

Aggregate Uncertainty Can Lead to Incorrect Herds

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Abstract

A continuum of homogeneous rational agents choose between two competing technologies. Agents observe a private signal and sample others' previous choices. Signals have an aggregate component of uncertainty, so aggregate behavior does not necessarily reflect the true state of nature. Nonetheless, agents still find others' choices informative, and base their decisions partly on others' behavior. Consequently, bad choices can be perpetuated: aggregate uncertainty makes agents herd on the inferior technology with positive probability. I derive the optimal decision rule when agents sample exactly two individuals. I also present examples with herds on the inferior technology for arbitrarily large sample sizes.

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1. Introduction

In many circumstances, individuals have heterogeneous private information about the environment they face. Consequently, as long as there is some correlation in players' valuations, others' behavior provides useful information. In these situations, rational agents base their decisions at least partially on the decisions of others. This kind of behavior is known as observational learning.

Observational learning is present in many economic situations. When a financial institution must decide whether or not to lend money to a firm, it can supplement its own analysis by seeing how other financial institutions behave. Similarly, when deciding where to have dinner, consumers often take into account other consumers' decisions. Similar stories apply to exploration for natural resources and research and development decisions.

When two competing new technologies are introduced in a market, will private information aggregate and lead agents to pick the superior one? In this paper I focus on a negative side of observational learning. I say that *incorrect herds* occur when the inferior technology is chosen in the long run with positive probability.

The seminal contributions to the observational learning literature are Banerjee [1992] and Bikhchandani, Hirshleifer, and Welch [1992]. In these papers, a set of rational agents choose sequentially between two technologies. The payoff from this decision is common but unknown to all. In each period, one agent receives a signal and observes what all agents before him have chosen. Next, this agent makes a once-and-for-all decision between the two technologies. Given that each agent knows that the signal he has received is no better than the signal other players have received, agents follow the behavior of others. As a result, incorrect herds occur.

In Banerjee and Fudenberg [2004] a continuum of agents choose between two competing technologies. In each period, a fraction of agents is replaced. The newcomers make once-and-for-all decisions after observing a finite sample of previous-period agents' choices and receiving an informative signal. Banerjee and Fuden-

berg [2004] show that if the signal is sufficiently informative and agents sample more than one agent's choice, then agents choose the superior technology in the long run. Consequently, in contrast to the models discussed above, incorrect herds do not occur in Banerjee and Fudenberg [2004].

In this paper, I incorporate aggregate uncertainty into an observational learning model and show how this alters some of the model's fundamental results. I argue that aggregate uncertainty is a realistic feature, and explain how it can lead to incorrect herds. Say two new competing software packages are released: one from a well established firm and the other from a new company. The quality of the software from the well established firm is well-known. The software from the new company is of higher quality, but consumers do not know this. What would happen if the first version of the new company's software contained a bug that seemed problematic but could be fixed with a simple update? In that case, a high fraction of the first potential buyers receive low-quality signals for this software. In other words, aggregate uncertainty implies that the realized distribution of agents' signals does not reflect the true quality of the new technology. This causes a high fraction of early buyers to choose the technology from the established company. Later potential buyers observe this. Consequently, even if the bug is rapidly fixed, word-of-mouth spreads that the software is not very good. As a result, individuals tend to buy the lower-quality software instead of the higher-quality software from the new company. Therefore, bad shocks in early periods can be perpetuated, leading to incorrect herds.

I present a model of technology adoption with a setup similar to that in Banerjee and Fudenberg [2004]. A continuum of agents learn about technologies' qualities from idiosyncratic signals and a sample of others' behavior. Unlike in Banerjee and Fudenberg [2004], an i.i.d. aggregate shock affects the precision of signals in each period. This randomness generates aggregate uncertainty and implies that aggregate behavior is not a deterministic function of the true state of the world. I show next how this can lead to incorrect herds, in contrast to Banerjee and Fudenberg

[2004].

Proposition 1 derives the optimal decision rule when each individual observes the behavior of exactly two agents from the previous period. I show that if both sampled agents chose the *same* technology, the individual follows the sample, disregarding his own signal. If instead sampled agents chose *different* technologies, the individual follows his own private signal. Proposition 1 provides a closed-form solution for the evolution of the fraction of adopters of the superior technology over time. In Proposition 2 I show how incorrect herds can arise due to aggregate uncertainty. I present necessary and sufficient conditions for incorrect herds to occur when exactly two agents are observed.

When a sample of more than two agents is observed the decision rule may change over time: agents facing the same sample and signal may react differently in distinct periods.¹ Furthermore, optimal decision rules may exhibit other counterintuitive traits: observing *more* agents choosing a given technology may make an agent choose the *other* technology. I present examples that illustrate these counterintuitive traits. I do not provide a full characterization of agents' optimal behavior when more than two agents are observed. I show in Propositions 3 and 4 simple environments in which incorrect herds occur for all odd sample sizes greater than two.

1.1 Related Literature

There is by now a large literature on social learning. Several papers study how different kinds of (aggregate) uncertainty about the environment can disrupt information aggregation. Smith and Sørensen [2000] show that with heterogeneous preferences, there is uncertainty about the informational content of the observed history, and so learning may fail.

Other papers stress that the observability of past actions is not perfect. In Smith and Sørensen [2008] agents do not know the positions of the individuals sampled.

¹When the sample size is either one or two, agents' decisions are independent of calendar time.

In spite of that, complete learning occurs. Monzón and Rapp [2014] present conditions for information aggregation when agents may be uncertain both about their own position in the sequence and about the positions of those they observe. In Callander and Hörner [2009] each agent is aware of his own position in the sequence, observes the aggregate number of adopters, but does not know the order of the decisions.² Callander and Hörner show the counterintuitive result that it is sometimes optimal to follow the decision of a minority.

The “wisdom of the minority” result in Callander and Hörner [2009] occurs when there are some agents who are better informed than others. Example 1 in Section 3.3 shows a case in which this happens too in my model, although in my setup there are no types with more precise signals. Galeotti and Goyal [2010] present a related result in a model of endogenous information acquisition and network formation. They show that in equilibrium a small number of agents acquire information while the rest get connected to them and thus learn from them. Finally, Lobel and Sadler [2015] introduce correlation on the set of sampled agents. They show that (aggregate) uncertainty on the network topology can prevent learning.

The rest of the paper is organized as follows. I present the model in Section 2. I solve the model when exactly one agent is observed in Section 3.1. I characterize the optimal decision rule when the sample size is 2 and show that incorrect herds occur in Section 3.2. In Section 3.3 I discuss the difficulties of providing a full characterization for higher sample sizes and show that incorrect herds can occur. Section 4 concludes.

2. The Model

There is a unit mass continuum of agents. Each agent lives for one period. Once they die, they are replaced by another continuum of agents of mass one. Time

²In related papers, Herrera and Hörner [2013] and Guarino, Harmgart, and Huck [2011] focus on the interesting case where only one decision (to invest) is observable whereas the other one (not to invest) is not.

is discrete, and periods are indexed by $t = 1, 2, \dots$. During each period, each agent chooses one of two different technologies: a and b . There are two states of the world: $\theta \in \Theta = \{A, B\}$. I assume that agents obtain a payoff of 1 when the action matches the state of the world, and 0 otherwise. Moreover, I assume that $\Pr(\theta = A) = \Pr(\theta = B) = 1/2$.

