Symbols Defined

Term	Description	Notation
Intensity constant	Illumination at surface seen by pixel, calibrated at 1m for albedo 1 and reflection angle 0	I
Albedo	Reflectivity of surface, [0,1]	ρ
Reflection Angle	Angle between surface normal and incident angle	β
Wrap State	Number of phase wraps	S
Observed Brightness	Brightness of pixel	B
Observed phase	Phase of signal observed by sensor, is true phase modulo 2π	$\bar{\theta}$
Phase	Actual phase	$\theta = \bar{\theta} + 2\pi S$
Modulation Frequency	Modulation frequency of signal	f_m
Distance	Distance from sensor to surface	D

Formulation of Data Term

We are interested in finding the probability of a potential wrap state given the observations: the wrapped phase and the brightness.

The distance of the surface can be determined by the observed phase, the wrap state and the modulation frequency.

$$D = \frac{c \cdot (\bar{\theta} + 2\pi S)}{4\pi f_m} \tag{1}$$

It is important to note that we make the simplifying assumption that the light source and camera come from the same location, which is a reasonable approximation when the surface is far enough from the camera. With this assumption, we can say that the surface observed by the sensor at any given pixel is illuminated with the same power no matter what distance. This is because while the illumination density decreases with the square of the distance, the area observed in each pixel according increases with the square of the distance. The illumination for each pixel is pre-determined during camera calibration. This gives us an equation relating illumination with the observed brightness, albedo and reflection angle.

$$B = \frac{I \cdot \rho \cdot \cos\beta}{D^2} \tag{2}$$

We can put together equations (1) and (2) to directly relate brightness to the measured phase$\bar{\theta}$ and a proposed state S.

$$B = \frac{16\pi^2 f_m^2 \cdot I \cdot \rho \cdot \cos\beta \cdot 16}{c^2 \cdot (\bar{\theta} + 2\pi S)^2} \tag{3}$$

We now derive the probability of the wrap state, given the brightness and observed phase.

$$p(S|\bar{\theta}, B) = \frac{p(\mathrm{B}|\bar{\theta}, S)p(S|\bar{\theta})}{p(\mathrm{B}|\bar{\theta})} \tag{4}$$

We can start attacking the denominator by integrating over the space of Wrap States, or since they are discrete states then sum over the states.

$$p(S|\bar{\theta}, B) = \frac{p(\mathrm{B}|\bar{\theta}, S)p(S|\bar{\theta})}{p(\mathrm{B}|\bar{\theta})} = \frac{p(\mathrm{B}|\bar{\theta}, S)p(S|\bar{\theta})}{p(S|\bar{\theta}) \sum_S p(\mathrm{B}|\bar{\theta}, S)} \tag{5}$$

The observed wrapped phase does not influence the wrap state, and if we assume all states are equally likely then $p(S|\bar{\theta}) = 1/K$, where K is the number of possible wrap states.

Now we see that the denominator has a similar form as the first term of the numerator. For this term, we can integrate over the space of the albedo and reflection angle.

$$p(\mathrm{B}|\bar{\theta}, S) = \int_{\rho=0}^{1} p(\mathrm{B}|\bar{\theta}, S, \rho)p(\rho)d\rho = \int_{\rho=0}^{1} \int_{\beta=0}^{\pi/2} p(\mathrm{B}|\bar{\theta}, S, \rho, \beta)p(\rho|\beta)p(\beta)d\beta d\rho \tag{6}$$

Since the albedo and reflection angle are independent, then $p(\rho|\beta) = p(\rho)$. If we assume the distribution for the albedo is uniform, this term is a constant which will appear in the numerator and denominator and cancels out. The distribution for the reflection angle β is of the shape $2\sin(\beta)\cos(\beta)$.

