Supplementary Material to:
IMU Preintegration on Manifold for Efficient
Visual-Inertial Maximum-a-Posteriori Estimation

Technical Report GT-IRIM-CP&R-2015-001

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This report provides additional derivations and implementation details to support the paper [4]. Therefore, it should not be considered a self-contained document, but rather regarded as an appendix of [4], and cited as:

> "C. Forster, L. Carlone, F. Dellaert, and D. Scaramuzza, *IMU preintegration on manifold for efficient visual-inertial maximum-a-posteriori estimation*, (supplementary material, GT-IRIM-CP&R-2015-001), In Robotics: Science and Systems (RSS), 2015."

Across this report, references in the form "(x)", e.g., (11), recall equations from the main paper [4], while references "(A.x)", e.g., (A.1), refer to equations within this appendix.

Contents

1 IMU Preintegration: Noise Propagation and Bias Updates

1.1 Iterative Noise Propagation

In this section we provide a complete derivation of the preintegrated measurements covariance $\mathbf{\Sigma}_{ij}$ (*cf.* Section V-B in [4]), which is such that:

$$\boldsymbol{\eta}^{\Delta}_{ij} \doteq [\delta\boldsymbol{\phi}^{\mathsf{T}}_{ij}, \delta\mathbf{v}^{\mathsf{T}}_{ij}, \delta\mathbf{p}^{\mathsf{T}}_{ij}]^{\mathsf{T}} \sim \mathcal{N}(\mathbf{0}_{9\times 1}, \mathbf{\Sigma}_{ij}). \tag{A.1}$$

We call $\boldsymbol{\eta}^{\Delta}_{ij}$ the *preintegration noise vector.*

Let us rewrite explicitly what the preintegration errors $\delta\boldsymbol{\phi}_{ij}, \delta\mathbf{v}_{ij}, \delta\mathbf{p}_{ij}$ are (*cf.* Eqs. (28)-(29)-(30)):

$$\begin{aligned}
\delta\boldsymbol{\phi}_{ij} &\simeq \sum_{k=i}^{j-1} \Delta\tilde{\mathtt{R}}^{\mathsf{T}}_{k+1j} \mathtt{J}^{k}_{r} \boldsymbol{\eta}^{gd}_{k} \Delta t \\
\delta\mathbf{v}_{ij} &\simeq \sum_{k=i}^{j-1} \left[-\Delta\tilde{\mathtt{R}}_{ik} \left(\tilde{\mathbf{a}}_k - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ik} \Delta t + \Delta\tilde{\mathtt{R}}_{ik} \boldsymbol{\eta}^{ad}_{k} \Delta t \right] \\
\delta\mathbf{p}_{ij} &\simeq \sum_{k=i}^{j-1} \left[-\frac{3}{2} \Delta\tilde{\mathtt{R}}_{ik} \left(\tilde{\mathbf{a}}_k - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ik} \Delta t^2 + \frac{3}{2} \Delta\tilde{\mathtt{R}}_{ik} \boldsymbol{\eta}^{ad}_{k} \Delta t^2 \right]
\end{aligned} \tag{A.2}$$

where the relations are valid up to the first order. Now, one way to derive the covariance of $\boldsymbol{\eta}^{\Delta}_{ij}$ is to substitute the expression of $\delta\boldsymbol{\phi}_{ij}$ back into $\delta\mathbf{v}_{ij}$ and $\delta\mathbf{p}_{ij}$. The result of this would be a linear expression that relates $\boldsymbol{\eta}^{\Delta}_{ij}$ to the raw measurements noise $\boldsymbol{\eta}^{gd}_{k}, \boldsymbol{\eta}^{ad}_{k}$, on which a (tedious) noise propagation can be carried out.

To avoid this long procedure, we prefer to write (A.2) in iterative form, and then to carry out noise propagation on the resulting (simpler) expressions. In order to write $\boldsymbol{\eta}^{\Delta}_{ij}$ in iterative form we first note that:

$$\begin{aligned}
\delta\boldsymbol{\phi}_{ij} &\simeq \sum_{k=i}^{j-1} \Delta\tilde{\mathtt{R}}^{\mathsf{T}}_{k+1j} \mathtt{J}^{k}_{r} \boldsymbol{\eta}^{gd}_{k} \Delta t = \sum_{k=i}^{j-2} \Delta\tilde{\mathtt{R}}^{\mathsf{T}}_{k+1j} \mathtt{J}^{k}_{r} \boldsymbol{\eta}^{gd}_{k} \Delta t + \overbrace{\Delta\tilde{\mathtt{R}}^{\mathsf{T}}_{jj}}^{=\mathbf{I}_3} \mathtt{J}^{j-1}_{r} \boldsymbol{\eta}^{gd}_{j-1} \Delta t \\
&= \sum_{k=i}^{j-2} (\Delta\tilde{\mathtt{R}}_{k+1j-1} \Delta\tilde{\mathtt{R}}_{j-1j})^{\mathsf{T}} \mathtt{J}^{k}_{r} \boldsymbol{\eta}^{gd}_{k} \Delta t + \mathtt{J}^{j-1}_{r} \boldsymbol{\eta}^{gd}_{j-1} \Delta t = \Delta\tilde{\mathtt{R}}^{\mathsf{T}}_{j-1j} \sum_{k=i}^{j-2} \Delta\tilde{\mathtt{R}}^{\mathsf{T}}_{k+1j-1} \mathtt{J}^{k}_{r} \boldsymbol{\eta}^{gd}_{k} \Delta t + \mathtt{J}^{j-1}_{r} \boldsymbol{\eta}^{gd}_{j-1} \Delta t \\
&= \Delta\tilde{\mathtt{R}}^{\mathsf{T}}_{j-1j} \delta\boldsymbol{\phi}_{ij-1} + \mathtt{J}^{j-1}_{r} \boldsymbol{\eta}^{gd}_{j-1} \Delta t
\end{aligned} \tag{A.3}$$

where we simply took the last term ($k = j-1$) out of the sum and conveniently rearranged the terms.

Repeating the same process for $\delta\mathbf{v}_{ij}$:

$$\begin{aligned}
\delta\mathbf{v}_{ij} &= \sum_{k=i}^{j-1} \left[-\Delta\tilde{\mathtt{R}}_{ik} \left(\tilde{\mathbf{a}}_k - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ik} \Delta t + \Delta\tilde{\mathtt{R}}_{ik} \boldsymbol{\eta}^{ad}_{k} \Delta t \right] \\
&= \sum_{k=i}^{j-2} \left[-\Delta\tilde{\mathtt{R}}_{ik} \left(\tilde{\mathbf{a}}_k - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ik} \Delta t + \Delta\tilde{\mathtt{R}}_{ik} \boldsymbol{\eta}^{ad}_{k} \Delta t \right] - \Delta\tilde{\mathtt{R}}_{ij-1} \left(\tilde{\mathbf{a}}_{j-1} - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ij-1} \Delta t + \Delta\tilde{\mathtt{R}}_{ij-1} \boldsymbol{\eta}^{ad}_{j-1} \Delta t \\
&= \delta\mathbf{v}_{ij-1} - \Delta\tilde{\mathtt{R}}_{ij-1} \left(\tilde{\mathbf{a}}_{j-1} - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ij-1} \Delta t + \Delta\tilde{\mathtt{R}}_{ij-1} \boldsymbol{\eta}^{ad}_{j-1} \Delta t
\end{aligned} \tag{A.4}$$