The timing of this game is as follows. First, nature chooses $\theta \in \Theta$, which does not change over time. Afterwards, period-one agents are born. The true state of the world is not revealed directly to agents. Instead, each agent receives a noisy signal about the true state of the world. This signal has both an idiosyncratic and an aggregate component of uncertainty. Next, each agent chooses between a and b without observing other agents' decisions. Payoffs are collected and agents die. In this way, each agent's decision is once and for all. Then, period $t = 2$ starts. This period is different from period one since now there are past decisions available to observe. A new group of agents is born. Each of these period-two agents observes a signal s and an independent random sample of the decisions of period-one agents. As I describe later, the sample is characterized by ζ , the number of individuals in the sample choosing technology a , and N , the sample size. With the signal and sample, each agent decides between a and b , collects payoffs and dies. The timing of each subsequent period t is identical to that of period $t = 2$: agents observe a signal s and a sample of the decisions from the previous period $t - 1$.

Agents receive noisy information about the true state of the world via a signal $s \in \{\alpha, \beta\}$, distributed $\Pr(\alpha | A) = \Pr(\beta | B) = q_t$. In each period some agents receive signal α while others receive signal β . Therein lies the *idiosyncratic* nature of the signal. To introduce *aggregate uncertainty*, I let the fraction q_t itself be random. The aggregate shock q_t may be good or bad: $q_t \in Q = \{l, h\}$ with $0 < l < 1/2 < h < 1$.³ The draws of q_t are i.i.d. across periods with $p \equiv \Pr(q_t = h)$. The realization of the sequence $\mathbf{q}^t = (q_1, q_2, \dots, q_t)$ is not observed by agents. I assume the signal is informative on average: $E[q] = ph + (1 - p)l > 1/2$. Signals are i.i.d. across

³In the present model there would be no incorrect herds with $l \geq 1/2$.

individuals conditional on the true state of the world and on the realization of q_t .

Period-one agents have no information other than the signal, so their beliefs about the state are summarized by $\Pr(A | \alpha) = \Pr(B | \beta) = E[q]$. It is useful to define the signal likelihood ratio $\psi(s)$ by

$$\psi(s) \equiv \frac{\Pr(s | A)}{\Pr(s | B)} = \begin{cases} E[q] / (1 - E[q]) & \text{if } s = \alpha \\ (1 - E[q]) / E[q] & \text{if } s = \beta. \end{cases}$$

Let x denote the fraction of agents choosing the superior technology. For example, if A is the true state of the world, x is the fraction of agents choosing a . At the beginning of his lifetime, each individual observes a sample of N individuals' choices in the previous period. I define ζ to be the number of those individuals that chose a . Sampling is assumed to be proportional. As a result, the likelihood of each sample is given by

$$\Pr(\zeta | x, \theta) = \begin{cases} \binom{N}{\zeta} x^\zeta (1 - x)^{N - \zeta} & \text{if } \theta = A \\ \binom{N}{\zeta} (1 - x)^\zeta x^{N - \zeta} & \text{if } \theta = B. \end{cases}$$

For notational simplicity, I define $\Pr(\zeta | x) \equiv \Pr(\zeta | x, A)$. This implies that $\Pr(\zeta | x, B) = \Pr(\zeta | 1 - x)$, which I use later.

Since there is a continuum of agents, the fraction of individuals observing sample ζ is exactly $\Pr(\zeta | x, \theta)$. In the same way, the fraction of individuals receiving signal s is exactly $\Pr(s | \theta)$.⁴ In every period t , the fraction of agents choosing the superior technology $x_t : Q^t \rightarrow [0, 1]$ is a deterministic function of the state vector $\mathbf{q}^t = (q_1, q_2, \dots, q_t)$.

Agents base their decisions on the information provided by the sample ζ and the signal s . An agent born in period $t + 1$ takes into account the likelihood ratio

⁴This involves the standard abuse of the law of large numbers. See Judd [1985] for a discussion.

$\Pr(A | \zeta, s) / \Pr(B | \zeta, s)$, which is a product of the two sources of information:

$$\frac{\Pr(A | \zeta, s)}{\Pr(B | \zeta, s)} = \psi(s) \frac{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t) \Pr[\zeta | x_t(\mathbf{q}^t)]}{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t) \Pr[\zeta | 1 - x_t(\mathbf{q}^t)]}$$

Agents choose technology a whenever $\Pr(A | \zeta, s) > \Pr(B | \zeta, s)$. Then, their decision is given by,

$$D_{t+1}(\zeta, s) = \begin{cases} 1 & \text{if } \frac{\Pr(A|\zeta,s)}{\Pr(B|\zeta,s)} > 1 \\ \sigma(\zeta, s) & \text{if } \frac{\Pr(A|\zeta,s)}{\Pr(B|\zeta,s)} = 1 \\ 0 & \text{if } \frac{\Pr(A|\zeta,s)}{\Pr(B|\zeta,s)} < 1 \end{cases}$$

If the agent is indifferent, he randomizes: technology a is chosen with probability $\sigma(\zeta, s)$.⁵

The model is symmetric with respect to the identity of the superior technology. From now on, let $\theta = A$. In that case, in period $t + 1$, a fraction $\Pr[\zeta | x_t(\mathbf{q}^t)]$ observes sample ζ . Of those, a fraction q_{t+1} receives signal α and a fraction $(1 - q_{t+1})$ receives signal β . Consequently the fraction of agents choosing the superior technology in $t + 1$ is given by

$$x_{t+1}(\mathbf{q}^{t+1}) = \sum_{\zeta=0}^N \Pr[\zeta | x_t(\mathbf{q}^t)] \left[q_{t+1} D_{t+1}(\zeta, \alpha) + (1 - q_{t+1}) D_{t+1}(\zeta, \beta) \right]. \quad (1)$$

I focus on the long run behavior of (1). In particular, could it happen that agents end up choosing the inferior technology? I say *incorrect herds occur* if $\Pr\{x_t \rightarrow 0 | A\} > 0$. In Banerjee and Fudenberg [2004], q_t is deterministic and incorrect herds do not occur. In what follows, I show how the randomness of q_t may lead to incorrect herds.

⁵I assume that the tie-breaking rule is symmetric: $\sigma(N - \zeta, \beta) = 1 - \sigma(\zeta, \alpha)$, so that the decision function is symmetric for all pairs (ζ, s) . See Lemma 1 for details.

3. Solution

I provide a closed-form solution for the evolution of $x_t(\mathbf{q}^t)$ when N is 1 or 2. When $N = 1$, agents do not benefit from the observation of others; the sample is as informative as the signal. Complete learning does not arise. When $N = 2$, a sample of two agents choosing the same technology (either $\zeta = 0$ or $\zeta = 2$) is more informative than the signal. A sample $\zeta = 1$ provides no information about the state of the world. Thus agents follow their signal when $\zeta = 1$ and follow the sample when $\zeta = 0$ or $\zeta = 2$. I give a simple, closed-form characterization of the evolution of $x_t(\mathbf{q}^t)$. This characterization illustrates the dynamics that lead to incorrect herds. I present necessary and sufficient conditions for incorrect herds to occur when $N = 2$.

Whenever $N > 2$ agents' decisions are hard to derive. The decision function may change over time and may be not monotonic in ζ for some values of l, h and p . I do not provide a general solution for the evolution of $x_t(\mathbf{q}^t)$ if $N \geq 3$. Nevertheless, I show that incorrect herds can occur for any odd $N \geq 3$.

