18.600: Lecture 37
Review: practice problems

Scott Sheffield

MIT
Eight athletic teams are ranked 1 through 8 after season one, and ranked 1 through 8 again after season two. Assume that each set of rankings is chosen uniformly from the set of 8! possible rankings and that the two rankings are independent. Let $N$ be the number of teams whose rank does not change from season one to season two. Let $N_+$ the number of teams whose rank improves by exactly two spots. Let $N_-$ be the number whose rank declines by exactly two spots. Compute the following:

$$E[N], E[N_+], E[N_-]$$

$$\text{Var}[N], \text{Var}[N_+]$$
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$E[N]$, $E[N_+]$, and $E[N_-]$
Expectation and variance

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- $\text{Var}[N]$
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- $E[N]$, $E[N_+]$, and $E[N_-]$
- $Var[N]$
- $Var[N_+]$
Let $N_i$ be 1 if team ranked $i$th first season remains $i$th second seasons. Then $E[N] = E[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1$. Similarly, $E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4$.
Let \( N_i \) be 1 if team ranked \( i \)th first season remains \( i \)th second season. Then \( E[N] = E[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1 \). Similarly, \( E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4 \).

\[
\text{Var}[N] = E[N^2] - E[N]^2 \quad \text{and} \quad E[N^2] = E[\sum_{i=1}^{8} \sum_{j=1}^{8} N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2.
\]
Let $N_i$ be 1 if team ranked $i$th first season remains $i$th second seasons. Then $E[N] = E[\sum_{i=1}^{8} N_i] = 8 \cdot \frac{1}{8} = 1$. Similarly, $E[N_+] = E[N_-] = 6 \cdot \frac{1}{8} = 3/4$.

$\text{Var}[N] = E[N^2] - E[N]^2$ and $E[N^2] = E[\sum_{i=1}^{8} \sum_{j=1}^{8} N_i N_j] = 8 \cdot \frac{1}{8} + 56 \cdot \frac{1}{56} = 2$.

$N_i^j$ be 1 if team ranked $i$th has rank improve to $(i - 2)$th for second seasons. Then $E[(N_+)^2] = E[\sum_{3=1}^{8} \sum_{3=1}^{8} N_i^i N_+^j] = 6 \cdot \frac{1}{8} + 30 \cdot \frac{1}{56} = 9/7$, so $\text{Var}[N_+] = 9/7 - (3/4)^2$. 
Roll ten dice. Find the conditional probability that there are exactly 4 ones, given that there are exactly 4 sixes.
Conditional distributions — answers

- Straightforward approach: \( P(A|B) = \frac{P(AB)}{P(B)} \).
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Numerator: is \( \frac{10}{4} \cdot \frac{6}{4} \cdot 4^2 \). Denominator is \( \frac{10}{4} \cdot \frac{5}{6} \cdot 6^{10} \).
Conditional distributions — answers

- Straightforward approach: \( P(A|B) = \frac{P(AB)}{P(B)} \).
- Numerator: is \( \binom{10}{4} \binom{6}{4} \frac{4^2}{6^{10}} \). Denominator is \( \binom{10}{4} \binom{5}{6} \frac{6^{10}}{6^{10}} \).
- Ratio is \( \binom{6}{4} \frac{4^2}{5^6} = \binom{6}{4} \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^2 \).
Conditional distributions — answers

- Straightforward approach: $P(A|B) = P(AB)/P(B)$.
- Numerator: is $\binom{10}{4}\binom{6}{4}4^2$. Denominator is $\binom{10}{4}5^6$.
- Ratio is $\binom{6}{4}4^2/5^6 = \binom{6}{4}(\frac{1}{5})^4(\frac{4}{5})^2$.
- Alternate solution: first condition on location of the 6’s and then use binomial theorem.
Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. Let $V$ be the length of time (in decades) until the first volcano eruption and $E$ the length of time (in decades) until the first earthquake. Compute the following:

- $\mathbb{E}[E^2]$ and $\text{Cov}[E, V]$. 
Suppose that in a certain town earthquakes are a Poisson point process, with an average of one per decade, and volcano eruptions are an independent Poisson point process, with an average of two per decade. The $V$ be length of time (in decades) until the first volcano eruption and $E$ the length of time (in decades) until the first earthquake. Compute the following:

- The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
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- $E[E^2]$ and $\text{Cov}[E, V]$.
- The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
- The probability density function of $\min\{E, V\}$. 

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**Poisson point processes**

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- $E[E^2]$ and $\text{Cov}[E, V]$.
- The expected number of calendar years, in the next decade (ten calendar years), that have no earthquakes and no volcano eruptions.
- The probability density function of $\min\{E, V\}$. 

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\[ E[E^2] = 2 \text{ and } \text{Cov}[E, V] = 0. \]
Poison point processes — answers

- $E[E^2] = 2$ and $\text{Cov}[E, V] = 0$.

- Probability of no earthquake or eruption in first year is $e^{-(2+1)\frac{1}{10}} = e^{-0.3}$ (see next part). Same for any year by memoryless property. Expected number of quake/eruption-free years is $10e^{-0.3} \approx 7.4$. 
\( E[E^2] = 2 \) and \( \text{Cov}[E, V] = 0. \)

Probability of no earthquake or eruption in first year is
\[ e^{-(2+1)\frac{1}{10}} = e^{-0.3} \] (see next part). Same for any year by memoryless property. Expected number of quake/eruption-free years is \( 10e^{-0.3} \approx 7.4. \)

Probability density function of \( \min\{E, V\} \) is \( 3e^{-(2+1)x} \) for \( x \geq 0 \), and 0 for \( x < 0. \).