorion, 1997 and [2] .)*

Correspondence: N N Taleb, email NNT1@nyu.edu

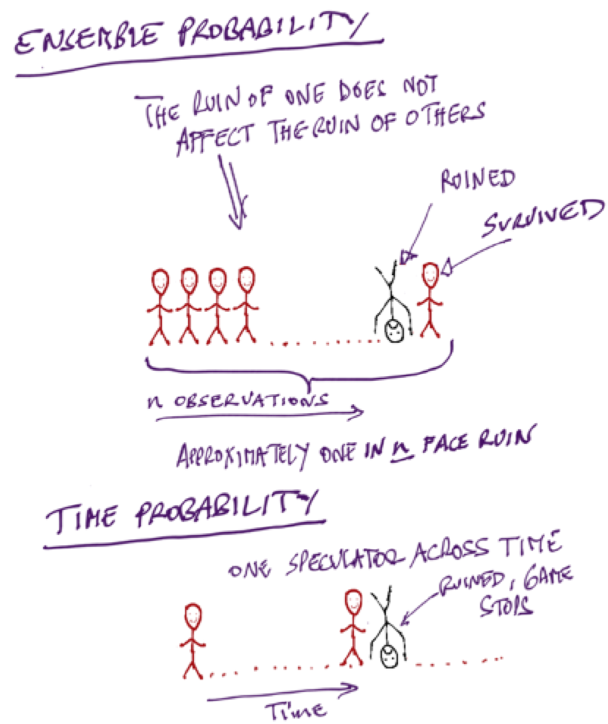

Fig. 1. Ensemble probability vs. time probability discussions, from Peters and Gell-Mann [1]. The treatment by option traders is done via the absorbing barrier. I have traditionally treated this in *Dynamic Hedging* and *Antifragile* as the conflation between X (a random variable) and f(x) a function of the r.v.; an option is path dependent and is not the underlying. This is the rationale behind the barbell strategy.

A. A simplified general case

Consider the extremely simplified example, the sequence of independent random variables $(X_i)_{i=1}^n = (X_1, X_2, \dots X_n)$ with support in the positive real numbers $(\mathbb{R}^+)$. The convergence theorems of classical probability theory address the behavior of the sum or average: $\lim_{n\to\infty} \frac{1}{n}\sum_i^n X_i = m$ by the (weak) law of large numbers (convergence in probability). As shown in Fig.1, n going to infinity produces convergence in probability to the true mean return m. Although the law of large number applies to draws i that can be strictly separated by time, it assumes (some) independence, and certainly path independence.

Now consider $(X_{i,t})_{t=1}^T = (X_{i,1}, X_{i,2}, \dots X_{i,T})$ where every state variable X_i is indexed by a unit of time $t : 0 < t < T$.

Assume that the "time events" are drawn from the exact same probability distribution: $P(X_i) = P(X_{i,t})$.

We define a time probability the evolution over time for a single agent i.

In the presence of terminal that is, irreversible, ruin, every observation is now conditional on some attribute of the preceding one, and what happens at period t depends on $t-1$, what happens at $t-1$ depends on $t-2$, etc. We now have path dependence.

Theorem 1 (space-time inequality)**.** *Assume that* $\forall t, P(X_t = 0) > 0$ *and* $X_0 > 0$, $\mathbb{E}_N(X_t) < \infty$ *the state space expectation for a static initial period t, and* $\mathbb{E}_T(X_i)$ *the time expectation for any agent i, both obtained through the weak law of large numbers. We have*

$$\mathbb{E}_N(X_t) \geq \mathbb{E}_T(X_i) \tag{1}$$

Proof.

$$\forall t, \lim_{n\to\infty} \frac{1}{n} \sum_i^n \mathbb{1}_{X_{i,t-1}>0} X_{i,t} = m \left(1 - \frac{1}{n} \sum_i^n \mathbb{1}_{X_{i,t-1}\leq 0}\right). \tag{2}$$

where $\mathbb{1}_{X_{t-1}>0}$ is the indicator function requiring survival at the previous period. Hence the limits of n for t show a decreasing temporal expectation: $\mathbb{E}_N(X_{t-1}) \leq \mathbb{E}_N(X_t)$.

We can actually prove divergence.

$$\forall i, \lim_{T\to\infty} \frac{1}{T} \sum_t^T \mathbb{1}_{X_{i,t-1}>0} X_{i,t} = 0. \tag{3}$$

As we can see by making $T < \infty$, by recursing the law of iterated expectations, we get the inequality for all T. ☐

In Eq. 3 we took the ensemble of risk takers expecting a return $m\left(1 - \frac{1}{n}\sum_i^n \mathbb{1}_{X_{i,t-1}=0}\right)$ in *any* period t, while every single risk taker is guaranteed to eventually go bust.

Note: we can also approach the proof more formally in a measure-theoretic way by showing that while space sets for "nonruin" $\mathcal{A}$ are disjoint, time sets are not. For a measure ν:

$$\nu\left(\bigcup_T \mathcal{A}_t \bigcap_{\leq t} \mathcal{A}_i^c\right) \text{ does not necessarily equal } \nu\left(\bigcup_T \mathcal{A}_t\right).$$

Commentary 3. *Almost all papers discussing the actuarial "overestimation" of tail risk via options, see review in [3] are void by the inequality in Theorem 1. Clearly they assume that an agent only exists for a single decision or exposure. Simply the original papers documenting the "bias" assume that the agents will never ever again make another decision in their remaining lives.*

The usual solution to this path dependence –if it depends on only ruin – is done by introducing a function of X to allow the ensemble (path independent) average to have the same properties as the time (path dependent) average – or survival conditioned mean. The natural logarithm seems a good candidate. Hence $S_n = \sum_{i=1}^n \log(X_i)$ and $S_T = \sum_{t=1}^T \log(X_t)$ belong to the same probabilistic class; hence a probability measure on one is invariant with respect to the other –what is called ergodicity. In that sense, when analyzing performance and risk, under conditions of ruin, it is necessary to use a logarithmic transformation of the variable [4], [1], or boundedness of the left tail[5] and [6] while maximizing opportunity in the right tail.