Before studying agents' behavior for specific sample sizes, I derive some properties of the decision rule that hold for any sample size. These properties simplify the study of agents' decisions. Period-one agents only observe a signal, since there are no agents from previous periods. Since the signal is informative ($E[q] > 1/2$), agents follow it. I also show that the symmetry of the model leads to symmetry in agents' decisions.⁶

LEMMA 1. *The decision function must satisfy:*

(a) $D_{t+1}(N - \zeta, \beta) = 1 - D_{t+1}(\zeta, \alpha)$

(b) *If N is even, $D_{t+1}(N/2, \alpha) = 1$ and $D_{t+1}(N/2, \beta) = 0$*

(c) $E[x_{t+1}(\mathbf{q}^{t+1})] \geq E[x_t(\mathbf{q}^t)]$

(d) $E[x_1(\mathbf{q}^1)] = E[q]$ and $E[x_t(\mathbf{q}^t)] \geq E[q]$

⁶The assumption on the symmetry of the tie-breaking rule guarantees that part (a) of Lemma 1 also holds when the agent is indifferent.

See Appendix A1 for the proof.

3.1 The Case $N = 1$

Lemma 2 describes agents' optimal behavior when $N = 1$. If the signal contradicts the sample, agents are indifferent between technologies.

LEMMA 2. *Let $N = 1$. For $t > 1$,*

$$D(\zeta, s) = \begin{cases} 1 & \text{if } (\zeta, s) = (1, \alpha) \\ \sigma(1, \beta) & \text{if } (\zeta, s) = (1, \beta) \\ 1 - \sigma(1, \beta) & \text{if } (\zeta, s) = (0, \alpha) \\ 0 & \text{if } (\zeta, s) = (0, \beta) \end{cases}$$

and $E[x_t(\mathbf{q}^t)] = E[q] \forall t$.

See Appendix A2 for the proof.

In this case, the informational content of the sample is always exactly as strong as the informational content of the signal. As a result, there is no complete learning. Next, I show that when $N = 2$ some samples are more informative than the signal.

3.2 The Case $N = 2$

In this case, there are six possible combinations of sample and signal (ζ, s) . Lemma 1(b) shows that a sample $\zeta = 1$ is uninformative. After observing a mixed sample (one that has the same number of people choosing a and b) agents base their decision solely on the signal. As a result, only samples with all agents choosing the same technology remain to be studied. What happens if the signal and sample provide contradictory information? Proposition 1 shows that a sample of two agents choosing the same technology is strictly more informative than the signal. The intuition behind this result is as follows. First, if only *one* agent was observed, his action would be at least as informative as the signal - see Lemma 1(d). Second, a sample

of *two* agents choosing the same technology provides (strictly) stronger information than a sample of only one agent. These two facts together guarantee that a full sample prevails over the signal in the determination of agents' decisions.

PROPOSITION 1. *Let $N = 2$. For $t > 1$,*

(a) *The decision function is given by:*

$$D(\zeta, s) = \begin{cases} 1 & \text{if } (\zeta, s) \in \{(2, \alpha), (2, \beta), (1, \alpha)\} \\ 0 & \text{if } (\zeta, s) \in \{(1, \beta), (0, \alpha), (0, \beta)\} \end{cases}$$

(b) *The evolution of the system is described by:*

$$x_{t+1}(\mathbf{q}^{t+1}) = 2q_{t+1}x_t(\mathbf{q}^t) + (1 - 2q_{t+1}) [x_t(\mathbf{q}^t)]^2 \quad (2)$$

(c) *Let $x_t(\mathbf{q}^t) \notin \{0, 1\}$. When $q_t = h$, then $x_{t+1}(\mathbf{q}^{t+1}) > x_t(\mathbf{q}^t)$, but when $q_t = l$, then $x_{t+1}(\mathbf{q}^{t+1}) < x_t(\mathbf{q}^t)$.*

That is, positive aggregate shocks increase the fraction of agents choosing the superior technology while negative aggregate shocks have the opposite effect.

See Appendix A3 for the proof.

Proposition 1 provides a closed-form solution for the evolution of the system. Next, I show that incorrect herds occur with positive probability. To do this, I utilize the following lemma from Ellison and Fudenberg [1995]:

LEMMA 3. LEMMA 1 IN ELLISON AND FUDENBERG [1995]. *Let x_t be a Markov process on $(0, 1)$ with:*

$$x_{t+1} = \begin{cases} H_1(x_t) & \text{with probability } p \\ H_2(x_t) & \text{with probability } 1 - p \end{cases}$$

Suppose that $H_i(x_t) = \gamma_i x_t + o(x_t)$, with $\gamma_2 < 1 < \gamma_1$.

(a) *If $E[\log(\gamma_i)] > 0$ then x_t cannot converge to 0 with positive probability.*

(b) If $E[\log(\gamma_i)] < 0$ then $\Pr\{x_t \rightarrow 0 \mid x_0 \leq \delta\} \geq \epsilon$ for some strictly positive δ and ϵ .

To see why this lemma holds, consider the simpler case of a linear Markov process $\tilde{x}_{t+1} = \gamma_t \tilde{x}_t$ (where the random variable $\gamma_t = \gamma_1$ with probability p and $\gamma_t = \gamma_2$ with probability $1-p$). To analyze the long run behavior of \tilde{x}_t , define the associated log-process $\log(\tilde{x}_{t+1}) = \log(\tilde{x}_t) + \log(\gamma_t)$ so that $\log(\tilde{x}_t) = \log(\tilde{x}_0) + \sum_{\tau=0}^{t-1} \log(\gamma_\tau)$. By the strong law of large numbers, $\log(\tilde{x}_t) \rightarrow -\infty$ with probability 1 if $E[\log(\gamma_t)] = p \log(\gamma_1) + (1-p) \log(\gamma_2) < 0$, which implies that $\tilde{x}_t \rightarrow 0$ with probability 1. Non-linear processes x_t that are approximately linear when $x \approx 0$ converge to 0 with positive probability if x_0 is sufficiently close to 0 and $E[\log(\gamma_t)] < 0$. With this lemma, I can state my condition for incorrect herds.