Doing the same for $\delta\mathbf{p}_{ij}$, and noting that $\delta\mathbf{p}_{ij}$ can be written as a function of $\delta\mathbf{v}_{ij}$ (this can be easily seen from the expression of $\delta\mathbf{v}_{ij}$ in (A.2)):

$$\begin{aligned}
\delta\mathbf{p}_{ij} &= \sum_{k=i}^{j-1} \left[\delta\mathbf{v}_{ik} \Delta t - \frac{1}{2} \Delta\tilde{\mathtt{R}}_{ik} \left(\tilde{\mathbf{a}}_k - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ik} \Delta t^2 + \frac{1}{2} \Delta\tilde{\mathtt{R}}_{ik} \boldsymbol{\eta}^{ad}_{k} \Delta t^2 \right] \\
&= \sum_{k=i}^{j-2} \left[\delta\mathbf{v}_{ik} \Delta t - \frac{1}{2} \Delta\tilde{\mathtt{R}}_{ik} \left(\tilde{\mathbf{a}}_k - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ik} \Delta t^2 + \frac{1}{2} \Delta\tilde{\mathtt{R}}_{ik} \boldsymbol{\eta}^{ad}_{k} \Delta t^2 \right] \\
&+ \delta\mathbf{v}_{ij-1} \Delta t - \frac{1}{2} \Delta\tilde{\mathtt{R}}_{ij-1} \left(\tilde{\mathbf{a}}_{j-1} - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ij-1} \Delta t^2 + \frac{1}{2} \Delta\tilde{\mathtt{R}}_{ij-1} \boldsymbol{\eta}^{ad}_{j-1} \Delta t^2 \\
&= \delta\mathbf{p}_{ij-1} + \delta\mathbf{v}_{ij-1} \Delta t - \frac{1}{2} \Delta\tilde{\mathtt{R}}_{ij-1} \left(\tilde{\mathbf{a}}_{j-1} - \mathbf{b}^a_i\right)^{\wedge} \delta\boldsymbol{\phi}_{ij-1} \Delta t^2 + \frac{1}{2} \Delta\tilde{\mathtt{R}}_{ij-1} \boldsymbol{\eta}^{ad}_{j-1} \Delta t^2
\end{aligned} \tag{A.5}$$

From (A.3), (A.4), and (A.5), it follows that that (A.2) can be written in iterative form:

$$
\begin{aligned}
\delta\boldsymbol{\phi}_{ik+1} &= \Delta\tilde{\mathrm{R}}_{kk+1}^{\mathsf{T}}\delta\boldsymbol{\phi}_{ik} + \mathrm{J}_r^k \boldsymbol{\eta}_k^{gd}\Delta t \\
\delta\mathbf{v}_{ik+1} &= \delta\mathbf{v}_{ik} - \Delta\tilde{\mathrm{R}}_{ik}\left(\tilde{\mathbf{a}}_k - \mathbf{b}_i^a\right)^{\wedge}\delta\boldsymbol{\phi}_{ik}\Delta t + \Delta\tilde{\mathrm{R}}_{ik}\boldsymbol{\eta}_k^{ad}\Delta t \\
\delta\mathbf{p}_{ik+1} &= \delta\mathbf{p}_{ik} + \delta\mathbf{v}_{ik}\Delta t - \frac{1}{2}\Delta\tilde{\mathrm{R}}_{ik}\left(\tilde{\mathbf{a}}_k - \mathbf{b}_i^a\right)^{\wedge}\delta\boldsymbol{\phi}_{ik}\Delta t^2 + \frac{1}{2}\Delta\tilde{\mathrm{R}}_{ik}\boldsymbol{\eta}_k^{ad}\Delta t^2
\end{aligned} \tag{A.6}
$$

for $k = i, \ldots, j$, with initial conditions $\delta\boldsymbol{\phi}_{ii} = \delta\mathbf{v}_{ii} = \delta\mathbf{p}_{ii} = \mathbf{0}_3$.

Eq. (A.6) can be conveniently written in matrix form:

$$
\begin{bmatrix} \delta\boldsymbol{\phi}_{ik+1} \\ \delta\mathbf{v}_{ik+1} \\ \delta\mathbf{p}_{ik+1} \end{bmatrix} = \begin{bmatrix} \Delta\tilde{\mathrm{R}}_{kk+1}^{\mathsf{T}} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ -\Delta\tilde{\mathrm{R}}_{ik}\left(\tilde{\mathbf{a}}_k - \mathbf{b}_i^a\right)^{\wedge}\Delta t & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ -\frac{1}{2}\Delta\tilde{\mathrm{R}}_{ik}\left(\tilde{\mathbf{a}}_k - \mathbf{b}_i^a\right)^{\wedge}\Delta t^2 & \mathbf{I}_{3\times3}\Delta t & \mathbf{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\phi}_{ik} \\ \delta\mathbf{v}_{ik} \\ \delta\mathbf{p}_{ik} \end{bmatrix} + \begin{bmatrix} \mathrm{J}_r^k\Delta t & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \Delta\tilde{\mathrm{R}}_{ik}\Delta t \\ \mathbf{0}_{3\times3} & \frac{1}{2}\Delta\tilde{\mathrm{R}}_{ik}\Delta t^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_k^{gd} \\ \boldsymbol{\eta}_k^{ad} \end{bmatrix} \tag{A.7}
$$

or more simply:

$$
\boldsymbol{\eta}_{ik+1}^{\Delta} = \mathbf{A}\boldsymbol{\eta}_{ik}^{\Delta} + \mathbf{B}\boldsymbol{\eta}_k^d \tag{A.8}
$$

where $\boldsymbol{\eta}_k^d \doteq [\boldsymbol{\eta}_k^{gd} \;\; \boldsymbol{\eta}_k^{ad}]$.

From the linear model (A.8) and given the covariance $\boldsymbol{\Sigma}_{\boldsymbol{\eta}} \in \mathbb{R}^{6\times6}$ of the raw IMU measurements noise $\boldsymbol{\eta}_k^d$, it's now possible to compute the covariance iteratively:

$$
\boldsymbol{\Sigma}_{ik+1} = \mathbf{A}\boldsymbol{\Sigma}_{ik+1}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\mathbf{B}^{\mathsf{T}} \tag{A.9}
$$

starting from initial conditions $\boldsymbol{\Sigma}_{ii} = \mathbf{0}_{9\times9}$.

Note that the fact that the covariance can be computed iteratively is convenient, computationally, as it means that we can easily *update* the covariance after integrating a new measurement.