Commentary 4. *What we showed here is that unless one takes a logarithmic transformation (or a similar –smooth –function producing* $-\infty$ *with ruin set at* $X = 0$*), both expectations diverge. The entire point of the precautionary principle is to avoid having to rely on logarithms or transformations by reducing the probability of ruin.*

Commentary 5. *In their magisterial paper, Peters and Gell-Mann[1] showed that the Bernouilli use of the logarithm wasn't for a concave "utility" function, but, as with the Kelly criterion, to restore ergodicity. A bit of history:*

- *Bernouilli discovers logarithmic risk taking under the illusion of "utility".*
- *Kelly and Thorp recovered the logarithm for maximal growth criterion as an optimal gambling strategy. Nothing to do with utility.*
- *Samuelson disses logarithm as aggressive, not realizing that semi-logarithm (or partial logarithm), i.e. on partial of wealth can be done. From Menger to Arrow, via Chernoff and Samuelson, many in decision theory are shown to be making the mistake of ergodicity.*
- *Pitman in 1975 [7] shows that a Brownian motion subjected to an absorbing barrier at 0, with censored absorbing paths, becomes a three-dimensional Bessel process. The drift of the surviving paths is* $\frac{1}{x}$*, which integrates to a logarithm.*
- *Peters and Gell-Mann recovers the logarithm for ergodicity and, in addition, put the Kelly-Thorpe result on rigorous physical grounds.*
- *With Cirillo, this author [8] discovers the log as unique smooth transformation to create a dual of the distribution in order to remove one-tail compact support to allow the use of extreme value theory.*
- *We can show (Briys and Taleb, in progress) the necessity of logarithmic transformation as simple ruin avoidance, which happens to be a special case of the HARA utility class.*

B. Adaptation of Theorem 1 to Brownian Motion

The implications of simplified discussion does not change whether one uses richer models, such as a full stochastic process subjected to an absorbing barrier. And of course in a natural setting the eradication of all previous life can happen (i.e. X_t can take extreme negative value), not just a stopping condition.

> The Peters and Gell-Mann argument [9] also cancels the so-called equity premium puzzle if you add fat tails (hence outcomes vastly more severe pushing some level equivalent to ruin) and absence of the fungibility of temporal and emsemble. There is no puzzle.

The problem is invariant in real life if one uses a Brownian-motion style stochastic process subjected to an absorbing barrier. In place of the simplified representation we would have, for an process subjected to L, an absorbing barrier from below, in the arithmetic version:

$$\forall i, X_{i,t} = \begin{cases} X_{i,t-1} + Z_{i,t}, & X_{i,t-1} > L \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

or, for a geometric process

$$\forall i, X_{i,t} = \begin{cases} X_{i,t-1}(1 + Z_{i,t}) \approx X_{i,t-1}e^{Z_{i,t}}, & X_{i,t-1} > L \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where Z is a random variable.

Going to continuous time, and considering the geometric case, let $\tau = \{\inf t : X_{i,t} > L\}$ be the stopping time. The idea is to have the simple expectation of the stopping time match the remaining lifespan –or remain in the same order.

Working with a function of X under stopping time: Let us digress to see how we can work with functions of ruin as an absorbing barrier. This is a bit more complicated but quite useful for calibration of ruin off a certain known shelf life.

Assume a finite expectation for the stopping time $E^L(\tau) < \infty$ for barrier level $L < X_0$. Let $Af(X)$ be the infinitesimal generator for the one-dimensional Brownian motion: $Af(t,x) = \frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x)$. Dynkin's operator gives the expected value of any suitably smooth statistic of X at a stopping time, for a square integrable function $f : \mathbb{R}^+ \to \mathbb{R}$:

$$\mathbf{E}^L\left(f(X_\tau)\right) = f(X_0) + \mathbf{E}^L\left(\int_0^\tau Af(X_s)\,\mathrm{d}s\right).$$

For the more general Lévy process (i.e. fat tails), we can obtain slightly more complicated results by the standard subordination, i.e., adding a Poisson to the Brownian motion above.

> **Commentary 6.** *We switched the focus from probability to the mismatch between stopping time τ for ruin and the remaining lifespan.*

II. Principle of Probabilistic Sustainability

Principle 1. *A unit needs to take any risk as if it were going to take it repeatedly –at a specified frequency – over its remaining lifespan.*

The principle of sustainability is necessary for the following argument. While academic studies are static (we saw the confusion between the state-space and the temporal), life is continuous. If you incur a tiny probability of ruin as a

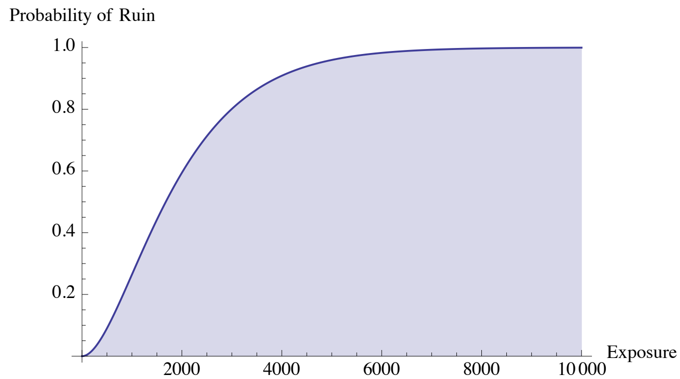

Fig. 2. **Why Ruin is not a Renewable Resource**. No matter how small the probability, in time, something bound to hit the ruin barrier is about guaranteed to hit it. No risk should be considered a "one-off" event.

"one-off" risk, survive it, then do it again (another "one-off" deal), you will eventually go bust with probability 1. Confusion arises because it may seem that the "one-off" risk is reasonable, but that also means that an additional one is reasonable. (see Fig. 2). The good news is that some classes of risk can be deemed to be practically of probability zero: the earth survived trillions of natural variations daily over 3 billion years, otherwise we would not be here. We can use conditional probability arguments (adjusting for the survivorship bias) so back-out the ruin probability in a system.

Now, we do not have to take $t \to \infty$ nor do is permanent sustainability necessary. We can just extent shelf time. The longer the t, the more the expectation operators diverge.

Consider the unconditional expected stopping time to ruin in a discrete and simplified model: $\mathbb{E}(\tau \wedge T) \approx \mathbb{E}(\tau) = \sum_{i=1}^{\lambda N} i\left(\frac{p}{\lambda}\left(1-\frac{p}{\lambda}\right)^{i-1}\right)$, where λ is the number of exposures per time period, T is the overall remaining lifespan and p is the ruin probability, both over that same time period for fixing p. Since $\mathbb{E}(\tau) = \frac{\lambda}{p}$, we can calibrate the risk under repetition. The longer the life expectancy T (expressed in time periods), the more serious the ruin problem. Humans and plant have a short shelf life, nature doesn't –at least for t of the order of 10^8 years hence annual ruin probabilities of $O(10^{-8})$ and (for a tighter increments) local ruin probabilities of at most $O(10^{-50})$. The higher up in the hierarchy individual-species-ecosystem, the more serious the ruin problem. This duality hinges on $t \to \infty$; hence requirement is not necessary for items that are not permanent, that have a finite shelf life.

