Discretization of First-order Gauss-Markov Processes

Luca Carlone
August 24, 2013

A Gauss-Markov process is a stochastic process that satisfies the Markov property and whose random variables follow a Normal distribution. We start by considering the following first-order Gauss-Markov process in continuous time:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{\omega}_c(t) \tag{1}$$

where $\boldsymbol{\omega}_c$ is a random vector having zero mean $\mathbb{E}[\boldsymbol{\omega}_c(t)] = \boldsymbol{0}$; moreover, we assume $\mathbb{E}[\boldsymbol{\omega}_c(t)\boldsymbol{\omega}_c(\tau)^{\mathsf{T}}] = \boldsymbol{Q}_c\delta(t-\tau)$, where $\delta(\cdot)$ is the delta function and $\boldsymbol{Q}_c$ is a given matrix (the *spectral density* matrix).

In this short document we show how to discretize the continuous-time equation (1). Assume we discretize the continuous-time line into an infinite set of times $t_0, t_1, \ldots$. Considering a generic time interval $[t_k, t_{k+1}]$, we can integrate both members of equation (1) obtaining:

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{x}(t_k) + \int_{t_k}^{t_{k+1}} [\boldsymbol{A}\boldsymbol{x}(\tau) + \boldsymbol{B}\boldsymbol{\omega}_c(\tau)]\, \mathrm{d}\tau = \boldsymbol{\Phi}(t_{k+1}, t_k)\boldsymbol{x}(t_k) + \int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \tau)\boldsymbol{B}\boldsymbol{\omega}_c(\tau)\mathrm{d}\tau \tag{2}$$

where $\boldsymbol{\Phi}(\cdot,\cdot)$ is the state transition matrix. Since our system is time-invariant, we have $\boldsymbol{\Phi}(t, t_0) = e^{\boldsymbol{A}(t-t_0)}$. We can then rewrite the previous equation as:

$$\boldsymbol{x}(t_{k+1}) = e^{\boldsymbol{A}(t_{k+1}-t_k)}\boldsymbol{x}(t_k) + \boldsymbol{\omega}_d(t_k) \tag{3}$$

where $\boldsymbol{\omega}_d(t_k) = \int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \tau)\boldsymbol{B}\boldsymbol{\omega}_c(\tau)\mathrm{d}\tau$. We can then compute the mean and covariance for $\boldsymbol{\omega}_d$ as follows:

$$\mathbb{E}[\boldsymbol{\omega}_d(t_k)] = \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \tau)\boldsymbol{B}\boldsymbol{\omega}_c(\tau)\mathrm{d}\tau\right] = \int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \tau)\boldsymbol{B}\mathbb{E}\left[\boldsymbol{\omega}_c(\tau)\right]\mathrm{d}\tau = \boldsymbol{0} \tag{4}$$

$$\mathbb{E}[\boldsymbol{\omega}_d(t_k)\boldsymbol{\omega}_d(t_k)^{\mathsf{T}}] = \mathbb{E}\left[\left(\int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \tau)\boldsymbol{B}\boldsymbol{\omega}_c(\tau)\mathrm{d}\tau\right)\left(\int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \alpha)\boldsymbol{B}\boldsymbol{\omega}_c(\alpha)\mathrm{d}\alpha\right)^{\mathsf{T}}\right] = \tag{5}$$

$$= \mathbb{E}\left[\iint_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \tau)\boldsymbol{B}\boldsymbol{\omega}_c(\tau)\boldsymbol{\omega}_c(\alpha)^{\mathsf{T}}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{\Phi}(t_{k+1}, \alpha)^{\mathsf{T}}\mathrm{d}\tau\mathrm{d}\alpha\right] = \tag{6}$$

$$= \iint_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1}, \tau)\boldsymbol{B}\mathbb{E}\left[\boldsymbol{\omega}_c(\tau)\boldsymbol{\omega}_c(\alpha)^{\mathsf{T}}\right]\boldsymbol{B}^{\mathsf{T}}\boldsymbol{\Phi}(t_{k+1}, \alpha)^{\mathsf{T}}\mathrm{d}\tau\mathrm{d}\alpha \tag{7}$$

Now recalling that $\mathbb{E}[\boldsymbol{\omega}_c(t)\boldsymbol{\omega}_c(\tau)^{\mathsf{T}}] = \boldsymbol{Q}_c\delta(t-\tau)$ we further develop the previous expression:

$$\mathbb{E}[\boldsymbol{\omega}_d(t_k)\boldsymbol{\omega}_d(t_k)^\mathsf{T}] = \iint_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1},\tau)\boldsymbol{B}\boldsymbol{Q}_c\delta(\tau-\alpha)\boldsymbol{B}^\mathsf{T}\boldsymbol{\Phi}(t_{k+1},\alpha)^\mathsf{T}\mathrm{d}\tau\mathrm{d}\alpha = \tag{8}$$

$$= \int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1},\tau)\boldsymbol{B}\boldsymbol{Q}_c\boldsymbol{B}^\mathsf{T}\boldsymbol{\Phi}(t_{k+1},\tau)^\mathsf{T}\mathrm{d}\tau \tag{9}$$