The same iterative computation is possible for the preintegrated measurements themselves:

$$
\begin{aligned}
\Delta\tilde{\mathrm{R}}_{ik+1} &= \Delta\tilde{\mathrm{R}}_{ik}\mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_i^g\right)\Delta t\right) \\
\Delta\tilde{\mathbf{v}}_{ik+1} &= \Delta\tilde{\mathbf{v}}_{ik} + \Delta\tilde{\mathrm{R}}_{ik}(\tilde{\mathbf{a}}_k - \mathbf{b}_i^a)\,\Delta t \\
\Delta\tilde{\mathbf{p}}_{ik+1} &= \Delta\tilde{\mathbf{p}}_{ik} + \Delta\tilde{\mathbf{v}}_{ik}\Delta t + \frac{1}{2}\Delta\tilde{\mathrm{R}}_{ik}(\tilde{\mathbf{a}}_k - \mathbf{b}_i^a)\,\Delta t^2
\end{aligned} \tag{A.10}
$$

which easily follows from Eqs. (28)-(29)-(30) of the main document.

1.2 Bias Correction via First-Order Updates

In this section we provide a complete derivation of the first-order bias correction proposed in Section V-C of [4]. Let us start by recalling the expression of the preintegrated measurements $\Delta\tilde{\mathrm{R}}_{ij}$, $\Delta\tilde{\mathbf{v}}_{ij}$, $\Delta\tilde{\mathbf{p}}_{ij}$, given in Eqs. (28)-(29)-(30) of the main document:

$$
\begin{aligned}
\Delta\tilde{\mathrm{R}}_{ij} &= \prod_{k=i}^{j-1}\mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \mathbf{b}_i^g\right)\Delta t\right) \\
\Delta\tilde{\mathbf{v}}_{ij} &= \sum_{k=i}^{j-1}\Delta\tilde{\mathrm{R}}_{ik}\left(\tilde{\mathbf{a}}_k - \mathbf{b}_i^a\right)\Delta t \\
\Delta\tilde{\mathbf{p}}_{ij} &= \sum_{k=i}^{j-1}\frac{3}{2}\Delta\tilde{\mathrm{R}}_{ik}\left(\tilde{\mathbf{a}}_k - \mathbf{b}_i^a\right)\Delta t^2
\end{aligned} \tag{A.11}
$$

Assume now that we have computed the preintegrated variables at a given bias estimate $\bar{\mathbf{b}}_i \doteq [\bar{\mathbf{b}}_i^g \;\; \bar{\mathbf{b}}_i^a]$, and let us denote the corresponding preintegrated measurements as $\Delta\tilde{\mathrm{R}}_{ij}(\bar{\mathbf{b}}_i)$, $\Delta\tilde{\mathbf{v}}_{ij}(\bar{\mathbf{b}}_i)$, $\Delta\tilde{\mathbf{p}}_{ij}(\bar{\mathbf{b}}_i)$. In this section we want to devise an expression to "update" $\Delta\tilde{\mathrm{R}}_{ij}(\bar{\mathbf{b}}_i)$, $\Delta\tilde{\mathbf{v}}_{ij}(\bar{\mathbf{b}}_i)$, $\Delta\tilde{\mathbf{p}}_{ij}(\bar{\mathbf{b}}_i)$ when our bias estimate changes.

Consider the case in which we get a new estimate $\hat{\mathbf{b}}_i \leftarrow \bar{\mathbf{b}}_i + \delta\mathbf{b}_i$, where $\delta\mathbf{b}_i$ is a *small* correction w.r.t. the previous estimate $\bar{\mathbf{b}}_i$. Using the new bias estimate $\hat{\mathbf{b}}_i$ in (A.11) we get the updated preintegrated measurements:

$$
\begin{aligned}
\Delta\tilde{\mathrm{R}}_{ij}(\hat{\mathbf{b}}_i) &= \prod_{k=i}^{j-1}\mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \hat{\mathbf{b}}_i^g\right)\Delta t\right) = \prod_{k=i}^{j-1}\mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \bar{\mathbf{b}}_i^g - \delta\mathbf{b}_i^g\right)\Delta t\right) \\
\Delta\tilde{\mathbf{v}}_{ij}(\hat{\mathbf{b}}_i) &= \sum_{k=i}^{j-1}\Delta\tilde{\mathrm{R}}_{ik}(\hat{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \hat{\mathbf{b}}_i^a\right)\Delta t = \sum_{k=i}^{j-1}\Delta\tilde{\mathrm{R}}_{ik}(\hat{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a - \delta\mathbf{b}_i^a\right)\Delta t \\
\Delta\tilde{\mathbf{p}}_{ij}(\hat{\mathbf{b}}_i) &= \sum_{k=i}^{j-1}\frac{3}{2}\Delta\tilde{\mathrm{R}}_{ik}\left(\tilde{\mathbf{a}}_k - \hat{\mathbf{b}}_i^a\right)\Delta t^2 = \sum_{k=i}^{j-1}\frac{3}{2}\Delta\tilde{\mathrm{R}}_{ik}(\hat{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a - \delta\mathbf{b}_i^a\right)\Delta t^2
\end{aligned} \tag{A.12}
$$

A naive solution would be to recompute the preintegrated measurements at the new bias estimate as prescribed in (A.12). In this section, instead, we show how to update the preintegrated measurements without repeating the integration.

Let us start from the preintegrated rotation measurement $\Delta\tilde{\mathtt{R}}_{ij}(\hat{\mathbf{b}}_i)$. We assumed that the bias correction is small, hence we use the first-order approximation (7) for each term in the product:

$$\Delta\tilde{\mathtt{R}}_{ij}(\hat{\mathbf{b}}_i) \simeq \prod_{k=i}^{j-1} \left[\mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \bar{\mathbf{b}}_i^g\right)\Delta t\right)\mathrm{Exp}\left(-\mathtt{J}_r^k\;\delta\mathbf{b}_i^{gd}\;\Delta t\right)\right]$$

where we defined $\mathtt{J}_r^k \doteq \mathtt{J}_r(\tilde{\boldsymbol{\omega}}_k - \bar{\mathbf{b}}_i^g)$ ($\mathtt{J}_r$ is the right Jacobian for SO(3) given in Eq.(8) of the paper). We rearrange the terms in the product, by "moving" the terms including $\delta\mathbf{b}$ to the end, using the relation (11):

$$\Delta\tilde{\mathtt{R}}_{ij}(\hat{\mathbf{b}}_i) = \Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\prod_{k=i}^{j-1}\left[\mathrm{Exp}\left(-\Delta\tilde{\mathtt{R}}_{k+1j}(\bar{\mathbf{b}}_i)^{\mathsf{T}}\mathtt{J}_r^k\;\delta\mathbf{b}_i^g\;\Delta t\right)\right] \tag{A.13}$$

where we used the fact that $\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i) = \prod_{k=i}^{j-1}\left[\mathrm{Exp}\left(\left(\tilde{\boldsymbol{\omega}}_k - \bar{\mathbf{b}}_i^g\right)\Delta t\right)\right]$ by definition.

Repeated application of the first-order approximation (9) (recall that $\delta\mathbf{b}_i^g$ is small, hence the right Jacobians are close to the identity) produces:

$$\Delta\tilde{\mathtt{R}}_{ij}(\hat{\mathbf{b}}_i) \simeq \Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\mathrm{Exp}\left(\sum_{k=i}^{j-1}\left[-\Delta\tilde{\mathtt{R}}_{k+1j}(\bar{\mathbf{b}}_i)^{\mathsf{T}}\mathtt{J}_r^k\Delta t\right]\delta\mathbf{b}_i^g\right) = \Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\mathrm{Exp}\Big(\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g\Big) \tag{A.14}$$

which corresponds to eq. (36) in [4]. The Jacobian $\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}$ can be precomputed during preintegration. This can be done in analogy with Section 1.1 since the structure of $\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}$ is essentially the same as the one multiplying the noise in (A.2). Using (A.14) we can update the previous preintegrated measurement $\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)$ to get $\Delta\tilde{\mathtt{R}}_{ij}(\hat{\mathbf{b}}_i)$.