PROPOSITION 2. *Incorrect herds occur whenever $p \log(h) + (1-p) \log(l) < \log(1/2)$. If instead $p \log(h) + (1-p) \log(l) > \log(1/2)$, then incorrect herds never occur.*

Proof. I rewrite (2) as follows:

$$x_{t+1} = \begin{cases} H_1(x_t) = 2hx_t + (1-2h)x_t^2 & \text{with probability } p \\ H_2(x_t) = 2lx_t + (1-2l)x_t^2 & \text{with probability } 1-p \end{cases}$$

so that $\gamma_2 = 2l$ and $\gamma_1 = 2h$. Note that $(1-2q)x_t^2 = o(x_t)$. First, assume $p \log(h) + (1-p) \log(l) < \log(1/2)$. By Lemma 3, $\Pr\{x_t \rightarrow 0 \mid x_0 \leq \delta\} \geq \epsilon$ for some positive ϵ and δ . Note that x_t can get below any $x \in (0, 1)$ in finite time with positive probability from any $x_1 \neq \{0, 1\}$. Then, no matter the starting point, $\Pr\{x_t \rightarrow 0\} > 0$. Next, assume $p \log(h) + (1-p) \log(l) > \log(1/2)$. Again, by Lemma 3, $\Pr\{x_t \rightarrow 0\} = 0$. ■

To understand Proposition 2, suppose that almost everyone chooses technology b (i.e., $x \approx 0$), so that $\zeta = 0$ is the most likely sample. Agents who observe it choose b , so they move with the herd. Next, note that the likelihood of $\zeta = 2$ relative to $\zeta = 1$ approaches 0 as x gets close to 0. Thus, I can disregard agents who observe $\zeta = 2$ and focus only on those who see $\zeta = 1$. Agents get $\zeta = 1$ with a probability of approximately $2x_t$, and in this event choose a only after observing signal α . As

a result, $x_{t+1}(\mathbf{q}^{t+1}) \approx 2q_{t+1}x_t(\mathbf{q}^t)$. As discussed above, it is $\log(l)$ and $\log(h)$ that determine the behavior of x_t . When $q_t = h$, x_t can at most double (if h is close to 1). But when $q_t = l$, a low l reduces x_t by a larger proportion (for example, if $l = 0.1$, x_t is divided by approximately 5 when $q_t = l$). In other words, a high h has a weaker effect than a low l .

An alternative way to see this herding condition is to write $\log [x_{t+1}(\mathbf{q}^{t+1})] \approx \log(2q_{t+1}) + \log [x_t(\mathbf{q}^t)]$. The long run value of $\log(x_t)$ depends on $E[\log(2q)]$. A low enough value for l makes $E[\log(2q)] < 0$; in this case $\log(x_t) \rightarrow -\infty$ with positive probability, which implies $x_t \rightarrow 0$.

The following straightforward corollary shows incorrect herds may occur for any *average* precision for the signal.

COROLLARY 1. For any $E[q] < 1$, there exist parameters p, h, l such that incorrect herds occur.

Proof. Fix some value $E[q] < 1$ and let $h = 1$ so that p and l are related by $p = (E[q] - l)/(1 - l)$. Incorrect herds occur with positive probability if $(E[q] - l) \log(2) + (1 - E[q]) \log(2l) < 0$. As $l \rightarrow 0$, the first term converges to $E[q] \log(2)$ whereas the second converges to $-\infty$. Consequently, there always exist p, h, l such that incorrect herds occur. ■

3.3 The Case $N \geq 3$

When $N \leq 2$ the optimal decision rule is independent of calendar time. However, if each agent observes more than 2 individuals, the optimal decision rule may change over time; furthermore, it may exhibit counterintuitive traits. I do not provide a general characterization of agents' optimal behavior if $N \geq 3$. In what follows, I present two examples to illustrate the difficulties of obtaining a general characterization of the decision rule and then describe a symmetric environment in which incorrect herds occur for sample sizes greater than two.

EXAMPLE 1. *A sample with more people choosing a given technology can make the agent less inclined to choose that same technology ($N = 3, l = 0.1, h = 0.95$ and $p = 1/2$).*

Agents follow their signal in the first period. In period $t = 2$, the optimal decision rule for Example 1 is given by:

$$D_2(\zeta, s) = \begin{cases} 1 & \text{if } \zeta \in \{1, 3\} \\ 0 & \text{if } \zeta \in \{0, 2\} \end{cases}$$

In period $t = 2$, observing a sample with only *one* person choosing a leads the agent to choose a , whereas observing *two* people choosing a leads the agent to choose b .⁷

To understand this behavior, note first that when $N = 2$, $\Pr(\zeta = 2 \mid x)$ is strictly increasing in x . Now, when $N \geq 3$, $\Pr(\zeta = 2 \mid x)$ is no longer monotonic in x . Indeed, for $N = 3$, $\Pr(\zeta = 2 \mid x = 0) = 0$; then the probability increases until it attains a maximum at $x = 2/3$ and then decreases until $\Pr(\zeta = 2 \mid x = 1) = 0$. In this way, when $N = 3$, a higher fraction in the population choosing a may actually make observing $\zeta = 2$ less likely.

The reason for the counterintuitive behavior in period $t = 2$ can now be explained as follows. On average there is a higher fraction choosing a when A is the true state of the world than there is when B is the true state of the world. This may cause the agent to choose b after observing $\zeta = 2$. This is the case in Example 1. When a is the superior technology, the fraction of the population choosing a is either 0.95 or 0.10, with equal probability. Symmetrically, if b is the superior technology, the fraction of agents choosing a is either 0.05 or 0.90. The agent observes $\zeta = 2$ and infers the fraction of people choosing a . The likelihood of being in a state in which x is 0.05 or 0.10 is practically negligible. A state with $x = 0.90$ is almost twice as likely as one with $x = 0.95$, which makes state of the world B significantly more likely than A . This leads agents to choose b after observing $\zeta = 2$ in

⁷In this example, the lack of monotonicity in ζ occurs in period $t = 2$. In the periods that follow, the decision function varies.

the second period.⁸ Finally, by Lemma 1(a), agents choose a after observing $\zeta = 1$.

EXAMPLE 2. *The optimal decision rule $D_t(\zeta, s)$ may reverse over time for the same (ζ, s) ($N = 3, l = 0.4, h = 0.8$ and $p = 1/2$).*

Table 1: Decision Function for Example 2

(ζ, s)	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$
$(3, s)$ or $(2, \alpha)$	1	1	1	1	1	1
$(2, \beta)$	0	1	0	1	0	1
$(1, \alpha)$	1	0	1	0	1	0
$(1, \beta)$ or $(0, s)$	0	0	0	0	0	0

Table 1 presents the optimal decision rule for Example 2. I numerically calculate the optimal decision rule for the first 7 periods.⁹ In this case, the decision rule changes over time. A sample $\zeta = 2$ is more informative than the signal for periods 3, 5 and 7. For periods 2, 4 and 6, the opposite holds. As time goes by, the unconditional expectation $E[x_t(\mathbf{q}^t)]$ increases. However, this increase does not necessarily make the agent more willing to follow a sample $\zeta = 2$. The lack of monotonicity of $\Pr(\zeta = 2 | x)$ in x is once again the reason the decision rule fluctuates.

Examples 1 and 2 illustrate that for general sample sizes the decision rule is hard to obtain.¹⁰ In spite of this, I present in Propositions 3 and 4 simple environments where the decision rule can be characterized and incorrect herds occur.

PROPOSITION 3. *Let $N \geq 3$ be odd, $h > 1 - 1/N$ and $l < (1 - h)h$. Then there exists $\bar{p} < 1$ such that for all $p \geq \bar{p}$,*

- (a) *In period $t = 1$, agents follow the signal. In all subsequent periods, agents follow the most observed choice in their sample: $D(\zeta, s) = 1$ if $\zeta \geq (N + 1)/2$ and $D(\zeta, s) = 0$*

⁸This also explains why the signal does not affect the decision $D_2(\zeta, s)$ in period 2: while $\Pr(A | 2) / \Pr(B | 2) \approx 0.65$, the information provided by the signal is weaker, $E[q]/(1 - E[q]) \approx 1.10$.

⁹In each period, there are 2^t possible values for $x_t(\mathbf{q}^t)$. As a result, numerically calculating the optimal decision rule is computationally demanding, even for low values of t . In numerical calculations, I computed the 20 first periods. The code can be downloaded at <https://sites.google.com/site/imonzon/research/#herds>.