Remark 2 (The fat tails argument)**.** *The more a system is capable of delivering large deviations, the worse the ruin problem.*

We will cover the fat tails problem more extensively. Clearly the variance of the process matters; but overall deviations that do not exceed the ruin threshold do not matter.

A. Logarithmic transformation

Under the axiom of sustainability, i.e., that "one should take risks as if you were going to do it forever", only a logarithmic (or similar) transformation applies.

Fattailedness is a property that is typically worrisome under absence of compact support for the random variable, less so when the variables are bounded. But as we saw the need of using a logarithmic transformation, a random variable with support in $[0, \infty)$ now has support in $(-\infty, \infty)$, hence properties derived from extreme value theory can now apply to our analysis. Likewise, if harm is defined as a positive number with an upper bound H which corresponds to ruin, it is possible to transform it from $[0, H]$ to $[0, \infty)$.

Cramér and Lundberg, in insurance analysis discovered the difficulty in [TK]

III. How Kahneman-Tversky's S-Curve is compatible with Bernoulli's Log

As shown in Fig. 3, we can summarize to expand later:

- Anything entailing ruin as reduction from an infinite life has effectively an infinite cost (or equivalent, a high price). Hence a logarithmic transformation is the unique solution as we can show log is the only smooth function that produces a penalty of negative infinity at the depletion of the entity and remains myopic. This justifies Bernoulli's logarithmic "utility" function. It is not a utility per se, but a straightforward cost function.
- As an individual, I have a finite shelf life so the costs are not as onerous as the loss of humanity. Hence for items such as wealth (or something renewable), the "S curve", convex-concave applies *necessarily*. Why the "S curve"? Because of emotional saturation on both the right and the left (an argument TK). Kahneman-Tversky doesn't go to negative infinity but one would expect the extension to have a sigmoidal form, i.e. to be bounded.
- The loss of one's life cannot be the worst case *if* there is worse: one's death << one's death + loss of family << ...+ loss of tribe << ecocide.

IV. Why Increase in "Benefits" Usually Increases the Risk of Ruin

Commentary 7. *We show that the probability of ruin increases upon an action to "improve" on a system as the trade-off benefits-risk increases in the tails. In other words, anything that increases benefits, if it increases uncertainty, leads to the violation of the equality state space-time expectations.*

This is a general precautionary method for iatrogenics.

Why should uncertainty about climate models lead to a more conservative, more cautious, "greener" stance, even –or especially – if one disbelieved the models?

Why do super-rich gain more from inequality than from gains in GDP, in proportion to how rich they are?

Why do derivatives "in the tails" depend more on changes in expected volatility than from changes in the mean (not well known by practitioners of mathematical finance (derivatives) who get periodically harmed by it)?

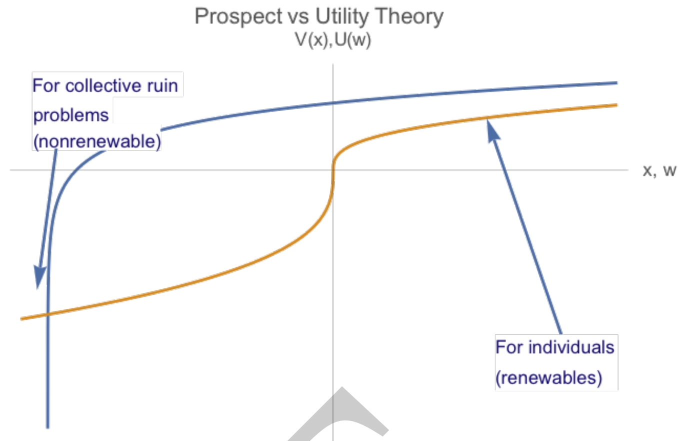

Fig. 3. For something that is supposed to have an infinite life, a logarithmic transformation is necessary to analyze harm –only a logarithmic function works there. For things that are bounded (or near-bounded) on the left and the right, an S-curve can be shown to be necessary. In preparation is a theorem showing how utilities depend on the hierarchy.

Why increase risk of deflation *also* necessarily increases the risk of hyperinflation.

Why should worry about GMOs even if one accepted their purported benefits?

It is a necessarily mathematical relation that remote parts of the distribution —the tails— are less sensitive to changes in the mean, and more sensitive to other parameters controlling the dispersion and scale (which in the special case of class of finite-variance distributions would be the variance) or those controlling the tail exponent.

Definition 1. *Let Φ be a twice derivable continuous probability CDF with at least one unbounded tail $\Phi \in C^2 : \mathfrak{D} \to [0,1]$, with $s > 0$, where λ is a slowly varying function with respect to x: $\forall t > 0, \lim_{x\to\pm\infty} \frac{\lambda(tx)}{\lambda(x)} = 1$:*

We have either

$$\Phi(x; l, s, \alpha) \triangleq \lambda(\frac{x-l}{s}, \alpha)\, Z\left(\frac{x-l}{s}, \alpha\right), \mathfrak{D} = [-\infty, x_0]$$

or

$$\overline{\Phi}(x; l, s, \alpha) \triangleq 1 - \lambda(\frac{x-l}{s}, \alpha)\, Z\left(\frac{x-l}{s}, \alpha\right), \mathfrak{D} = [x_0, \infty) \tag{6}$$

where x_0 is the (maximum) minimum value for the representation of the distribution, l is the location and $s \in (0, \infty)$ the scale.

Intuitively we are using any probability distribution and mapping the random variable $x \mapsto \frac{x-l}{s}$ and only focusing on the tails. Thanks to such focus on the tails only, the distribution does not necessarily need to be in the location-scale family (i.e., retain their properties after the transformation). We are factorizing the CDF into a two functions, one of which becomes a constant for "large" values of |x|, given that we aim at isolating tail probabilities and disregard other portions of the distribution.

The distribution in (6) can accommodate $x_0 = \pm\infty$, in which the function is whole; but all is required is for it to smooth and have the expressions above in either of the open "tail segments" concerned, which we define as values below (above) K.