Let us now focus on the preintegrated velocity $\Delta\tilde{\mathbf{v}}_{ij}(\hat{\mathbf{b}}_i)$. We substitute $\Delta\tilde{\mathtt{R}}_{ij}(\hat{\mathbf{b}}_i)$ back into (A.12):

$$\Delta\tilde{\mathbf{v}}_{ij}(\hat{\mathbf{b}}_i) = \sum_{k=i}^{j-1}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\mathrm{Exp}\Big(\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g\Big)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a - \delta\mathbf{b}_i^a\right)\Delta t \tag{A.15}$$

Recalling that the correction $\delta\mathbf{b}_i^g$ is small, we use the first-order approximation (4):

$$\Delta\tilde{\mathbf{v}}_{ij}(\hat{\mathbf{b}}_i) \simeq \sum_{k=i}^{j-1}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\left(\mathbf{I} + \Big(\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g\Big)^{\wedge}\right)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a - \delta\mathbf{b}_i^a\right)\Delta t \tag{A.16}$$

Developing the previous expression and dropping higher-order terms:

$$\Delta\tilde{\mathbf{v}}_{ij}(\hat{\mathbf{b}}_i) \simeq \sum_{k=i}^{j-1}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a\right)\Delta t + \sum_{k=i}^{j-1}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\left(-\delta\mathbf{b}_i^a\right)\Delta t + \sum_{k=i}^{j-1}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\Big(\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g\Big)^{\wedge}\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a\right)\Delta t \tag{A.17}$$

Recalling that $\Delta\tilde{\mathbf{v}}_{ij}(\bar{\mathbf{b}}_i) = \sum_{k=i}^{j-1}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a\right)\Delta t$ and using property (2):

$$\begin{aligned}\Delta\tilde{\mathbf{v}}_{ij}(\hat{\mathbf{b}}_i) &= \Delta\tilde{\mathbf{v}}_{ij}(\bar{\mathbf{b}}_i) + \sum_{k=i}^{j-1} -\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\Delta t\delta\mathbf{b}_i^a + \sum_{k=i}^{j-1} -\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a\right)^{\wedge}\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\Delta t\;\delta\mathbf{b}_i^g \\ &= \Delta\tilde{\mathbf{v}}_{ij}(\bar{\mathbf{b}}_i) + \frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^a}\delta\mathbf{b}_i^a + \frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g\end{aligned} \tag{A.18}$$

which corresponds to the second expression in Eq. (36) of the paper.

Finally, repeating the same derivation for $\Delta\tilde{\mathbf{p}}_{ij}(\hat{\mathbf{b}}_i)$

$$
\begin{aligned}
\Delta\tilde{\mathbf{p}}_{ij}(\hat{\mathbf{b}}_i) &= \sum_{k=i}^{j-1} \frac{3}{2}\Delta\tilde{\mathtt{R}}_{ik}(\hat{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a - \delta\mathbf{b}_i^a\right)\Delta t^2 \\
&\overset{\text{Eq. (A.14)}}{=} \sum_{k=i}^{j-1} \frac{3}{2}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\mathrm{Exp}\Big(\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g\Big)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a - \delta\mathbf{b}_i^a\right)\Delta t^2 \\
&\overset{\text{Eq. (4)}}{\simeq} \sum_{k=i}^{j-1} \frac{3}{2}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\left(\mathbf{I} + \Big(\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g\Big)^\wedge\right)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a - \delta\mathbf{b}_i^a\right)\Delta t^2 \\
&= \Delta\tilde{\mathbf{p}}_{ij}(\bar{\mathbf{b}}_i) + \sum_{k=i}^{j-1} -\frac{3}{2}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\delta\mathbf{b}_i^a\Delta t^2 + \sum_{k=i}^{j-1} \frac{3}{2}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\Big(\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g\Big)^\wedge\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a\right)\Delta t^2 \\
&\overset{\text{Eq. (2)}}{=} \Delta\tilde{\mathbf{p}}_{ij}(\bar{\mathbf{b}}_i) + \sum_{k=i}^{j-1} -\frac{3}{2}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\Delta t^2\delta\mathbf{b}_i^a + \sum_{k=i}^{j-1} -\frac{3}{2}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a\right)^\wedge\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\Delta t^2\delta\mathbf{b}_i^g \\
&= \Delta\tilde{\mathbf{p}}_{ij}(\bar{\mathbf{b}}_i^g, \bar{\mathbf{b}}_i^a) + \frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^a}\delta\mathbf{b}_i^a + \frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}_i^g
\end{aligned}
\tag{A.19}
$$

which corresponds to the last expression in Eq.(36).

To summarize, the Jacobians used for a-posteriori bias update are (*cf.* eqs. (A.14)-(A.18)-(A.19)):

$$
\begin{aligned}
\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g} &= -\sum_{k=i}^{j-1}\left[\Delta\tilde{\mathtt{R}}_{k+1j}(\bar{\mathbf{b}}_i)^{\mathsf{T}}\ \mathtt{J}_r^k\ \Delta t\right] \\
\frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^a} &= -\sum_{k=i}^{j-1}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\Delta t \\
\frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^g} &= -\sum_{k=i}^{j-1}\Delta\tilde{\mathtt{R}}_{ij}\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a\right)^\wedge\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\Delta t \\
\frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^a} &= -\sum_{k=i}^{j-1}\frac{3}{2}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\Delta t^2 \\
\frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^g} &= -\sum_{k=i}^{j-1}\frac{3}{2}\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i)\left(\tilde{\mathbf{a}}_k - \bar{\mathbf{b}}_i^a\right)^\wedge\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\Delta t^2
\end{aligned}
\tag{A.20}
$$

Repeating the same derivation of Section 1.1, it is possible to show that (A.20) can be computed incrementally, as new measurements arrive.

