18.600: Lecture 36
Risk Neutral Probability and Black-Scholes

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MIT
Black-Scholes

Call quotes and risk neutral probability
Black-Scholes

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Will spend time giving financial *interpretations* of the math.

Can interpret this lecture as a sophisticated story problem, illustrating an important application of the probability we have learned in this course (involving probability axioms, expectations, cumulative distribution functions, risk neutral probability, etc.)
If \( r \) is risk free interest rate, then by definition, price of a contract paying dollar at time \( T \) if \( A \) occurs is \( P_{RN}(A)e^{-rT} \).
Interest discounted asset prices as martingales

- If $r$ is risk free interest rate, then by definition, price of a contract paying dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.
- If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?

Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.
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- Generally, in absence of arbitrage, price of contract that pays $X$ at time $T$ should be $E_{RN}(X)e^{-rT}$ where $E_{RN}$ denotes expectation with respect to the risk neutral probability.
- Example: if a non-divided paying stock will be worth $X$ at time $T$, then its price today should be $E_{RN}(X)e^{-rT}$.
- Risk neutral probability basically defined so price of asset today is $e^{-rT}$ times risk neutral expectation of time $T$ price.
- In particular, the risk neutral expectation of tomorrow’s (interest discounted) stock price is today’s stock price.
- Implies fundamental theorem of asset pricing, which says discounted price $X(n)A(n)$ (where $A$ is a risk-free asset) is a martingale with respected to risk neutral probability.
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Implies **fundamental theorem of asset pricing**, which says discounted price $\frac{X(n)}{A(n)}$ (where $A$ is a risk-free asset) is a martingale with respected to risk neutral probability.
Black-Scholes: main assumption and conclusion

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Assumption:
the log of an asset price \( X \) at fixed future time \( T \) is a normal random variable (call it \( N \)) with some known variance (call it \( T\sigma^2 \)) and some mean (call it \( \mu \)) with respect to risk neutral probability.

Observation:
\[ N \text{ normal (} \mu, T\sigma^2 \text{)} \text{ implies } E[e^{N}] = e^{\mu + T\sigma^2/2}. \]

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If \( X_0 \) is the current price then
\[ X_0 = E[RN\{X\}e^{-rT}] = E[RN\{e^{N}\}e^{-rT}] = e^{\mu + (\sigma^2/2 - r)T}. \]

Observation:
This implies \( \mu = \log X_0 + (r - \sigma^2/2)T. \)

General Black-Scholes conclusion:
If \( g \) is any function then the price of a contract that pays \( g(X) \) at time \( T \) is
\[ E[RN\{g(X)\}e^{-rT}] = E[RN\{g(e^{N})\}e^{-rT}] \text{ where } N \text{ is normal with mean } \mu \text{ and variance } T\sigma^2. \]
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**General Black-Scholes conclusion:** If $g$ is any function then the price of a contract that pays $g(X)$ at time $T$ is

$$E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$$

where $N$ is normal with mean $\mu$ and variance $T\sigma^2$. 
Black-Scholes example: European call option

A **European call option** on a stock at **maturity date** $T$, **strike price** $K$, gives the holder the right (but not obligation) to purchase a share of stock for $K$ dollars at time $T$.

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Write this as

$$e^{-rT} E_{RN}[\max\{0, e^N - K\}] = e^{-rT} E_{RN}[(e^N - K)1_{N \geq \log K}]$$

$$= \frac{e^{-rT}}{\sigma \sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) \, dx.$$
The famous formula

Let $T$ be time to maturity, $X_0$ current price of underlying asset, $K$ strike price, $r$ risk free interest rate, $\sigma$ the volatility.
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Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function $\Phi$. 

Price of European call is $\Phi(d_1) X_0 - \Phi(d_2) Ke^{-rT}$ where $d_1 = \ln(X_0/K) + (r + \sigma^2/2)T/\sigma\sqrt{T}$ and $d_2 = \ln(X_0/K) + (r - \sigma^2/2)T/\sigma\sqrt{T}$. 

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Outline

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If $C(K)$ is price of European call with strike price $K$ and $f = f_X$ is risk neutral probability density function for $X$ at time $T$, then $C(K) = e^{-rT} \int_{-\infty}^{\infty} f(x) \max\{0, x - K\} \, dx$. 

Differentiating under the integral, we find that $e^{rT} C'(K) = \int f(x) (-1) 1_{x > K} \, dx = -P_{RN}\{X > K\} = F_X(K) - 1$, and $e^{rT} C''(K) = f(K)$. 

We can look up $C(K)$ for a given stock symbol (say GOOG) and expiration time $T$ at cboe.com and work out approximately what $F_X$ and hence $f_X$ must be.

Try this out for near term option (so $e^{rT}$ is essentially one).
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Determining risk neutral probability from call quotes

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Perspective: implied volatility

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- “Implied volatility” is the value of $\sigma$ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.
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“Implied volatility” is the value of $\sigma$ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.

If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices. In practice, when the implied volatility is viewed as a function of strike price (sometimes called the “volatility smile”), it is not constant.
Main Black-Scholes assumption: risk neutral probability densities are lognormal.
Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.

- **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large $T$.
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Replicating portfolio point of view: in the simple binary tree models (or continuum Brownian models), we can transfer money back and forth between the stock and the risk free asset to ensure our wealth at time $T$ equals the option payout. Option price is required initial investment, which is risk neutral expectation of payout. “True probabilities” are irrelevant.
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- **Fixes:** variable volatility, random interest rates, Lévy jumps....