A. Scale vs. Location

Let $\upsilon(K)_- \triangleq \frac{\partial \Phi(.)}{\partial s}|_{x=K}$ and $\upsilon(K)_+ \triangleq \frac{\partial \bar{\Phi}(.)}{\partial s}|_{x=K}$ be the "vega", that is sensitivity to scale and $\delta(K)_- \triangleq \frac{\partial \Phi(.)}{\partial l}|_{x=K}$ and $\delta(K)_+ \triangleq \frac{\partial \bar{\Phi(.)}}{\partial l}|_{x=K}$ be the "delta" that is sensitivity to location for positive and negative tail, respectively. Consider the tail probabilities at level K, defined as $P(x<K) \triangleq \Phi(K)$ and $P(x>K) \triangleq \overline{\Phi}(K)$.

We have the "vega", that is sensitivity to scale

$$\upsilon(K)_- = -\frac{1}{s^2}\left((K-l)\left(\lambda\left(\frac{K-l}{s},\alpha\right)Z^{(1,0)}\left(\frac{K-l}{s},\alpha\right) + \lambda^{(1,0)}\left(\frac{K-l}{s},\alpha\right)Z\left(\frac{K-l}{s},\alpha\right)\right)\right) \tag{7}$$

For clarity we are using the slot notation: $Z^{(1,0)}(.,.)$ refers the first partial derivative of the function Z with respect of the first argument (not the variable under concern), and $Z^{(0,1)}(.,.)$ that with respect of the second (by the chain ruke, $Z^{(1,0)}\left(\frac{K-l}{s},\alpha\right) = s\frac{\partial Z(.,.)}{\partial s}$).

$$\delta(K)_- = -\frac{1}{s}\left(\lambda\left(\frac{K-l}{s},\alpha\right)Z^{(1,0)}\left(\frac{K-l}{s},\alpha\right) + \lambda^{(1,0)}\left(\frac{K-l}{s},\alpha\right)Z\left(\frac{K-l}{s},\alpha\right)\right). \tag{8}$$

Thanks to the Karamata Tauberian theorem, we can refine $\upsilon(.)$ and $\delta(.)$. given that $\lambda(.)$ is a slowly varying function, we have:

$$\lambda(\frac{K-l}{s},\alpha) \to \lambda,$$

and

$$\lambda^{(1,0)}(\frac{K-l}{s},\alpha) \to 0.$$

Hence: $\upsilon(K)_- = -\frac{1}{s^2}(K-l)\lambda Z^{(1,0)}\left(\frac{K-l}{s},\alpha\right)$ and $\delta(K)_- = -\frac{1}{s}\lambda\left(\frac{K-l}{s},\alpha\right)Z^{(1,0)}\left(\frac{K-l}{s},\alpha\right)$, which allows us to prove the following theorem:

Theorem 2 (scale/location tradeoff)**.** *There exists, for any distribution with an one unbounded tail, in C^2, twice differentiable below (above) level K^* s.t. for the unbounded tail under concern, $|K|\geq|K^*|, r = \frac{\upsilon(K)_-}{\delta(K)_-} = \frac{\upsilon(K)_+}{\delta(K)_+} \to \frac{K-l}{s}$.*

The theorem is trivial once we set up the probability distributions as we did in 6.

The physical interpretation of the theorem is that in the tail, ruin probability is vastly more reactive to s, that is the ratio of dispersion, than l, a shifting of the mean.

We note that our result is general –some paper (cite) saw the effect and applied to specific tail distributions from extreme value theory when our result does not require the particulars of the distribution.

1) Examples: For a LogNormal Distribution with parameters μ and σ, we construct a change of variable $\frac{x-l}{s}$, ending with a CDF: $\frac{1}{2}\text{erfc}\left(\frac{\mu-\log\left(\frac{x-l}{s}\right)}{\sqrt{2}\sigma}\right)$. Now the right tail probability

$$\Phi(K) = \frac{1}{2}\left(\text{erf}\left(\frac{-\log(K-l)+\mu+\log(s)}{\sqrt{2}\sigma}\right)+1\right)$$

has for derivatives $\upsilon(K)_+ = \frac{e^{-\frac{(-\log(K-l)+\mu+\log(s))^2}{2\sigma^2}}}{\sqrt{2\pi}s\sigma}$, $\delta(K)_+ = \frac{e^{-\frac{(-\log(K-l)+\mu+\log(s))^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma(K-l)}$, and the ratio $r = \frac{K-l}{s}$.

B. Shape vs. location

Discussion: exponent determining the shape of the distribution, expression uncertainty by broadening the tails and having an effect on the variance for powerlaws (except for the Lévy Stable distribution).

As to the tail probability sensitivity to the exponent, it is to be taken in the negative: lower exponent means more uncertainty, so we reverse the sign.

$$\begin{aligned}\omega(K)_+ &= -\frac{\partial\overline{\Phi}(.)}{\partial\alpha}|_{x=K} \\ &= \lambda(K-l,\alpha)Z^{(0,1)}\left(\frac{K-l}{s},\alpha\right) \\ &\quad + \lambda^{(0,1)}(K-l,\alpha)Z\left(\frac{K-l}{s},\alpha\right)\end{aligned} \tag{9}$$

Theorem 3 (tail shape/location tradeoff)**.** *There exists, for a distribution with an one unbounded tail, a level K^* s.t. for an unbounded tail, $|K|\geq|K^*|$,*

$$r = \frac{\omega(K)_+}{\delta(K)_+} = \frac{1}{\alpha}(K-l(s+1))\left(\log\left(\frac{K-l(s+1)}{s^2}\right)-1\right)$$

Proof. For large positive deviations, with $K > ls$ we can write, by Karamata's result for slowly moving functions, $\overline{\Phi}(K) \approx \lambda\left(\frac{K-l}{s}\right)^{-\alpha}$, hence $Z(.) = \left(\frac{K-l}{s}\right)^{-\alpha}$.

$$\frac{\omega(K)_+}{\delta(K)_+} = \frac{sZ\left(\frac{K-l}{s},\alpha\right)}{Z^{(1,0)}\left(\frac{K-l}{s},\alpha\right)} + \frac{sZ^{(0,1)}\left(\frac{K-l}{s},\alpha\right)}{Z^{(1,0)}\left(\frac{K-l}{s},\alpha\right)} \tag{10}$$