2 IMU Factors: Residual Errors and Jacobians

In this section we provide analytic expressions for the Jacobian matrices of the residual errors introduced in Section V-D of [4]. We start from the expression of the residual errors for the preintegrated IMU measurements (*c.f.*, Eq. (37)):

$$
\begin{aligned}
\mathbf{r}_{\Delta\mathtt{R}_{ij}} &\doteq \mathrm{Log}\left(\left(\Delta\tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\mathrm{Exp}\left(\frac{\partial\Delta\bar{\mathtt{R}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}^g\right)\right)^{\mathsf{T}}\mathtt{R}_i^{\mathsf{T}}\mathtt{R}_j\right) \\
\mathbf{r}_{\Delta\mathbf{v}_{ij}} &\doteq \mathtt{R}_i^{\mathsf{T}}\left(\mathbf{v}_j - \mathbf{v}_i - \mathbf{g}\Delta t_{ij}\right) - \left[\Delta\tilde{\mathbf{v}}_{ij}(\bar{\mathbf{b}}_i^g, \bar{\mathbf{b}}_i^a) + \frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}^g + \frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^a}\delta\mathbf{b}^a\right] \\
\mathbf{r}_{\Delta\mathbf{p}_{ij}} &\doteq \mathtt{R}_i^{\mathsf{T}}\left(\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i\Delta t_{ij} - \frac{1}{2}\mathbf{g}\Delta t_{ij}^2\right) - \left[\Delta\tilde{\mathbf{p}}_{ij}(\bar{\mathbf{b}}_i^g, \bar{\mathbf{b}}_i^a) + \frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^g}\delta\mathbf{b}^g + \frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}_a}\delta\mathbf{b}^a\right]
\end{aligned}
\tag{A.21}
$$

"Lifting" the cost function consists in substituting the following retraction:

$$
\mathbf{p}_i \leftarrow \mathbf{p}_i + \mathtt{R}_i\delta\mathbf{p}_i, \qquad \mathtt{R}_i \leftarrow \mathtt{R}_i\ \mathrm{Exp}(\delta\boldsymbol{\phi}_i), \qquad \mathbf{p}_j \leftarrow \mathbf{p}_j + \mathtt{R}_j\delta\mathbf{p}_j, \qquad \mathtt{R}_j \leftarrow \mathtt{R}_j\ \mathrm{Exp}(\delta\boldsymbol{\phi}_j), \tag{A.22}
$$

while, since velocity and biases already live in a vector space the corresponding retraction reduces to:

$$
\mathbf{v}_i \leftarrow \mathbf{v}_i + \delta\mathbf{v}_i, \qquad \mathbf{v}_j \leftarrow \mathbf{v}_j + \delta\mathbf{v}_i, \qquad \delta\mathbf{b}_i^g \leftarrow \delta\mathbf{b}_i^g + \tilde{\delta}\mathbf{b}_{g_i}, \qquad \delta\mathbf{b}_i^a \leftarrow \delta\mathbf{b}_i^a + \tilde{\delta}\mathbf{b}_{ai}, \tag{A.23}
$$

The process of lifting makes the residual errors a function defined on a vector space, on which it is easy to compute Jacobians. We derive the Jacobians w.r.t. the vectors $\delta\boldsymbol{\phi}_i, \delta\mathbf{p}_i, \delta\mathbf{v}_i, \delta\boldsymbol{\phi}_j, \delta\mathbf{p}_j, \delta\mathbf{v}_j, \tilde{\delta}\mathbf{b}_{g_i}, \tilde{\delta}\mathbf{b}_{ai}$ in the following sections.

2.1 Jacobians of $\mathbf{r}_{\Delta \mathbf{p}_{ij}}$

$$\begin{aligned}\mathbf{r}_{\Delta \mathbf{p}_{ij}}(\mathbf{p}_i + \mathtt{R}_i \delta \mathbf{p}_i) &= \mathtt{R}_i^\mathsf{T}\left(\mathbf{p}_j - \mathbf{p}_i - \mathtt{R}_i \delta \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \frac{1}{2}\mathbf{g}\Delta t_{ij}^2\right) - \left[\Delta\tilde{\mathbf{p}}_{ij} + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &= \mathbf{r}_{\Delta \mathbf{p}_{ij}}(\mathbf{p}_i) + (-\mathbf{I}_{3\times 1})\delta \mathbf{p}_i\end{aligned} \tag{A.24}$$

$$\begin{aligned}\mathbf{r}_{\Delta \mathbf{p}_{ij}}(\mathbf{p}_j + \mathtt{R}_j \delta \mathbf{p}_j) &= \mathtt{R}_i^\mathsf{T}\left(\mathbf{p}_j + \mathtt{R}_j \delta \mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \frac{1}{2}\mathbf{g}\Delta t_{ij}^2\right) - \left[\Delta\tilde{\mathbf{p}}_{ij} + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &= \mathbf{r}_{\Delta \mathbf{p}_{ij}}(\mathbf{p}_j) + (\mathtt{R}_i^\mathsf{T}\mathtt{R}_j)\delta \mathbf{p}_j\end{aligned} \tag{A.25}$$

$$\begin{aligned}\mathbf{r}_{\Delta \mathbf{p}_{ij}}(\mathbf{v}_i + \delta \mathbf{v}_i) &= \mathtt{R}_i^\mathsf{T}\left(\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \delta\mathbf{v}_i \Delta t_{ij} - \frac{1}{2}\mathbf{g}\Delta t_{ij}^2\right) - \left[\Delta\tilde{\mathbf{p}}_{ij} + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &= \mathbf{r}_{\Delta \mathbf{p}_{ij}}(\mathbf{v}_i) + (-\mathtt{R}_i^\mathsf{T}\Delta t_{ij})\delta \mathbf{v}_i\end{aligned} \tag{A.26}$$

$$\begin{aligned}\mathbf{r}_{\Delta \mathbf{p}_{ij}}(\mathtt{R}_i \ \mathrm{Exp}(\delta\boldsymbol{\phi}_i)) &= (\mathtt{R}_i \mathrm{Exp}(\delta\boldsymbol{\phi}_i))^\mathsf{T}\left(\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \frac{1}{2}\mathbf{g}\Delta t_{ij}^2\right) - \left[\Delta\tilde{\mathbf{p}}_{ij} + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &\overset{\text{Eq. (4)}}{\simeq} (\mathbf{I} - \delta\boldsymbol{\phi}_i^\wedge)\mathtt{R}_i^\mathsf{T}\left(\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \frac{1}{2}\mathbf{g}\Delta t_{ij}^2\right) - \left[\Delta\tilde{\mathbf{p}}_{ij} + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &= \mathbf{r}_{\Delta \mathbf{p}_{ij}}(\mathtt{R}_i) + \left(\mathtt{R}_i^\mathsf{T}\left(\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \frac{1}{2}\mathbf{g}\Delta t_{ij}^2\right)\right)^\wedge \delta\boldsymbol{\phi}_i \quad (\text{we used } a^\wedge b = -b^\wedge a)\end{aligned} \tag{A.27}$$

In summary, the jacobians of $\mathbf{r}_{\Delta \mathbf{p}_{ij}}$ are:

$$\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \boldsymbol{\phi}_i} &= \left(\mathtt{R}_i^\mathsf{T}\left(\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \tfrac{1}{2}\mathbf{g}\Delta t_{ij}^2\right)\right)^\wedge \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{p}_i} &= -\mathbf{I}_{3\times 1} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{v}_i} &= -\mathtt{R}_i^\mathsf{T}\Delta t_{ij} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \boldsymbol{\phi}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathtt{R}_i^\mathsf{T}\mathtt{R}_j \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \tilde{\mathbf{b}}_a} &= -\frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^a} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \tilde{\mathbf{b}}_g} &= -\frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}_i^g}
\end{aligned}$$