¹⁰Interestingly, these difficulties disappear when there is no aggregate uncertainty. With $l = h > 1/2$ the decision function satisfies monotonicity in ζ , which makes the evolution of the system tractable.

if $\zeta \leq (N - 1)/2$.

(b) *Incorrect herds occur when $q_1 = l$, and so have probability $1 - p$.*

See Appendix A4 for the proof.

Proposition 3 presents an environment where agents find it optimal to disregard their signals after period $t = 1$. In the first period, the level of aggregate uncertainty determines the fraction choosing the superior technology. Starting in period $t = 2$, agents follow the most observed choice in their sample, so in each period there are only two possible values for x_t : $x_t(h)$ and $x_t(l)$, determined by the level of aggregate uncertainty in the first period. If $q_1 = h$, more than half of the agents in the population choose the superior technology in the first period. In subsequent periods, agents follow the most observed choice in their sample, so the fraction choosing the superior technology increases over time and converges to 1. On the other side, if $q_1 = l$, the fraction choosing the inferior technology increases and converges to 1. Since more than half of the first-period agents choose the inferior technology and subsequent agents follow the most observed choice in their sample, the effect of a bad shock in the first period “snowballs”, creating an incorrect herd. Incorrect herds occur with probability $1 - p$, which is the probability that $q_1 = l$.

Agents disregard their private signal when the informational content of the sample outweighs that of the signal. When a is chosen by most but not all in the sample ($(N + 1)/2 \leq \zeta \leq N - 1$), the lack of monotonicity of $\Pr(\zeta | x)$ makes the analysis hard, as shown in Example 1.¹¹ The information given by the sample is captured by the likelihood ratio

$$\frac{\Pr(A | \zeta)}{\Pr(B | \zeta)} = \frac{p \Pr[\zeta | x_t(h)] + (1 - p) \Pr[\zeta | x_t(l)]}{p \Pr[\zeta | 1 - x_t(h)] + (1 - p) \Pr[\zeta | 1 - x_t(l)]}. \quad (3)$$

In spite of the lack of monotonicity, in the environment of Proposition 3 the information contained in any sample in which a is observed more often than b out-

¹¹The discussion that follows focuses on $\zeta \neq N$. When $\zeta = N$, $\Pr(\zeta | x)$ is indeed monotonic in x . I cover the case $\zeta = N$ in Appendix A4.

weighs the information contained in the signal $s = \beta$. To see why this is true, disregard the term $(1 - p) \Pr[\zeta | x_t(l)]$ in the numerator of (3), which only makes (3) larger:

$$\frac{\Pr(A | \zeta)}{\Pr(B | \zeta)} > \frac{p}{p \frac{\Pr[\zeta | 1 - x_t(h)]}{\Pr[\zeta | x_t(h)]} + (1 - p) \frac{\Pr[\zeta | 1 - x_t(l)]}{\Pr[\zeta | x_t(h)]}} \quad (4)$$

Next, focus on the term $\Pr[\zeta | 1 - x_t(l)] / \Pr[\zeta | x_t(h)]$ in the denominator of (4). I show that when $x > 1 - 1/N$, $\Pr(\zeta | x)$ is strictly decreasing in x , for any $(N + 1)/2 \leq \zeta \leq N - 1$. Moreover, the assumption $1 - l > h > 1 - 1/N$ guarantees that $1 - x_t(l) > x_t(h) > 1 - 1/N$.¹² These two facts together imply that $\Pr[\zeta | 1 - x_t(l)] / \Pr[\zeta | x_t(h)] < 1$. Finally, it is easy to show that

$$\frac{\Pr[\zeta | 1 - x_t(h)]}{\Pr[\zeta | x_t(h)]} = \left(\frac{1 - x_t(h)}{x_t(h)} \right)^{2\zeta - N} < \frac{1 - x_t(h)}{x_t(h)} < \frac{1 - h}{h}.$$

As a result, I obtain the following lower bound for the likelihood ratio in (3):

$$\frac{\Pr(A | \zeta)}{\Pr(B | \zeta)} > \frac{p}{p \frac{1-h}{h} + (1 - p)}$$

In Appendix A4, I show that for a large enough p , the information provided in the sample outweighs the information provided by the signal $s = \beta$, and therefore that agents choose technology a when it is the most observed choice, regardless of the signal. By symmetry, agents choose b after observing a sample where b is the most observed choice. In this way, I conclude that the signal plays no role starting in period $t = 2$.

Proposition 3 relies on high values for h and $1 - l$ to show how incorrect herds can occur for sample sizes $N \geq 3$. The next proposition instead highlights that incorrect herds can occur for moderate values of q .

PROPOSITION 4. *Let $N \geq 3$ be odd and $h = 1 - l$. Then, incorrect herds occur when $q_1 = l$, and so have probability $1 - p$.*

¹² $x_t(h)$ increases over time and $x_1(h) = h > 1 - 1/N$. Moreover, $1 - l = 1 - x_1(l) > x_1(h) = h$. Since x_{t+1} depends only on x_t , then it is true that in every period $1 - x_t(l) > x_t(h)$.

Proposition 4 introduces symmetry to the aggregate shock q_t . The fraction h receiving the *good* signal when the aggregate shock is *good* equals the fraction $1 - l$ receiving the *bad* signal when the aggregate shock is *bad*. In this environment agents also disregard their signals after period $t = 1$, as I show next. As before, in each period there are only two possible values for x_t : $x_t(h)$ and $x_t(l) = 1 - x_t(h)$, determined by the level of aggregate uncertainty in the first period. Then, the expression for the information provided by a sample can be simplified as follows:

$$\frac{\Pr(A | \zeta)}{\Pr(B | \zeta)} = \frac{1 + \frac{p}{1-p} \left(\frac{x_t(h)}{1-x_t(h)} \right)^{2\zeta-N}}{\frac{p}{1-p} + \left(\frac{x_t(h)}{1-x_t(h)} \right)^{2\zeta-N}} \quad (5)$$

The likelihood ratio $\Pr(A | \zeta) / \Pr(B | \zeta)$ in equation (5) strictly increases in ζ and in $x_t(h)$. When $x_t(h) = h$, then

$$\frac{\Pr(A | \frac{N+1}{2}, \beta)}{\Pr(B | \frac{N+1}{2}, \beta)} = \frac{\Pr(A | \frac{N+1}{2})}{\Pr(B | \frac{N+1}{2})} \frac{1 - E[q]}{E[q]} = 1.$$

As $x_t(h)$ grows, $\Pr(A | \zeta, \beta) / \Pr(B | \zeta, \beta) > 1$ for all $\zeta \geq (N + 1)/2$, so agents follow the most observed choice in their sample.¹³ As in Proposition 3, the effect of a bad shock in the first period has long-lasting consequences. More than half the agents choose the inferior technology in the first period with probability $1 - p$. In the following periods agents disregard their private information and follow the most observed choice. This leads to an incorrect herd.¹⁴

¹³I assume that $\sigma((N + 1)/2, \beta) = 1$, so agents indifferent after observing a sample $\zeta = (N + 1)/2$ follow the most observed choice in the sample. This simplifies the decision rule in the second period.

¹⁴I provide code that calculates the evolution of the system for general values of N , p , l and h at <https://sites.google.com/site/imonzon/research/#herds>. Numerical calculations show that incorrect herds occur for parameter values other than those in Propositions 3 and 4.