□

C. Shape vs Scale

We note the role of shape vs scale:

$$\frac{\upsilon(K)_+}{\upsilon(K)_+} = cs(K-l(s+1))\left(\log\left(\frac{K-l(s+1)}{s^2}\right)-1\right)\alpha(K-l)$$

Commentary 8. *We can show the effect of changes in the tails on, say, "fragile" items, and the **inaplicability of the average**. For instance, what matters for the climate is **not** the average, but the distribution of the extremes. Here average effect is l and extremes from s and, worse, α.*

I am working on the distribution of risk based on *"stochastic α"*

V. Technical definition of Fat Tails

Probability distributions range between extreme thin-tailed (Bernoulli) and extreme fat tailed [**?**]. Among the categories of distributions that are often distinguished due to the convergence properties of moments are: 1) Having a support that is compact but not degenerate, 2) Subgaussian, 3) Gaussian, 4) Subexponential, 5) Power law with exponent greater than 3, 6) Power law with exponent less than or equal to 3 and greater than 2, 7) Power law with exponent less than or equal to 2. In particular, power law distributions have a finite mean only if the exponent is greater than 1, and have a finite variance only if the exponent exceeds 2.

Our interest is in distinguishing between cases where tail events dominate impacts, as a formal definition of the boundary between the categories of distributions to be considered as Mediocristan and Extremistan. The natural boundary between these occurs at the subexponential class which has the following property:

Let $\mathbf{X} = (X_i)_{1\leq i\leq n}$ be a sequence of independent and identically distributed random variables with support in ($\mathbb{R}^+$), with cumulative distribution function F. The subexponential class of distributions is defined by [10],[11].

$$\lim_{x\to+\infty} \frac{1-F^{*2}(x)}{1-F(x)} = 2$$

where $F^{*2} = F' * F$ is the cumulative distribution of X_1+X_2, the sum of two independent copies of X. This implies that the probability that the sum X_1+X_2 exceeds a value x is twice the probability that either one separately exceeds x. Thus, every time the sum exceeds x, for large enough values of x, the value of the sum is due to either one or the other exceeding x—the maximum over the two variables—and the other of them contributes negligibly.

More generally, it can be shown that the sum of n variables is dominated by the maximum of the values over those variables in the same way. Formally, the following two properties are equivalent to the subexponential condition [12],[13]. For a given $n \geq 2$, let $S_n = \Sigma_{i=1}^n x_i$ and $M_n = \max_{1\leq i\leq n} x_i$

a) $\lim_{x\to\infty} \frac{P(S_n>x)}{P(X>x)} = n$,

b) $\lim_{x\to\infty} \frac{P(S_n>x)}{P(M_n>x)} = 1$.

Thus the sum S_n has the same magnitude as the largest sample M_n, which is another way of saying that tails play the most important role.

Intuitively, tail events in subexponential distributions should decline more slowly than an exponential distribution for which large tail events should be irrelevant. Indeed, one can show that subexponential distributions have no exponential moments:

$$\int_0^\infty \mathbf{e}^{\epsilon x}\, dF(x) = +\infty$$

for all values of ε greater than zero. However,the converse isn't true, since distributions can have no exponential moments, yet not satisfy the subexponential condition.

We note that if we choose to indicate deviations as negative values of the variable x, the same result holds by symmetry for extreme negative values, replacing $x \to +\infty$ with $x \to -\infty$. For two-tailed variables, we can separately consider positive and negative domains.

VI. On the Unreliability of Hypothesis testing for Risk Analysis

The derivations in this section were motivated during the GMO debate by discussions of "evidence", based on p-values of around $\approx .05$. Not only we need much lower than .05 for any safety assessment of repeated exposure, but the p-value method and, sadly, the "power of test" have acute stochasticities reminiscent of fat tail problems.

Commentary 9. *Where we show that p-values are unreliable for risk analysis, hence say* ***nothing*** *about ruin probabilities. The so-called "scientific" studies are too speculative to be of any use for tail risk (which shows in their continuous evolution). The exception, of course, is negative empiricism.*

Assume that we knew the "true" p-value, p_s, what would its realizations look like across various attempts on statistically identical copies of the phenomena? By true value p_s, we mean its expected value by the law of large numbers across an m ensemble of possible samples for the phenomenon under scrutiny, that is $\frac{1}{m}\sum_{\leq m} p_i \xrightarrow{P} p_s$ (where $\xrightarrow{P}$ denotes convergence in probability). A similar convergence argument can be also made for the corresponding "true median" p_M. The main result of the paper is that the the distribution of n small samples can be made explicit (albeit with special inverse functions), as well as its parsimonious limiting one for n large, with no other parameter than the median value p_M. We were unable to get an explicit form for p_s but we go around it with the use of the median. Finally, the distribution of the minimum p-value under can be made explicit, in a parsimonious formula allowing for the understanding of biases in scientific studies.

It turned out, as we can see in Fig. 5 the distribution is extremely asymmetric (right-skewed), to the point where 75% of the realizations of a "true" p-value of .05 will be <.05 (a borderline situation is 3× as likely to pass than fail a given protocol), and, what is worse, 60% of the true p-value of .12 will be below .05.

Although with compact support, the distribution exhibits the attributes of extreme fat-tailedness. For an observed p-value of, say, .02, the "true" p-value is likely to be >.1 (and very possibly close to .2), with a standard deviation >.2 (sic) and a mean deviation of around .35 (sic, sic). Because of the excessive skewness, measures of dispersion in L^1 and L^2 (and higher norms) vary hardly with p_s, so the standard deviation is not proportional, meaning

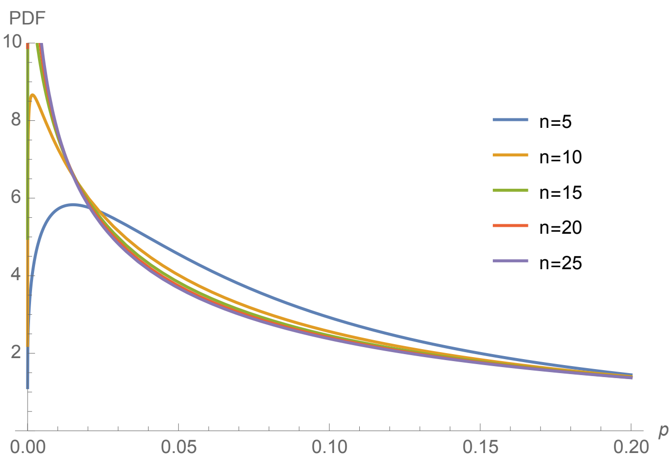

Fig. 4. The different values for Equ. 11 showing convergence to the limiting distribution.

an in-sample .01 p-value has a significant probability of having a true value > .3.