$$p(S|\bar{\theta}, B) = \frac{\int_{\rho=0}^{1} \int_{\beta=0}^{\pi/2} p(\mathrm{B}|\bar{\theta}, S, \rho, \beta) \sin(\beta)\cos(\beta) d\beta d\rho}{\sum_S \int_{\rho=0}^{1} \int_{\beta=0}^{\pi/2} p(\mathrm{B}|\bar{\theta}, S, \rho, \beta) \sin(\beta)\cos(\beta)\, d\beta d\rho} \tag{7}$$

Given the wrap state, wrapped phase, albedo and reflection angle, then the brightness can be determined. So we can define the probability of brightness as follows:

$$p(\mathrm{B}|\bar{\theta}, S, \rho, \beta) = \begin{cases} 1 & if\ B = \dfrac{I \cdot \rho \cdot \cos\beta}{D^2} \\ 0 & otherwise \end{cases}$$

(8)

Given this, we can write the probability of brightness using a Dirac delta function. Also, since equation (1) relates the distance with measured phase and proposed state, we can state that $p(\mathrm{B}|\bar{\theta}, S, \rho, \beta) = p(\mathrm{B}|D, \rho, \beta)$.

$$p(S|\bar{\theta}, B) = \frac{\int_{\rho=0}^{1}\int_{\beta=0}^{\pi/2} \delta\left(B - \frac{I \cdot \rho \cdot \cos\beta}{D^2}\right)\sin(\beta)\cos(\beta)d\beta d\rho}{\sum_D \int_{\rho=0}^{1}\int_{\beta=0}^{\pi/2} \delta\left(B - \frac{I \cdot \rho \cdot \cos\beta}{D^2}\right)\sin(\beta)\cos(\beta)\,d\beta d\rho}$$

(9)

So when integrating over the albedo and reflection angle, the only non-zero part of the space is along the line: $\rho = \frac{B \cdot D^2}{I \cdot \cos\beta}$. With the uniform density at each point, the probability will be proportional to the length of the line over the proper bounds. Given that the maximum value for ρ is 1, then the maximum value for β is $\cos^{-1}\frac{B \cdot D^2}{I}$. Thus the line extends from $\beta = [0, \cos^{-1}\frac{B \cdot D^2}{I}]$. Note that if $B \cdot D^2 > I$, then there is no solution for that distance (this is the case that the surface is so bright that it must be somewhat near the camera).

The formula for the length of a line given a function f(x) is $s = \int_a^b \sqrt{1 + (f'(x))^2}dx$. The function we need to integrate is as follows:

$$\rho = \frac{B \cdot D^2}{I \cdot \cos\beta}$$

$$\frac{d\rho}{d\beta} = \frac{B \cdot D^2 \cdot \sin\beta}{I \cdot \cos^2\beta}$$

$$p(\mathrm{B}|\bar{\theta}, S, \rho, \beta) \propto \int_{\beta=0}^{\cos^{-1}\frac{B \cdot D^2}{I}} \sqrt{1 + \left(\frac{B \cdot D^2 \cdot \sin\beta}{I \cdot \cos^2\beta}\right)^2}\,d\beta$$

This is not something I can identify as being easily solved symbolically, and my TI-89 was also not able to do the job. However, it can certainly be solved numerically in software. It is merely a (complicated?) function of $\frac{B \cdot D^2}{I}$, where $D = \frac{c \cdot (\bar{\theta} + 2\pi S)}{4\pi f_m}$. So a lookup table could be precomputed and at runtime all that that is need to find the index is to evaluate $C_i \cdot B \cdot (\bar{\theta} + 2\pi S)^2$, where C_i is a constant term equal to $\frac{c^2}{I_i 16(\pi f_m)^2}$ which can be determined for each pixel i after calibration. To simplify the notation, we'll call the function that solves this integral $G(B, \bar{\theta}, S)$. Then we have:

$$p(S|\bar{\theta}, B) = \frac{p(\mathrm{B}|\bar{\theta}, S)p(S|\bar{\theta})}{p(\mathrm{B}|\bar{\theta})} = \frac{G(B, \bar{\theta}, S)}{K\sum_S G(B, \bar{\theta}, S)}$$