2.2 Jacobians of $\mathbf{r}_{\Delta \mathbf{v}_{ij}}$

$$\begin{aligned}\mathbf{r}_{\Delta \mathbf{v}_{ij}}(\mathbf{v}_i + \delta \mathbf{v}_i) &= \mathtt{R}_i^\mathsf{T}\left(\mathbf{v}_j - \mathbf{v}_i - \delta\mathbf{v}_i - \mathbf{g}\Delta t_{ij}\right) - \left[\Delta\tilde{\mathbf{v}}_{ij} + \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &= \mathbf{r}_{\Delta \mathbf{v}}(\mathbf{v}_i) - \mathtt{R}_i^\mathsf{T}\delta \mathbf{v}_i\end{aligned} \tag{A.28}$$

$$\begin{aligned}\mathbf{r}_{\Delta \mathbf{v}_{ij}}(\mathbf{v}_j + \delta \mathbf{v}_j) &= \mathtt{R}_i^\mathsf{T}\left(\mathbf{v}_j + \delta\mathbf{v}_j - \mathbf{v}_i - \mathbf{g}\Delta t_{ij}\right) - \left[\Delta\tilde{\mathbf{v}}_{ij} + \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &= \mathbf{r}_{\Delta \mathbf{v}}(\mathbf{v}_j) + \mathtt{R}_i^\mathsf{T}\delta \mathbf{v}_j\end{aligned} \tag{A.29}$$

$$\begin{aligned}\mathbf{r}_{\Delta \mathbf{v}_{ij}}(\mathtt{R}_i \ \mathrm{Exp}(\delta\boldsymbol{\phi}_i)) &= (\mathtt{R}_i \ \mathrm{Exp}(\delta\boldsymbol{\phi}_i))^\mathsf{T}\left(\mathbf{v}_j - \mathbf{v}_i - \mathbf{g}\Delta t_{ij}\right) - \left[\Delta\tilde{\mathbf{v}}_{ij} + \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &\overset{\text{Eq. (4)}}{\simeq} (\mathbf{I} - \delta\boldsymbol{\phi}_i^\wedge)\mathtt{R}_i^\mathsf{T}\left(\mathbf{v}_j - \mathbf{v}_i - \mathbf{g}\Delta t_{ij}\right) - \left[\Delta\tilde{\mathbf{v}}_{ij} + \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^g}\delta \mathbf{b}_i^g + \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^a}\delta \mathbf{b}_i^a\right] \\ &= \mathbf{r}_{\Delta \mathbf{v}}(\mathtt{R}_i) + \left(\mathtt{R}_i^\mathsf{T}\left(\mathbf{v}_j - \mathbf{v}_i - \mathbf{g}\Delta t_{ij}\right)\right)^\wedge \delta\boldsymbol{\phi}_i \quad (\text{we used } a^\wedge b = -b^\wedge a)\end{aligned} \tag{A.30}$$

In summary, the jacobians of $\mathbf{r}_{\Delta \mathbf{v}_{ij}}$ are:

$$\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \boldsymbol{\phi}_i} &= \left(\mathtt{R}_i^\mathsf{T}\left(\mathbf{v}_j - \mathbf{v}_i - \mathbf{g}\Delta t_{ij}\right)\right)^\wedge \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{p}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{v}_i} &= -\mathtt{R}_i^\mathsf{T} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \boldsymbol{\phi}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathtt{R}_i^\mathsf{T} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \tilde{\delta} \mathbf{b}_a} &= -\frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^a} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \tilde{\delta} \mathbf{b}_g} &= -\frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}_i^g}
\end{aligned}$$

2.3 Jacobians of $\mathbf{r}_{\Delta \mathtt{R}_{ij}}$

$$\begin{aligned}
\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\mathtt{R}_i \operatorname{Exp}(\delta \boldsymbol{\phi}_i)) &= \operatorname{Log}\left(\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}^g\right)\right)^\mathsf{T}\left(\mathtt{R}_i \operatorname{Exp}(\delta \boldsymbol{\phi}_i)\right)^\mathsf{T}\mathtt{R}_j\right) \\
&= \operatorname{Log}\left(\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}^g\right)\right)^\mathsf{T}\operatorname{Exp}(-\delta \boldsymbol{\phi}_i)\mathtt{R}_i^\mathsf{T}\mathtt{R}_j\right) \\
&\overset{\text{Eq.(11)}}{=} \operatorname{Log}\left(\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}^g\right)\right)^\mathsf{T}\mathtt{R}_i^\mathsf{T}\mathtt{R}_j\operatorname{Exp}(-\mathtt{R}_j^\mathsf{T}\mathtt{R}_i\delta \boldsymbol{\phi}_i)\right) \\
&\overset{\text{Eq.(9)}}{\simeq} \operatorname{Log}\left(\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}^g\right)\right)^\mathsf{T}\mathtt{R}_i^\mathsf{T}\mathtt{R}_j\right) \\
&+ \mathtt{J}_r^{-1}\left(\operatorname{Log}\left(\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}^g\right)\right)^\mathsf{T}\mathtt{R}_i^\mathsf{T}\mathtt{R}_j\right)\right)\left(-\mathtt{R}_j^\mathsf{T}\mathtt{R}_i\delta \boldsymbol{\phi}_i\right) \\
&= \mathbf{r}_{\Delta \mathtt{R}}(\mathtt{R}_i) - \mathtt{J}_r^{-1}(\mathbf{r}_{\Delta \mathtt{R}}(\mathtt{R}_i))\mathtt{R}_j^\mathsf{T}\mathtt{R}_i\delta \boldsymbol{\phi}_i
\end{aligned} \tag{A.31}$$

$$\begin{aligned}
\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\mathtt{R}_j \operatorname{Exp}(\delta \boldsymbol{\phi}_j)) &= \operatorname{Log}\left(\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}^g\right)\right)^\mathsf{T}\mathtt{R}_i^\mathsf{T}\left(\mathtt{R}_j \operatorname{Exp}(\delta \boldsymbol{\phi}_j)\right)\right) \\
&\overset{\text{Eq.(9)}}{\simeq} \mathbf{r}_{\Delta \mathtt{R}}(\mathtt{R}_j) + \mathtt{J}_r^{-1}(\mathbf{r}_{\Delta \mathtt{R}}(\mathtt{R}_j))\delta \boldsymbol{\phi}_j
\end{aligned} \tag{A.32}$$

$$\begin{aligned}
\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g + \tilde{\delta}\mathbf{b}_{g_i}) &= \operatorname{Log}\left(\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\left(\delta \mathbf{b}_i^g + \tilde{\delta}\mathbf{b}_{g_i}\right)\right)\right)^\mathsf{T}\mathtt{R}_i^\mathsf{T}\mathtt{R}_j\right) \\
&\overset{\text{Eq.(7)}}{\simeq} \operatorname{Log}\left(\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g\right)\operatorname{Exp}\left(\mathtt{J}_r\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g\right)\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\tilde{\delta}\mathbf{b}_{g_i}\right)\right)^\mathsf{T}\mathtt{R}_i^\mathsf{T}\mathtt{R}_j\right) \\
&= \operatorname{Log}\left(\operatorname{Exp}\left(-\mathtt{J}_r\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g\right)\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\tilde{\delta}\mathbf{b}_{g_i}\right)\left(\Delta \tilde{\mathtt{R}}_{ij}(\bar{\mathbf{b}}_i^g)\operatorname{Exp}\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g\right)\right)^\mathsf{T}\mathtt{R}_i^\mathsf{T}\mathtt{R}_j\right) \\
&= \operatorname{Log}\left(\operatorname{Exp}\left(-\mathtt{J}_r\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g\right)\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\tilde{\delta}\mathbf{b}_{g_i}\right)\operatorname{Exp}\left(\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g)\right)\right) \\
&\overset{\text{Eq.(11)}}{=} \operatorname{Log}\left(\operatorname{Exp}\left(\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g)\right)\operatorname{Exp}\left(-\operatorname{Exp}\left(\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g)\right)^\mathsf{T}\mathtt{J}_r\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g\right)\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\tilde{\delta}\mathbf{b}_{g_i}\right)\right) \\
&\overset{\text{Eq.(9)}}{\simeq} \mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g) - \mathtt{J}_r^{-1}\left(\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g)\right)\operatorname{Exp}\left(\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g)\right)^\mathsf{T}\mathtt{J}_r\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g\right)\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\tilde{\delta}\mathbf{b}_{g_i}
\end{aligned} \tag{A.33}$$