So clearly we don't know what we are talking about when we talk about p-values.

Earlier attempts for an explicit meta-distribution in the literature were found in [14] and [15], though for situations of Gaussian subordination and less parsimonious parametrization. The severity of the problem of *significance of the so-called "statistically significant"* has been discussed in [16] and offered a remedy via Bayesian methods in [17], which in fact recommends the same tightening of standards to p-values $\approx .01$. But the gravity of the extreme skewness of the distribution of p-values is only apparent when one looks at the meta-distribution.

For notation, we use n for the sample size of a given study and m the number of trials leading to a p-value.

A. Proofs and derivations

Proposition 1. *Let P be a random variable $\in [0,1]$) corresponding to the sample-derived one-tailed p-value from the paired T-test statistic (unknown variance) with median value $\mathbb{M}(P) = p_M \in [0,1]$ derived from a sample of n size. The distribution across the ensemble of statistically identical copies of the sample has for PDF*

$$\varphi(p;p_M) = \begin{cases} \varphi(p;p_M)_L & \text{for } p < \frac{1}{2} \\ \varphi(p;p_M)_H & \text{for } p > \frac{1}{2} \end{cases}$$

$$\varphi(p;p_M)_L = \lambda_p^{\frac{1}{2}(-n-1)} \sqrt{-\frac{\lambda_p\left(\lambda_{p_M}-1\right)}{\left(\lambda_p-1\right)\lambda_{p_M}-2\sqrt{\left(1-\lambda_p\right)\lambda_p}\sqrt{\left(1-\lambda_{p_M}\right)\lambda_{p_M}}+1}} \left(\frac{1}{\frac{1}{\lambda_p}-\frac{2\sqrt{1-\lambda_p}\sqrt{\lambda_{p_M}}}{\sqrt{\lambda_p}\sqrt{1-\lambda_{p_M}}}+\frac{1}{1-\lambda_{p_M}}-1}\right)^{n/2}$$

$$\varphi(p;p_M)_H = \left(1-\lambda'_p\right)^{\frac{1}{2}(-n-1)} \left(\frac{\left(\lambda'_p-1\right)\left(\lambda_{p_M}-1\right)}{\lambda'_p\left(-\lambda_{p_M}\right)+2\sqrt{\left(1-\lambda'_p\right)\lambda'_p}\sqrt{\left(1-\lambda_{p_M}\right)\lambda_{p_M}}+1}\right)^{\frac{n+1}{2}} \quad (11)$$

where $\lambda_p = I^{-1}_{2p}\left(\frac{n}{2},\frac{1}{2}\right)$, $\lambda_{p_M} = I^{-1}_{1-2p_M}\left(\frac{1}{2},\frac{n}{2}\right)$, $\lambda'_p = I^{-1}_{2p-1}\left(\frac{1}{2},\frac{n}{2}\right)$, and $I^{-1}_{(.)}(.,.)$ is the inverse beta regularized function.

Remark 3. *For $p=\frac{1}{2}$ the distribution doesn't exist in theory, but does in practice and we can work around it with the sequence $p_{m_k} = \frac{1}{2} \pm \frac{1}{k}$, as in the graph showing a convergence to the Uniform distribution on $[0,1]$ in Figure 6. Also note that what is called the "null" hypothesis is effectively a set of measure 0.*

Proof. Let Z be a random normalized variable with realizations ζ, from a vector $\vec{v}$ of n realizations, with sample mean m_v, and sample standard deviation s_v, $\zeta = \frac{m_v - m_h}{\frac{s_v}{\sqrt{n}}}$ (where m_h is the level it is tested against), hence assumed to $\sim$ Student T with n degrees of freedom, and, crucially, supposed to deliver a mean of $\bar{\zeta}$,

$$f(\zeta;\bar{\zeta}) = \frac{\left(\frac{n}{(\bar{\zeta}-\zeta)^2+n}\right)^{\frac{n+1}{2}}}{\sqrt{n}B\left(\frac{n}{2},\frac{1}{2}\right)}$$

where B(.,.) is the standard beta function. Let $g(.)$ be the one-tailed survival function of the Student T distribution with zero mean and n degrees of freedom:

$$g(\zeta) = \mathbb{P}(Z > \zeta) = \begin{cases} \frac{1}{2} I_{\frac{n}{\zeta^2+n}}\left(\frac{n}{2},\frac{1}{2}\right) & \zeta \geq 0 \\ \frac{1}{2}\left(I_{\frac{\zeta^2}{\zeta^2+n}}\left(\frac{1}{2},\frac{n}{2}\right)+1\right) & \zeta < 0 \end{cases}$$

where $I_{(.,.)}$ is the incomplete Beta function.

We now look for the distribution of $g \circ f(\zeta)$. Given that g(.) is a legit Borel function, and naming p the probability as a random variable, we have by a standard result for the transformation:

$$\varphi(p,\bar{\zeta}) = \frac{f\left(g^{(-1)}(p)\right)}{\left|g'\left(g^{(-1)}(p)\right)\right|}$$

We can convert $\bar{\zeta}$ into the corresponding median survival probability because of symmetry of Z. Since one half the observations fall on either side of $\bar{\zeta}$, we can ascertain that the transformation is median preserving: $g(\bar{\zeta}) = \frac{1}{2}$, hence $\varphi(p_M,.) = \frac{1}{2}$. Hence we end up having $\{\bar{\zeta} : \frac{1}{2} I_{\frac{n}{\bar{\zeta}^2+n}}\left(\frac{n}{2},\frac{1}{2}\right) = p_M\}$ (positive case) and $\{\bar{\zeta} : \frac{1}{2}\left(I_{\frac{\zeta^2}{\zeta^2+n}}\left(\frac{1}{2},\frac{n}{2}\right)+1\right) = p_M\}$ (negative case). Replacing we get Eq.11 and Proposition 1 is done.

