Introduction to Probability and Information Theory
Homework #5

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#1 $P(\{a,b,c\}) = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$
$P(\{a,b\}) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$
$P(\{b,c\}) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$
$P(\{a,c\}) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
$P(\{a\}) = \frac{1}{2}$
$P(\{b\}) = \frac{1}{3}$
$P(\{c\}) = \frac{1}{6}$
$P(\varnothing) = 0$

#7 By the sum rule:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= (1 - P(\bar{A})) + P(B) - P(A \cap B) \\ &= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} \\ &= \frac{11}{12} \end{aligned}$$

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#6 Since the absolute position around the table doesn't matter, let's just pick one person and arbitrarily call them 'first'. If there are n people in total, there are $n-1$ people who can sit to the right of the first, $n-2$ people who can set at that person's right, $n-3$ people who can set at that person's right, and so on. Once we're down to the very last person, they must sit to the penultimate person's right and the first person's left. So, the number of arrangements of people around a circular table is:

$$(n-1) \times (n-2) \times \cdots \times 1 = (n-1)!$$

#13 (a) $10^3 \times 26^3 = 17{,}576{,}000$
(b) $10^3 \times 26^3 \times \binom{6}{3} = 351{,}520{,}000$

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#4 (a) $P(\text{heart}|\text{red}) = \frac{P(\text{heart})}{P(\text{red})} = \frac{\frac{13}{52}}{\frac{26}{52}} = \frac{1}{2}$
(b) $P(\text{heart facecard}|\text{heart}) = \frac{P(\text{heart facecard})}{P(\text{heart})} = \frac{\frac{4}{52}}{\frac{13}{52}} = \frac{4}{13}$
(c) $P(\text{red jack}|\text{red}) = \frac{P(\text{red jack})}{P(\text{red})} = \frac{\frac{2}{52}}{\frac{26}{52}} = \frac{1}{13}$

#12 By Bayes' Theorem, we have:

$$\begin{aligned} P(\text{cancer}|+) &= \frac{P(+|\text{cancer}) \times P(\text{cancer})}{P(+)} \\ &= \frac{P(+|\text{cancer}) \times P(\text{cancer})}{P(+|\text{cancer}) \times P(\text{cancer}) + P(+|\text{not cancer}) \times P(\text{not cancer})} \end{aligned}$$

and we want to find the value of $P(\text{cancer})$ such that $P(\text{cancer}|+) = 0.5$. Filling in what we know, we get:

$$0.5 = \frac{0.99 \times P(\text{cancer})}{0.99 \times P(\text{cancer}) + 0.05 \times (1 - P(\text{cancer}))}$$

Solving for $P(\text{cancer})$, we get:

$$P(\text{cancer}) = 0.048$$

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#8 We want to know the probability that the first head will occur after the fifth toss, given that it has not occurred in the first two tosses. In other words, what's the probability of the third, fourth, and fifth tosses all being tails: $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

#16 If the chance of getting a call in one second is 0.01, then the probability of getting zero or one calls in a 5 minute period is:

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> pbinom(1,60*5,0.01)
[1] 0.1976497
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But suppose that we didn't have R. Since the cdf of the binomial distribution is pretty inconvenient to evaluation for $n = 300$, we could use the Poisson approximation instead. The parameter of the Poisson is λ, where:

$$\begin{aligned} \lambda &= n \times p \\ &= 60 \times 5 \times 0.01 \\ &= 3 \end{aligned}$$

So, we have:

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> ppois(1,3)
[1] 0.1991483
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which is pretty darn close to the exact[1] answer we get from the binomial.

[1]The result we get from the binomial distribution is only 'exact' to the extent that our model of incoming calls is correct. Since in reality the number of calls per second is almost certainly not a perfect Bernoulli trial, neither the binomial nor the Poisson model exactly reflects the way the world is.