In summary, the jacobians of $\mathbf{r}_{\Delta \mathtt{R}_{ij}}$ are:

$$
\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathtt{R}_{ij}}}{\partial \delta \boldsymbol{\phi}_i} &= -\mathtt{J}_r^{-1}(\mathbf{r}_{\Delta \mathtt{R}}(\mathtt{R}_i))\mathtt{R}_j^{\mathsf{T}}\mathtt{R}_i \\
\frac{\partial \mathbf{r}_{\Delta \mathtt{R}_{ij}}}{\partial \delta \mathbf{p}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathtt{R}_{ij}}}{\partial \delta \mathbf{v}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathtt{R}_{ij}}}{\partial \delta \boldsymbol{\phi}_j} &= \mathtt{J}_r^{-1}(\mathbf{r}_{\Delta \mathtt{R}}(\mathtt{R}_j)) \\
\frac{\partial \mathbf{r}_{\Delta \mathtt{R}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathtt{R}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathtt{R}_{ij}}}{\partial \delta \tilde{\mathbf{b}}_a} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathtt{R}_{ij}}}{\partial \delta \tilde{\mathbf{b}}_g} &= -\mathtt{J}_r^{-1}\left(\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g)\right) \mathrm{Exp}\left(\mathbf{r}_{\Delta \mathtt{R}_{ij}}(\delta \mathbf{b}_i^g)\right)^{\mathsf{T}} \mathtt{J}_r\left(\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g\right)\frac{\partial \Delta \bar{\mathtt{R}}_{ij}}{\partial \mathbf{b}^g}
\end{aligned}
$$

3 Vision factors: Schur Complement and Null Space Projection

Let us start from the linearized vision factors, given in Eq. (44) of the main document:

$$\sum_{l=1}^{L} \sum_{i \in \mathcal{X}(l)} \|\mathbf{F}_{il}\delta \mathbf{T}_i + \mathbf{E}_{il}\delta \rho_l - \mathbf{b}_{il}\|^2 \tag{A.34}$$

where $\delta \mathbf{T}_i \doteq [\delta \boldsymbol{\phi}_i \; \delta \mathbf{p}_i]^{\mathsf{T}} \in \mathbb{R}^6$ is a perturbation w.r.t. the linearization point of the pose at keyframe i and $\delta \rho_l$ is a perturbation w.r.t. the linerization point of landmark l. The vector $\mathbf{b}_{il} \in \mathbb{R}^2$ is the residual error at the linearization point. For brevity, we do not enter in the details of the Jacobians $\mathbf{F}_{il} \in \mathbb{R}^{2\times 6}, \mathbf{E}_{il} \in \mathbb{R}^{2\times 3}$.

Now, as done in the main document, we denote with $\delta \mathbf{T}_{\mathcal{X}(l)} \in \mathbb{R}^{6n}$ the vector stacking the perturbations $\delta \mathbf{T}_i$ for each of the n_l cameras observing landmark l. With this notation (A.34) can be written in matrix

$$\sum_{l=1}^{L} \|\mathbf{F}_l \; \delta \mathbf{T}_{\mathcal{X}(l)} + \mathbf{E}_l \; \delta \rho_l - \mathbf{b}_l\|^2 \tag{A.35}$$

where:

$$\mathbf{F}_l \doteq \begin{bmatrix} \ddots & \mathbf{0}_{2\times 6} & \dots & \mathbf{0}_{2\times 6} \\ \mathbf{0}_{2\times 6} & \mathbf{F}_{il} & \dots & \vdots \\ \vdots & \dots & \ddots & \mathbf{0}_{2\times 6} \\ \mathbf{0}_{2\times 6} & \dots & \mathbf{0}_{2\times 6} & \ddots \end{bmatrix} \in \mathbb{R}^{2n_l \times 6n_l}, \qquad \mathbf{E}_l \doteq \begin{bmatrix} \dots \\ \mathbf{E}_{il} \\ \dots \\ \dots \end{bmatrix} \in \mathbb{R}^{2n_l \times 3}, \qquad \mathbf{b}_l \doteq \begin{bmatrix} \vdots \\ \mathbf{b}_{il} \\ \vdots \\ \vdots \end{bmatrix} \in \mathbb{R}^{2n_l}. \tag{A.36}$$

Since a landmark l appears in a single term of the sum (A.35), for any given choice of the pose perturbation $\delta \mathbf{T}_{\mathcal{X}(l)}$, the landmark perturbation $\delta \rho_l$ that minimizes the quadratic cost $\|\mathbf{F}_l \; \delta \mathbf{T}_{\mathcal{X}(l)} + \mathbf{E}_l \; \delta \rho_l - \mathbf{b}_l\|^2$ is:

$$\delta \rho_l = -(\mathbf{E}_l^{\mathsf{T}}\mathbf{E}_l)^{-1}\mathbf{E}_l^{\mathsf{T}}(\mathbf{F}_l \; \delta \mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l) \tag{A.37}$$

Substituting (A.37) back into (A.35) we can *eliminate* the landmarks from the optimization problem:

$$\sum_{l=1}^{L} \|(\mathbf{I} - \mathbf{E}_l(\mathbf{E}_l^{\mathsf{T}}\mathbf{E}_l)^{-1}\mathbf{E}_l^{\mathsf{T}})\left(\mathbf{F}_l \; \delta \mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right)\|^2 \tag{A.38}$$

which corresponds to eq. (46) of the main document. The structureless factors (A.38) only involve poses and allow to perform the optimization disregarding landmark positions.