4. Conclusion

I have presented a model of observational learning under aggregate uncertainty. A continuum of homogeneous rational agents choose between two competing technologies. Agents learn from past agents' behavior and from a signal whose quality depends on random aggregate shocks.

Because of aggregate uncertainty, aggregate behavior does not necessarily reflect the true state of nature. In spite of this, agents still find others' actions informative. Thus, inferior choices can have lasting effects. I say an incorrect herd occurs when the inferior action is chosen in the long run with positive probability. I show how incorrect herds can occur because of aggregate uncertainty.

I derive the agents' optimal decision rule when the sample size is either 1 or 2. In the former case, I show that there are no incorrect herds, but no complete learning. In the latter, I find necessary and sufficient conditions for agents to herd on the inferior technology with positive probability, and I show that incorrect herds can occur even with a high average quality of the signal.

Finally, I show that the decision function may change over time if the sample size exceeds two. Moreover, agents' optimal behavior may be counterintuitive: observing *more* agents choosing a technology may cause an individual to choose the other one. I present examples illustrating these traits. I do not fully characterize the decision function when the sample size exceeds two. However, I present simple environments where incorrect herds occur for any odd sample size greater than two.

References

- BANERJEE, A. (1992): "A Simple Model of Herd Behavior," *Quarterly Journal of Economics*, 107, 797–817.
- BANERJEE, A. AND D. FUDENBERG (2004): "Word-of-mouth Learning," *Games and*

- Economic Behavior*, 46, 1–22.
- BIKHCHANDANI, S., D. HIRSHLEIFER, AND I. WELCH (1992): “A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades,” *Journal of Political Economy*, 100, 992–1026.
- CALLANDER, S. AND J. HÖRNER (2009): “The Wisdom of the Minority,” *Journal of Economic Theory*, 144, 1421–1439.
- ELLISON, G. AND D. FUDENBERG (1995): “Word-of-Mouth Communication and Social Learning,” *The Quarterly Journal of Economics*, 110, 93–125.
- GALEOTTI, A. AND S. GOYAL (2010): “The Law of the Few,” *American Economic Review*, 100, 1468–92.
- GUARINO, A., H. HARMGART, AND S. HUCK (2011): “Aggregate Information Cascades,” *Games and Economic Behavior*, 73, 167–185.
- HERRERA, H. AND J. HÖRNER (2013): “Biased Social Learning,” *Games and Economic Behavior*, 80, 131 – 146.
- JUDD, K. L. (1985): “The Law of Large Numbers with a Continuum of IID Random Variables,” *Journal of Economic Theory*, 35, 19–25.
- LOBEL, I. AND E. SADLER (2015): “Information diffusion in networks through social learning,” *Theoretical Economics*, 10, 807–851.
- MONZÓN, I. AND M. RAPP (2014): “Observational Learning with Position Uncertainty,” *Journal of Economic Theory*, 154, 375 – 402.
- SMITH, L. AND P. SØRENSEN (2000): “Pathological Outcomes of Observational Learning,” *Econometrica*, 68, 371–398.
- (2008): “Rational Social Learning with Random Sampling,” Working Paper.

A. Appendix: Omitted Proofs

A.1 Proof of Lemma 1

Since $\Pr(\zeta | x) = \Pr(N - \zeta | 1 - x)$ and

$$\psi(\alpha) = \frac{E[q]}{1 - E[q]} = \left(\frac{1 - E[q]}{E[q]} \right)^{-1} = \frac{1}{\psi(\beta)}, \text{ then}$$

$$\begin{aligned} \frac{\Pr(A | \zeta, \alpha)}{\Pr(B | \zeta, \alpha)} &= \psi(\alpha) \frac{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t) \Pr[\zeta | x_t(\mathbf{q}^t)]}{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t) \Pr[\zeta | 1 - x_t(\mathbf{q}^t)]} \\ &= \frac{1}{\psi(\beta) \frac{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t) \Pr[N - \zeta | x_t(\mathbf{q}^t)]}{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t) \Pr[N - \zeta | 1 - x_t(\mathbf{q}^t)]}} = \left(\frac{\Pr(A | N - \zeta, \beta)}{\Pr(B | N - \zeta, \beta)} \right)^{-1}. \end{aligned}$$

This, together with the assumption that $\sigma(N - \zeta, \beta) = 1 - \sigma(\zeta, \alpha)$, shows (a). A sample $\zeta = N/2$ is uninformative since

$$\Pr\left[\frac{N}{2} | x_t(\mathbf{q}^t)\right] = \Pr\left[N - \frac{N}{2} | x_t(\mathbf{q}^t)\right] = \Pr\left[\frac{N}{2} | 1 - x_t(\mathbf{q}^t)\right].$$

Then, $\Pr(A | N/2, s) / \Pr(B | N/2, s) = \psi(s)$. This shows (b). Next, note that $E[x_t(\mathbf{q}^t)]$ represents not only the expected fraction of adopters of the superior technology in period t , but also the *expected utility* of a period t agent. By usual arguments in the literature (see for example Lemma 1 in Banerjee and Fudenberg [2004]) the expected utility must *weakly* increase over time.¹⁵ This shows (c). Finally, $E[x_1(\mathbf{q}^1)] = px_1(h) + (1 - p)x_1(l) = ph + (1 - p)l = E[q]$. Given (c), $E[x_t(\mathbf{q}^t)] \geq E[q]$ for all t . ■

¹⁵The argument (usually referred to as *improvement principle*) is simple. A period t agent observes a sample of period $t - 1$ choices. She can always follow a simple strategy: copy the action of a random agent from the sample. This strategy guarantees an expected utility of $E[x_{t-1}(\mathbf{q}^{t-1})]$. An optimal strategy must do weakly better, so $E[x_t(\mathbf{q}^t)] \geq E[x_{t-1}(\mathbf{q}^{t-1})]$.

A.2 Proof of Lemma 2

Assume $E[x_t(\mathbf{q}^t)] = E[q]$. Then

$$\frac{\Pr(A | 1, s)}{\Pr(B | 1, s)} = \psi(s) \frac{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t) x_t(\mathbf{q}^t)}{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t) [1 - x_t(\mathbf{q}^t)]} = \psi(s) \frac{E[q]}{1 - E[q]}.$$

Consequently, $\Pr(A | 1, \alpha) / \Pr(B | 1, \alpha) > 1$ and $\Pr(A | 1, \beta) / \Pr(B | 1, \beta) = 1$. By Lemma 1, $D(\zeta, s)$ is as stated in Lemma 2. Then,

$$\begin{aligned} x_{t+1}(\mathbf{q}^{t+1}) &= \Pr[1 | x_t(\mathbf{q}^t)] q_{t+1} D(1, \alpha) + \Pr[1 | x_t(\mathbf{q}^t)] (1 - q_{t+1}) D(1, \beta) \\ &\quad + \Pr[0 | x_t(\mathbf{q}^t)] q_{t+1} D(0, \alpha) + \Pr[0 | x_t(\mathbf{q}^t)] (1 - q_{t+1}) D(0, \beta) \\ &= x_t(\mathbf{q}^t) q_{t+1} + x_t(\mathbf{q}^t) (1 - q_{t+1}) \sigma(1, \beta) + [1 - x_t(\mathbf{q}^t)] q_{t+1} [1 - \sigma(1, \beta)] \\ &= \sigma(1, \beta) x_t(\mathbf{q}^t) + [1 - \sigma(1, \beta)] q_{t+1} \end{aligned}$$

Thus $E[x_{t+1}(\mathbf{q}^{t+1})] = E[q]$. Since $E[x_1(\mathbf{q}^1)] = E[q]$, Lemma 2 holds by induction.