Scalar case. For the case of a system with a single state, we have the simple continuous-time model:

$$\dot{x}(t) = -\frac{1}{T}x(t) + \omega_c(t) \tag{10}$$

in which x is a scalar; according to our derivation the previous system can be easily discretized as:

$$x(t_{k+1}) = e^{-\frac{1}{T}(t_{k+1}-t_k)}x(t_k) + \omega_d(t_k) \tag{11}$$

where $\omega_d(t_k)$ is a zero-mean random variable with variance:

$$\mathbb{E}[\omega_d(t_k)\omega_d(t_k)] = \int_{t_k}^{t_{k+1}} e^{-\frac{1}{T}(t_{k+1}-\tau)}Q_c e^{-\frac{1}{T}(t_{k+1}-\tau)}\mathrm{d}\tau = Q_c\int_{t_k}^{t_{k+1}} e^{-\frac{2}{T}(t_{k+1}-\tau)}\mathrm{d}\tau = Q_c e^{-\frac{2}{T}t_{k+1}}\int_{t_k}^{t_{k+1}} e^{\frac{2}{T}\tau}\mathrm{d}\tau \tag{12}$$

which can be written explicitly as:

$$\mathbb{E}[\omega_d(t_k)\omega_d(t_k)] = Q_c e^{-\frac{2}{T}t_{k+1}}\frac{T}{2}\left[e^{\frac{2}{T}t_{k+1}} - e^{\frac{2}{T}t_k}\right] = Q_c\frac{T}{2}\left[1 - e^{-\frac{2}{T}(t_{k+1}-t_k)}\right] \tag{13}$$

If we call Δt the sampling interval ($\Delta t = t_{k+1} - t_k$) the discrete noise covariance is then

$$Q_d \doteq \mathbb{E}[\omega_d(t_k)\omega_d(t_k)] = Q_c\frac{T}{2}\left[1 - e^{-\frac{2}{T}\Delta t}\right] \tag{14}$$

If Δt is small it is common to take a first-order approximation of the exponential function as:

$$\mathbb{E}[\omega_d(t_k)\omega_d(t_k)] \approx Q_c\frac{T}{2}\left[1 - (1 - \frac{2}{T}\Delta t)\right] = Q_c\Delta t \tag{15}$$

Given the derivation so far, we can also compute the covariance $P_{t_{k+1}}$ of $x(t_{k+1})$ in (10), given the covariance P_{t_k} of $x(t_k)$:

$$P_{t_{k+1}} = e^{-\frac{2}{T}\Delta t}P_{t_k} + Q_c\frac{T}{2}\left[1 - e^{-\frac{2}{T}\Delta t}\right] \tag{16}$$

Vector case. It is of practical interest the case in which the matrix $\boldsymbol{A}$ in (1) is diagonal, i.e. $\boldsymbol{A} = \mathrm{diag}\left(-\frac{1}{T_1},\ldots,-\frac{1}{T_n}\right)$, $\boldsymbol{B} = \mathbf{I}$ (identity matrix of suitable dimension), and $\boldsymbol{Q}_c$ is diagonal. In this case it is easy to verify that

$$\boldsymbol{Q}_d = \mathbb{E}[\boldsymbol{\omega}_d(t_k)\boldsymbol{\omega}_d(t_k)^\mathsf{T}] = \int_{t_k}^{t_{k+1}} \boldsymbol{\Phi}(t_{k+1},\tau)\boldsymbol{Q}_c\boldsymbol{\Phi}(t_{k+1},\tau)^\mathsf{T}\mathrm{d}\tau \tag{17}$$

is also diagonal and the i-th diagonal element is $Q_d^i = Q_c^i \frac{T_i}{2}\left[1 - e^{-\frac{2}{T_i}\Delta t}\right]$, where Q_c^i is the i-th diagonal entry in $\boldsymbol{Q}_c$. The predicted covariance $\boldsymbol{P}_{t_{k+1}}$, instead, preserves the structure:

$$\boldsymbol{P}_{t_{k+1}} = \left(e^{\boldsymbol{A}\Delta t}\right) \boldsymbol{P}_{t_k} \left(e^{\boldsymbol{A}\Delta t}\right)^{\mathsf{T}} + \boldsymbol{Q}_d \tag{18}$$

as the covariance $\boldsymbol{P}_{t_k}$ may contain off-diagonal terms. If $\boldsymbol{A} = \mathrm{diag}\left(-\frac{1}{T_1}, \ldots, -\frac{1}{T_n}\right)$, then $e^{\boldsymbol{A}\Delta t} = \mathrm{diag}\left(e^{-\frac{1}{T_1}\Delta t}, \ldots, e^{-\frac{1}{T_n}\Delta t}\right)$.