Introduction to Probability and Information Theory
Homework #1

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#4 Suppose each outcome is a sequence of coin tosses, where 'H' is heads and 'T' is tails. Then:

(a) the first toss is heads

(b) all three tosses are the same

(c) exactly one toss comes up tails

(d) at least one toss comes up tails

#5 Suppose each outcome is equally likely. Then:

(a) $P(E) = \frac{4}{8} = 0.5$

(b) $P(E) = \frac{2}{8} = 0.25$

(c) $P(E) = \frac{3}{8} = 0.375$

(d) $P(E) = \frac{7}{8} = 0.875$

#6 Let $d(s)$ be the number of dots on the upper face of the die for outcome s. The probability of each outcome is proportional to the number of dots, so:

$$p(s) = c \times d(s)$$

for some constant c. We also know that:

$$\sum_{s \in \Omega} p(s) = 1$$

Putting these together, we get:

$$\begin{aligned} c + 2c + 3c + 4c + 5c + 6c &= 1 \\ 21c &= 1 \\ c &= \frac{1}{21} \end{aligned}$$

Then:

$$\begin{aligned} P(\text{even}) &= \frac{2}{21} + \frac{4}{21} + \frac{6}{21} \\ &= \frac{12}{21} \\ &\approx 0.571 \end{aligned}$$

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#3 Each bit can take on one of two values and there are 32 bits, so the total number of words is 2^{32} or 4,294,967,296.

#8 We can represent a subset of Ω as a sequence of **indicator variables** x_i where $x_i = 1$ if the ith element of Ω is in the set, and $x_i = 0$ if it isn't. For example, if $\Omega = \{a, b, c\}$, then the subset $\{a, c\}$ would correspond to the sequence $\langle 1, 0, 1 \rangle$. Now then problem becomes finding the number of possible sequences of n 1's and 0's. As we saw above, that's 2^n.

#16 Suppose everyone has exactly three initials. The number of possible different initials (assuming a 26 letter alphabet) is $26^3 = 17{,}576$. If we consider that some people don't have middle names, then the number of possible initials is:

$$26^3 + 26^2 = 18{,}252$$

And, adding people with four initials ('George H. W. Bush'), we get:

$$26^4 + 26^3 + 26^2 = 475{,}228$$

Since there are more than 475,228 people in Philadelphia, at least two of them must share the same initials (this is sometimes called the **pigeonhole principle**: if you're putting things into bins and there are more things than there are bins, some bin is going to have to get more than one thing).

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#1 By Bayes' Theorem:

$$P(F|E) = \frac{P(E|F) \times P(F)}{P(E)}$$

Since $P(E|F) = P(E)$, we have:

$$\begin{aligned} P(F|E) &= \frac{P(E) \times P(F)}{P(E)} \\ &= P(F) \end{aligned}$$

#2 [UPDATE: The answers I had here originally were wrong. Sorry!] The best way of answering questions like these is to spell out all the possibilities. If we flip a coin three times, the possible outcomes are:

$$\Omega = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

So:

(a) there are four outcomes where the first flip comes up heads, and in two of those there are exactly two heads, so $p = \frac{1}{2}$

(b) $\frac{1}{4}$

(c) $\frac{1}{2}$

(d) 0

(e) $\frac{1}{2}$

#9 Suppose having children is like tossing a coin: for each child there's an equal chance that it will be a boy or a girl (which turns out to not be quite true...). Then for a family with two children, there are four equally likely outcomes:

$$\Omega = \{\text{GG}, \text{GB}, \text{BG}, \text{BB}\}$$

Then:

(a) There are three outcomes with at least one boy and one of those has exactly two boys, so $p = \frac{1}{3}$.

(a) There are two outcomes in which the first child is a boy and one of those has exactly two boys, so $p = \frac{1}{2}$.

#18 By Bayes' Theorem, the probability that the patient has d_i given a positive test result is:

$$P(d_i|+) = \frac{P(+|d_i) \times P(d_i)}{P(+)}$$

The doctor's prior assumption is that the patient has one of these diseases and that all three diseases are equally likely:

$$P(d_1) = P(d_2) = P(d_3) = \frac{1}{3}$$

And, we know the conditional probability of a positive test result given each of the diseases:

$$\begin{aligned} P(+|d_1) &= 0.8 \\ P(+|d_2) &= 0.6 \\ P(+|d_3) &= 0.4 \end{aligned}$$

That gives us the numerator, but what about the denominator? We know that there are only three ways to get a positive test, namely by having d_1, d_2, or d_3 (since we're assuming the patient has one of these diseases, there's no question of false positives). By the partition rule:

$$\begin{aligned} P(+) &= P(+|d_1)P(d_1) + P(+|d_2)P(d_2) + P(+|d_3)P(d_3) \\ &= 0.8 \times \frac{1}{3} + 0.6 \times \frac{1}{3} + 0.4 \times \frac{1}{3} \\ &= 0.6 \end{aligned}$$

Putting these values into Bayes' Theorem gives us:

$$
\begin{aligned}
P(d_1|+) &= 0.44 \\
P(d_2|+) &= 0.33 \\
P(d_3|+) &= 0.22
\end{aligned}
$$