■

A.3 Proof of Proposition 1

The key insight in this proposition is that a full sample (one with all agents choosing the same action, say $\zeta = 2$) is strictly more informative than an opposing signal (say $s = \beta$). The intuition behind this result is simple and consists of two parts. First, if an agent observed *only one* action, it would (weakly) prevail over the signal:

$$\frac{E[x_t(\mathbf{q}^t)]}{E[1 - x_t(\mathbf{q}^t)]} \geq \frac{E[q]}{1 - E[q]}$$

This is an immediate consequence of Lemma 1(d), which shows that the average action is at least as informative as the signal: $E[x_t(\mathbf{q}^t)] \geq E[q]$. Second, a sample of two agents who choose the same action is strictly more informative than a sample

of just one agent:¹⁶

$$\frac{E[x_t(\mathbf{q}^t)^2]}{E[[1-x_t(\mathbf{q}^t)]^2]} > \frac{E[x_t(\mathbf{q}^t)]}{E[1-x_t(\mathbf{q}^t)]}$$

These two parts together imply that

$$\frac{\Pr(A \mid 2, \beta)}{\Pr(B \mid 2, \beta)} = \frac{1 - E[q]}{E[q]} \frac{E[x_t(\mathbf{q}^t)^2]}{E[[1-x_t(\mathbf{q}^t)]^2]} > 1.$$

This makes $D_{t+1}(2, \beta) = 1$ and of course $D_{t+1}(2, \alpha) = 1$. By Lemma 1(b), $D_{t+1}(1, \alpha) = 1$ and $D_{t+1}(1, \beta) = 0$. Lemma 1(a) completes the proof of (a). Given this decision rule, (b) is straightforward:

$$\begin{aligned} x_{t+1}(\mathbf{q}^{t+1}) &= \Pr[2 \mid x_t(\mathbf{q}^t)] + q_{t+1} \Pr[1 \mid x_t(\mathbf{q}^t)] \\ &= [x_t(\mathbf{q}^t)]^2 + 2q_{t+1} [x_t(\mathbf{q}^t) - [x_t(\mathbf{q}^t)]^2] \\ &= (1 - 2q_{t+1}) [x_t(\mathbf{q}^t)]^2 + 2q_{t+1}x_t(\mathbf{q}^t) \end{aligned}$$

For (c), note that $x_{t+1}(\mathbf{q}^{t+1}) - x_t(\mathbf{q}^t) = (1 - 2q_{t+1}) [x_t(\mathbf{q}^t)]^2 - x_t(\mathbf{q}^t)$ and that $x_t(\mathbf{q}^t) - [x_t(\mathbf{q}^t)]^2 > 0 \forall x_t(\mathbf{q}^t) \notin \{0, 1\}$. ■

A.4 Proof of Proposition 3

In the example I present here, $x_t(\mathbf{q}^t)$ takes only two values in each period. As I show later, these values are determined by the level of aggregate uncertainty in the first period. Consequently, I denote those two values $x_t(h)$ and $x_t(l)$, depending on whether $q_1 = h$ or $q_1 = l$. Auxiliary Lemmas 4 and 5 present some useful properties.

LEMMA 4. *Let $N \geq 3$ be odd and $1 - 1/N < x_t(h) < 1 - x_t(l)$. Then*

$$(a) \text{ If } (N+1)/2 \leq \zeta \leq N-1, \text{ then } (1-x_t(l))^\zeta (x_t(l))^{N-\zeta} < (x_t(h))^\zeta (1-x_t(h))^{N-\zeta}.$$

¹⁶Equivalently, $E[x_t(\mathbf{q}^t)^2] [1 - E[x_t(\mathbf{q}^t)]] > E[x_t(\mathbf{q}^t)] E[1 + x_t(\mathbf{q}^t)^2 - 2x_t(\mathbf{q}^t)]$, which holds if and only if $E[x_t(\mathbf{q}^t)^2] [1 - 2E[x_t(\mathbf{q}^t)]] > E[x_t(\mathbf{q}^t)] E[1 - 2E[x_t(\mathbf{q}^t)]]$. Note that $E[x_t(\mathbf{q}^t)^2] \leq E[x_t(\mathbf{q}^t)]$ and $1 - 2E[x_t(\mathbf{q}^t)] < 0$ (since $1/2 < E[q] \leq E[x_t(\mathbf{q}^t)] < 1$). Then this strict inequality holds.

(b) If $x > 1/2$, then $\Pr(\zeta \geq (N+1)/2 \mid x) > x$.

Proof. Define $f(y) = (y)^\zeta (1-y)^{N-\zeta}$. Then,

$$\frac{\partial f}{\partial y} = \left[\frac{\zeta}{y} - \frac{N-\zeta}{1-y} \right] \left[(y)^\zeta (1-y)^{N-\zeta} \right].$$

Note that $y > \zeta/N \Rightarrow \zeta/y - (N-\zeta)/(1-y) < 0 \Rightarrow \partial f/\partial y < 0$. But $\zeta/N \leq 1 - 1/N < x_t(h) < 1 - x_t(l)$. So $\partial f/\partial y < 0$ for all $y \in [x_t(h), 1 - x_t(l)]$. Then $f(x_t(h)) > f(1 - x_t(l))$. This shows (a). Regarding (b), define $Y \sim B(N, x)$ with $E[Y] = Nx$. Then,¹⁷

$$\begin{aligned} \Pr\left(\zeta \geq \frac{N+1}{2} \mid x\right) &= \sum_{\zeta=\frac{N+1}{2}}^N \binom{N}{\zeta} x^\zeta (1-x)^{N-\zeta} \\ &= \frac{1}{N} \left[\sum_{\zeta=\frac{N+1}{2}}^N \binom{N}{\zeta} x^\zeta (1-x)^{N-\zeta} (N-\zeta) + \sum_{\zeta=\frac{N+1}{2}}^N \binom{N}{\zeta} x^\zeta (1-x)^{N-\zeta} \zeta \right] \\ &= \frac{1}{N} \left[\sum_{\zeta=0}^{\frac{N-1}{2}} \binom{N}{N-\zeta} x^{N-\zeta} (1-x)^\zeta \zeta + \sum_{\zeta=\frac{N+1}{2}}^N \binom{N}{\zeta} x^\zeta (1-x)^{N-\zeta} \zeta \right] \\ &> \frac{1}{N} \left[\sum_{\zeta=0}^{\frac{N-1}{2}} \binom{N}{N-\zeta} x^\zeta (1-x)^{N-\zeta} \zeta + \sum_{\zeta=\frac{N+1}{2}}^N \binom{N}{\zeta} x^\zeta (1-x)^{N-\zeta} \zeta \right] \\ &> \frac{1}{N} \sum_{\zeta=0}^N \binom{N}{\zeta} x^\zeta (1-x)^{N-\zeta} \zeta = \frac{1}{N} E[Y] = \frac{1}{N} Nx = x \blacksquare \end{aligned}$$

Auxiliary Lemma 5 presents properties for settings where $x_t(\mathbf{q}^t)$ takes only two values.