Computation can be further improved by the following linear algebra considerations. First, we note that $\mathbf{Q} \doteq (\mathbf{I} - \mathbf{E}_l(\mathbf{E}_l^{\mathsf{T}}\mathbf{E}_l)^{-1}\mathbf{E}_l^{\mathsf{T}}) \in \mathbb{R}^{2n_l \times 2n_l}$ is an *orthogonal projector* of $\mathbf{E}_l$. Roughly speaking, $\mathbf{Q}$ projects any vector in $\mathbb{R}^{2n_l}$ to the null space of the matrix $\mathbf{E}_l$. Moreover, any basis $\mathbf{E}_l^{\perp} \in \mathbb{R}^{2n_l \times 2n_l - 3}$ of the null space of $\mathbf{E}_l$ satisfies the following relation [5]:

$$\mathbf{E}_l^{\perp}\left((\mathbf{E}_l^{\perp})^{\mathsf{T}}\mathbf{E}_l^{\perp}\right)^{-1}(\mathbf{E}_l^{\perp})^{\mathsf{T}} = \mathbf{I} - \mathbf{E}_l(\mathbf{E}_l^{\mathsf{T}}\mathbf{E}_l)^{-1}\mathbf{E}_l^{\mathsf{T}} \tag{A.39}$$

A basis for the null space can be easily computed from $\mathbf{E}_l$ using SVD. Such basis is *unitary*, i.e., satisfies $(\mathbf{E}_l^\perp)^\mathsf{T}\mathbf{E}_l^\perp = \mathbf{I}$. Substituting (A.39) into (A.38), and recalling that $\mathbf{E}_l^\perp$ is a unitary matrix, we obtain:

$$\sum_{l=1}^{L} \|\mathbf{E}_l^\perp(\mathbf{E}_l^\perp)^\mathsf{T}\left(\mathbf{F}_l\,\delta\mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right)\|^2 = \sum_{l=1}^{L}\left(\mathbf{E}_l^\perp(\mathbf{E}_l^\perp)^\mathsf{T}\left(\mathbf{F}_l\,\delta\mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right)\right)^\mathsf{T}\left(\mathbf{E}_l^\perp(\mathbf{E}_l^\perp)^\mathsf{T}\left(\mathbf{F}_l\,\delta\mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right)\right)$$
$$= \sum_{l=1}^{L}\left(\mathbf{F}_l\,\delta\mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right)^\mathsf{T}\mathbf{E}_l^\perp(\mathbf{E}_l^\perp)^\mathsf{T}\mathbf{E}_l^\perp(\mathbf{E}_l^\perp)^\mathsf{T}\left(\mathbf{F}_l\,\delta\mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right)$$
$$= \sum_{l=1}^{L}\left(\mathbf{F}_l\,\delta\mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right)^\mathsf{T}\mathbf{E}_l^\perp(\mathbf{E}_l^\perp)^\mathsf{T}\left(\mathbf{F}_l\,\delta\mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right) = \sum_{l=1}^{L}\|(\mathbf{E}_l^\perp)^\mathsf{T}\left(\mathbf{F}_l\,\delta\mathbf{T}_{\mathcal{X}(l)} - \mathbf{b}_l\right)\|^2 \tag{A.40}$$

which is an alternative representation of our vision factors (A.38), and is usually preferable from a computational standpoint. A similar null space projection is used in [6] within a Kalman filter architecture, while a factor graph view on Schur complement is given in [2].

4 Angular Velocity and Right Jacobians for SO(3)

The right Jacobian matrix $\mathtt{J}_r(\boldsymbol{\phi})$ (also called body Jacobian [7]) relates rates of change in the parameter vector $\dot{\boldsymbol{\phi}}$ to the instantaneous body angular velocity ${}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}$:

$${}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}} = \mathtt{J}_r(\boldsymbol{\phi})\;\dot{\boldsymbol{\phi}}. \tag{A.41}$$

A closed-form expression of the right Jacobian is given in [3]:

$$\mathtt{J}_r(\boldsymbol{\phi}) = \mathbf{I} - \frac{1-\cos(\|\boldsymbol{\phi}\|)}{\|\boldsymbol{\phi}\|^2}\boldsymbol{\phi}^\wedge + \frac{\|\boldsymbol{\phi}\| - \sin(\|\boldsymbol{\phi}\|)}{\|\boldsymbol{\phi}^3\|}(\boldsymbol{\phi}^\wedge)^2. \tag{A.42}$$

Note that the Jacobian becomes the identity matrix for $\boldsymbol{\phi} = \mathbf{0}$.

Consider a direct cosine matrix $\mathtt{R}_{\mathrm{WB}}(\boldsymbol{\phi}) \in \mathrm{SO}(3)$, that rotates a point from body coordinates B to world coordinates W, and that is parametrized by the rotation vector $\boldsymbol{\phi}$. The relation between angular velocity and the derivative of a rotation matrix is [7]

$$\mathtt{R}_{\mathrm{WB}}^\mathsf{T}\;\dot{\mathtt{R}}_{\mathrm{WB}} = {}_{\mathrm{B}}\boldsymbol{\omega}_{\mathrm{WB}}^\wedge. \tag{A.43}$$

Hence, using (A.41) we can write the derivative of a rotation matrix at $\boldsymbol{\phi}$:

$$\dot{\mathtt{R}}_{\mathrm{WB}}(\boldsymbol{\phi}) = \mathtt{R}_{\mathrm{WB}}(\boldsymbol{\phi})\;(\mathtt{J}_r(\boldsymbol{\phi})\;\dot{\boldsymbol{\phi}})^\wedge. \tag{A.44}$$

Given a multiplicative perturbation $\mathrm{Exp}(\delta\boldsymbol{\psi})$ on the right hand side of an element of the group SO(3), we may ask what is the equivalent additive perturbation in the tangent space $\delta\boldsymbol{\phi} \in \mathfrak{so}(3)$ that results in the same compound rotation:

$$\mathrm{Exp}(\boldsymbol{\phi})\mathrm{Exp}(\delta\boldsymbol{\psi}) = \mathrm{Exp}(\boldsymbol{\phi} + \delta\boldsymbol{\phi}). \tag{A.45}$$

Computing the derivative with respect to the increments on both sides, using (A.44), and assuming that the increments are small, we find

$$\delta\boldsymbol{\psi} \approx \mathtt{J}_r(\boldsymbol{\phi})\delta\boldsymbol{\phi}, \tag{A.46}$$

leading to:

$$\mathrm{Exp}(\boldsymbol{\phi} + \delta\boldsymbol{\phi}) \approx \mathrm{Exp}(\boldsymbol{\phi})\;\mathrm{Exp}(\mathtt{J}_r(\boldsymbol{\phi})\delta\boldsymbol{\phi}). \tag{A.47}$$

A similar first-order approximation holds for the logarithm:

$$\mathrm{Log}\big(\;\mathrm{Exp}(\boldsymbol{\phi})\;\mathrm{Exp}(\delta\boldsymbol{\phi})\;\big) \approx \boldsymbol{\phi} + \mathtt{J}_r^{-1}(\boldsymbol{\phi})\delta\boldsymbol{\phi}. \tag{A.48}$$

This property follows directly from the *Baker–Campbell–Hausdorff* (BCH) formula under the assumption that $\delta\boldsymbol{\phi}$ is small [1]. An explicit expression for the inverse of the right Jacobian is given in [3]:

$$\mathtt{J}_r^{-1}(\boldsymbol{\phi}) = \mathbf{I} + \frac{1}{2}\boldsymbol{\phi}^\wedge + \left(\frac{1}{\|\boldsymbol{\phi}\|^2} + \frac{1+\cos(\|\boldsymbol{\phi}\|)}{2\|\boldsymbol{\phi}\|\sin(\|\boldsymbol{\phi}\|)}\right)(\boldsymbol{\phi}^\wedge)^2.$$

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