□

We note that n does not increase significance, since p-values are computed from normalized variables (hence the universality of the meta-distribution); a high n corresponds to an increased convergence to the Gaussian. For large n, we can prove the following proposition:

Proposition 2. *Under the same assumptions as above, the limiting distribution for $\varphi(.)$:*

$$\lim_{n\to\infty} \varphi(p;p_M) = e^{-erfc^{-1}(2p_M)\left(erfc^{-1}(2p_M)-2erfc^{-1}(2p)\right)} \quad (12)$$

where erfc(.) is the complementary error function and $erfc(.)^{-1}$ its inverse.

The limiting CDF $\Phi(.)$

$$\Phi(k;p_M) = \frac{1}{2} erfc\left(erf^{-1}(1-2k) - erf^{-1}(1-2p_M)\right) \quad (13)$$

Proof. For large n, the distribution of $Z = \frac{m_v}{\frac{s_v}{\sqrt{n}}}$ becomes that of a Gaussian, and the one-tailed survival function $g(.) = \frac{1}{2}\text{erfc}\left(\frac{\zeta}{\sqrt{2}}\right)$, $\zeta(p) \to \sqrt{2}\text{erfc}^{-1}(p)$. □

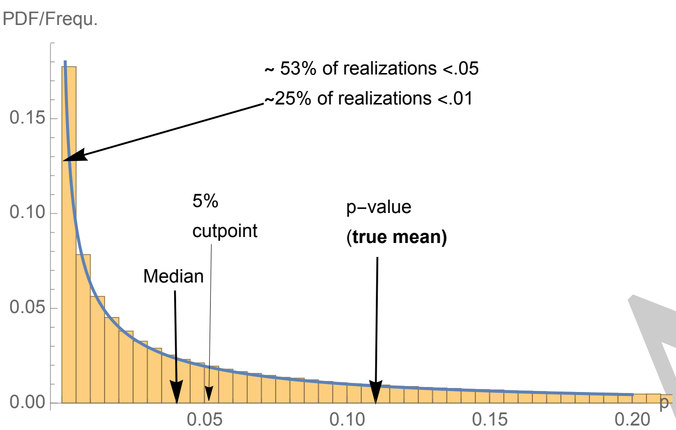

Fig. 5. The probability distribution of a one-tailed p-value with expected value .11 generated by Monte Carlo (histogram) as well as analytically with $\varphi(.)$ (the solid line). We draw all possible subsamples from an ensemble with given properties. The excessive skewness of the distribution makes the average value considerably higher than most observations, hence causing illusions of "statistical significance".

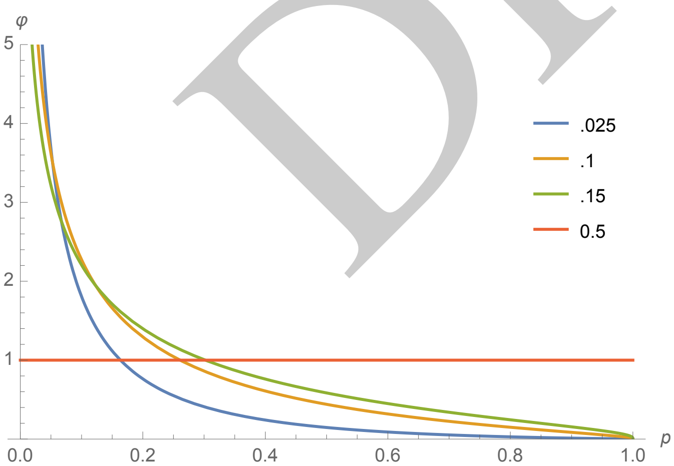

Fig. 6. The probability distribution of p at different values of p_M. We observe how $p_M = \frac{1}{2}$ leads to a uniform distribution.

This limiting distribution applies for paired tests with known or assumed sample variance since the test becomes a Gaussian variable, equivalent to the convergence of the T-test (Student T) to the Gaussian when n is large.

Remark 4. *For values of p close to 0, φ in Equ. 12 can be usefully calculated as:*

$$\varphi(p;p_M) = \sqrt{2\pi} p_M \sqrt{\log\left(\frac{1}{2\pi p_M^2}\right)} e^{\sqrt{-\log\left(2\pi\log\left(\frac{1}{2\pi p^2}\right)\right)-2\log(p)}\sqrt{-\log\left(2\pi\log\left(\frac{1}{2\pi p_M^2}\right)\right)-2\log(p_M)}} + O(p^2). \quad (14)$$

The approximation works more precisely for the band of relevant values $0 < p < \frac{1}{2\pi}$.

From this we can get numerical results for convolutions of φ using the Fourier Transform or similar methods.

We can and get the distribution of the minimum p-value per m trials across statistically identical situations thus get an idea of "p-hacking", defined as attempts by researchers to get the lowest p-values of many experiments, or try until one of the tests produces statistical significance.

Proposition 3. *The distribution of the minimum of m observations of statistically identical p-values becomes (under the limiting distribution of proposition 2):*

$$\varphi_m(p;p_M) = m\, e^{erfc^{-1}(2p_M)\left(2erfc^{-1}(2p)-erfc^{-1}(2p_M)\right)} \left(1 - \frac{1}{2}erfc\left(erfc^{-1}(2p) - erfc^{-1}(2p_M)\right)\right)^{m-1} \quad (15)$$

Proof. $P(p_1 > p, p_2 > p, \ldots, p_m > p) = \bigcap_{i=1}^{n} \Phi(p_i) = \bar{\Phi}(p)^m$. Taking the first derivative we get the result. □

Outside the limiting distribution: we integrate numerically for different values of m as shown in figure 7. So, more precisely, for m trials, the expectation is calculated as:

$$\mathbb{E}(p_{min}) = \int_0^1 -m\, \varphi(p;p_M) \left(\int_0^p \varphi(u,.)\, \mathrm{d}u\right)^{m-1} \mathrm{d}p$$

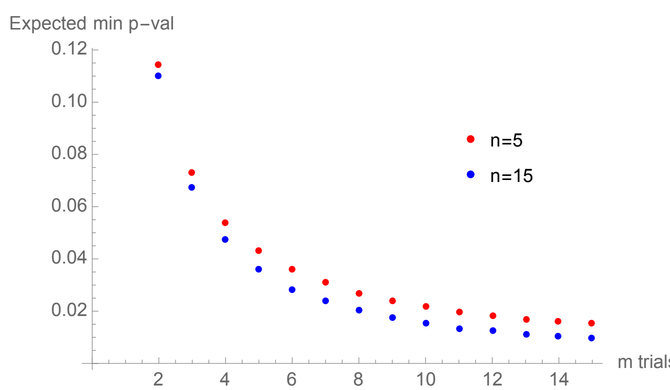

Fig. 7. The "p-hacking" value across m trials for $p_M = .15$ and $p_s = .22$.