LEMMA 5. Let $N \geq 3$ be odd, $1 - 1/N < h < 1 - l$, and $x_t(\mathbf{q}^t)$ be given by,

$$x_t(\mathbf{q}^t) = \begin{cases} x_t(h) & \text{if } q_1 = h \\ x_t(l) & \text{if } q_1 = l \end{cases}$$

¹⁷The inequality in the fourth line is a result of $(x/(1-x))^{N-2\zeta} > 1$ for $\zeta \leq (N-1)/2$ and $x > 1/2$.

with $h \leq x_t(h) < 1 - x_t(l)$ and $x_t(l) \leq l$. Then:

$$\frac{\Pr(A \mid \zeta, s)}{\Pr(B \mid \zeta, s)} > \frac{p}{p^{\frac{1-h}{h}} + (1-p)} \frac{1 - E[q]}{E[q]} \quad \text{for } \frac{N+1}{2} \leq \zeta \leq N-1 \quad (6)$$

$$\frac{\Pr(A \mid \zeta, s)}{\Pr(B \mid \zeta, s)} > \frac{ph^N}{p(1-h)^N + (1-p)} \frac{1 - E[q]}{E[q]} \quad \text{for } \zeta = N \quad (7)$$

Moreover, if $D_{t+1}(\zeta, s) = 1$ whenever $\zeta \geq (N+1)/2$ and zero otherwise, there are only two possible values for x_{t+1} :

$$x_{t+1}(\mathbf{q}^{t+1}) = \begin{cases} x_{t+1}(h) & \text{if } q_1 = h \\ x_{t+1}(l) & \text{if } q_1 = l \end{cases}$$

with $x_t(h) < x_{t+1}(h) < 1 - x_{t+1}(l)$ and $x_{t+1}(l) < x_t(l)$.

Proof. If $(N+1)/2 \leq \zeta \leq N-1$, the informational content of the sample is given by:

$$\begin{aligned} \frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} &= \frac{p [x_t(h)]^\zeta [1 - x_t(h)]^{(N-\zeta)} + (1-p) [x_t(l)]^\zeta [1 - x_t(l)]^{(N-\zeta)}}{p [1 - x_t(h)]^\zeta [x_t(h)]^{(N-\zeta)} + (1-p) [1 - x_t(l)]^\zeta [x_t(l)]^{(N-\zeta)}} \\ &> \frac{p}{p \left[\frac{1-x_t(h)}{x_t(h)} \right]^{2\zeta-N} + (1-p) \left[\frac{1-x_t(l)}{x_t(h)} \right]^\zeta \left[\frac{x_t(l)}{1-x_t(h)} \right]^{(N-\zeta)}} \\ &> \frac{p}{p^{\frac{1-x_t(h)}{x_t(h)}} + (1-p)} > \frac{p}{p^{\frac{1-h}{h}} + (1-p)} \end{aligned}$$

because of Lemma 4(a). Finally, since $\psi(\alpha) > \psi(\beta) = (1 - E[q])/E[q]$, equation (6) holds. If $\zeta = N$, the informational content of the sample is also bounded:

$$\frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} = \frac{p [x_t(h)]^N + (1-p) [x_t(l)]^N}{p [1 - x_t(h)]^N + (1-p) [1 - x_t(l)]^N} > \frac{p (h)^N}{p (1-h)^N + (1-p)}$$

The inequality is strict since $0 < x_t(l) < 1$. Since $\psi(\alpha) > \psi(\beta)$, equation (7) holds.

Next, if $D_{t+1}(\zeta, s) = 1$ whenever $\zeta \geq (N+1)/2$ and zero otherwise, then

$$x_{t+1}(\mathbf{q}^{t+1}) = \Pr \left(\zeta \geq \frac{N+1}{2} \mid x_t(\mathbf{q}^t) \right).$$

Since the decision does not depend on the signal, the aggregate level of uncertainty in period $t + 1$ does not have any impact on the evolution of the system in that period. Thus, x_{t+1} takes only two values. Note that

$$\begin{aligned} 1 - x_{t+1}(l) &= 1 - \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(l)\right) = \Pr\left(\zeta \leq \frac{N-1}{2} \mid x_t(l)\right) \\ &= \Pr\left(\zeta \geq \frac{N+1}{2} \mid 1 - x_t(l)\right). \end{aligned}$$

Since $\Pr(\zeta \geq (N+1)/2 \mid x)$ is increasing in x , then:

$$1 - x_{t+1}(l) = \Pr\left(\zeta \geq \frac{N+1}{2} \mid 1 - x_t(l)\right) > \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(h)\right) = x_{t+1}(h)$$

Lemma 4(b) guarantees $x_{t+1}(h) > x_t(h)$ and $x_{t+1}(l) < x_t(l)$. ■

I use Lemma 5 to prove Proposition 3.

Proof. In period $t = 1$, agents follow the signal they receive. As a result:

$$x_1 = \begin{cases} h & \text{with probability } p \\ l & \text{with probability } 1 - p \end{cases}$$

I need to show that agents choose technology a when it is the most observed choice, regardless of the signal. To do this, I show that the information contained in a sample $\zeta \geq (N+1)/2$ always outweighs the information provided by a signal $s = \beta$. Given equations (6) and (7), it suffices to show that:

$$\begin{aligned} \frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} &> \frac{ph^N}{p(1-h)^N + (1-p)} \frac{p(1-h) + (1-p)(1-l)}{ph + (1-p)l} \geq 1 \\ \frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} &> \frac{p}{p\frac{1-h}{h} + (1-p)} \frac{p(1-h) + (1-p)(1-l)}{ph + (1-p)l} \geq 1 \end{aligned}$$

There exists $\bar{p} < 1$, such that both inequalities hold for all $p \geq \bar{p}$.¹⁸

Finally, $x_{t+1}(h)$ is a continuous function of $x_t(h)$. Moreover, $x_{t+1}(h) > x_t(h)$

¹⁸After rearranging, the second inequality turns into: $p \geq \bar{p}^1 \equiv [(1-h)(1-l/h)]^{-1}l$. The assumption $(1-h)h > l$ guarantees that $\bar{p}^1 < 1$. Next, the first inequality is equivalent to the

for all $x_t(h) \in (1/2, 1)$ and $x_{t+1}(h) = x_t(h)$ for $x_t(h) \in \{1/2, 1\}$. Consequently, $\lim_{t \rightarrow \infty} x_t(h) = 1$. Note that $q_1 = h$ with probability p . Consequently, the fraction of agents choosing the superior technology converges to 1 with probability p . As a result, since $x_t(l) < 1 - x_t(h)$, $\Pr(\lim_{t \rightarrow \infty} x_t(\mathbf{q}^t) = 0) = 1 - p$. ■

quadratic inequality $(h - l)p^2 [1 - h^N - (1 - h)^N] + p [(1 - l)h^N - h - l(1 - h)^N + 2l] \geq l$. This inequality is not satisfied for $p = 0$. It holds with strict inequality for $p = 1$. Then there exists $\bar{p}^2 < 1$ such that this expression holds for all $p \geq \bar{p}^2$. So both inequalities hold for all $p \geq \bar{p} \equiv \max\{\bar{p}^1, \bar{p}^2\}$.