B. Inverse Power of Test

Let β be the power of a test for a given p-value p, for random draws X from unobserved parameter θ and a sample size of n. To gauge the reliability of β as a true measure of power, we perform an inverse problem:

$$\begin{array}{ccc} \beta & \longrightarrow & X_{\theta,p,n} \\ \Delta \updownarrow & \swarrow & \\ \beta^{-1}(X) & & \end{array}$$

Proposition 4. *Let β_c be the projection of the power of the test from the realizations assumed to be student T distributed and evaluated under the parameter θ. We have*

$$\Phi(\beta_c) = \begin{cases} \Phi(\beta_c)_L & \text{for } \beta_c < \frac{1}{2} \\ \Phi(\beta_c)_H & \text{for } \beta_c > \frac{1}{2} \end{cases}$$

where

$$\Phi(\beta_c)_L = \sqrt{1-\gamma_1}\gamma_1^{-\frac{n}{2}} \frac{\left(-\frac{\gamma_1}{2\sqrt{\frac{1}{\gamma_3}-1}\sqrt{-(\gamma_1-1)\gamma_1}-2\sqrt{-(\gamma_1-1)\gamma_1}+\gamma_1\left(2\sqrt{\frac{1}{\gamma_3}-1}-\frac{1}{\gamma_3}\right)-1}\right)^{\frac{n+1}{2}}}{\sqrt{-\left(\gamma_1-1\right)\gamma_1}} \tag{16}$$

$$\Phi(\beta_c)_H = \sqrt{\gamma_2}\left(1-\gamma_2\right)^{-\frac{n}{2}} B\left(\frac{1}{2},\frac{n}{2}\right) \frac{\left(\frac{1}{\frac{-2\left(\sqrt{-(\gamma_2-1)\gamma_2}+\gamma_2\right)\sqrt{\frac{1}{\gamma_3}-1}+2\sqrt{\frac{1}{\gamma_3}-1}+2\sqrt{-(\gamma_2-1)\gamma_2}-1}{\gamma_2-1}+\frac{1}{\gamma_3}}\right)^{\frac{n+1}{2}}}{\sqrt{-\left(\gamma_2-1\right)\gamma_2}B\left(\frac{n}{2},\frac{1}{2}\right)} \tag{17}$$

where $\gamma_1 = I^{-1}_{2\beta_c}\left(\frac{n}{2},\frac{1}{2}\right)$, $\gamma_2 = I^{-1}_{2\beta_c-1}\left(\frac{1}{2},\frac{n}{2}\right)$, *and* $\gamma_3 = I^{-1}_{(1,2p_s-1)}\left(\frac{n}{2},\frac{1}{2}\right)$.

C. Application and Conclusion

- One can safely see that under such stochasticity for the realizations of p-values and the distribution of its minimum, to get what people mean by 5% confidence (and the inferences they get from it), they need a p-value of at least one order of magnitude smaller.
- The "power" of a test has the same problem unless one either lowers p-values or sets the test at higher levels, such at .99.

SUMMARY AND CONCLUSIONS

We showed the fallacies committed in the name of "rationality" by various people such as Cass Sunstein or similar persons in the verbalistic "evidence based" category.

REFERENCES

[1] O. Peters and M. Gell-Mann, "Evaluating gambles using dynamics," *Chaos*, vol. 26, no. 2, 2016. [Online]. Available: http://scitation.aip.org/content/aip/journal/chaos/26/2/10.1063/1.4940236

[2] N. N. Taleb, "Black swans and the domains of statistics," *The American Statistician*, vol. 61, no. 3, pp. 198–200, 2007.

[3] N. Barberis, "The psychology of tail events: Progress and challenges," *American Economic Review*, vol. 103, no. 3, pp. 611–16, 2013.

[4] O. Peters, "The time resolution of the st petersburg paradox," *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 369, no. 1956, pp. 4913–4931, 2011.

[5] J. L. Kelly, "A new interpretation of information rate," *Information Theory, IRE Transactions on*, vol. 2, no. 3, pp. 185–189, 1956.

[6] D. Geman, H. Geman, and N. N. Taleb, "Tail risk constraints and maximum entropy," *Entropy*, vol. 17, no. 6, p. 3724, 2015. [Online]. Available: http://www.mdpi.com/1099-4300/17/6/3724

[7] J. W. Pitman, "One-dimensional brownian motion and the three-dimensional bessel process," *Advances in Applied Probability*, pp. 511–526, 1975.

[8] N. N. Taleb and P. Cirillo, "On the shadow moments of apparently infinite-mean phenomena," *arXiv preprint arXiv:1510.06731*, 2015.

[9] O. Peters and M. Gell-Mann, "Evaluating gambles using dynamics," *arXiv preprint arXiv:1405.0585*, 2014.

[10] J. L. Teugels, "The class of subexponential distributions," *The Annals of Probability*, vol. 3, no. 6, pp. 1000–1011, 1975.

[11] E. Pitman, "Subexponential distribution functions," *J. Austral. Math. Soc. Ser. A*, vol. 29, no. 3, pp. 337–347, 1980.

[12] V. Chistyakov, "A theorem on sums of independent positive random variables and its applications to branching random processes," *Theory of Probability & Its Applications*, vol. 9, no. 4, pp. 640–648, 1964.

[13] P. Embrechts, C. M. Goldie, and N. Veraverbeke, "Subexponentiality and infinite divisibility," *Probability Theory and Related Fields*, vol. 49, no. 3, pp. 335–347, 1979.

[14] H. J. Hung, R. T. O'Neill, P. Bauer, and K. Kohne, "The behavior of the p-value when the alternative hypothesis is true," *Biometrics*, pp. 11–22, 1997.

[15] H. Sackrowitz and E. Samuel-Cahn, "P values as random variables—expected p values," *The American Statistician*, vol. 53, no. 4, pp. 326–331, 1999.

[16] A. Gelman and H. Stern, "The difference between "significant" and "not significant" is not itself statistically significant," *The American Statistician*, vol. 60, no. 4, pp. 328–331, 2006.

[17] V. E. Johnson, "Revised standards for statistical evidence," *Proceedings of the National Academy of Sciences*, vol. 110, no. 48, pp. 19 313–19 317, 2013.