Outline

Central limit theorem

Proving the central limit theorem
Central limit theorem

Proving the central limit theorem
Recall: DeMoivre-Laplace limit theorem

Let $X_i$ be an i.i.d. sequence of random variables. Write $S_n = \sum_{i=1}^{n} X_i$.

Suppose each $X_i$ is 1 with probability $p$ and 0 with probability $q = 1 - p$.

DeMoivre-Laplace limit theorem: 
$$\lim_{n \to \infty} P\{a \leq S_n - np \sqrt{npq} \leq b\} \to \Phi(b) - \Phi(a).$$

Here $\Phi(b) - \Phi(a) = P\{a \leq Z \leq b\}$ when $Z$ is a standard normal random variable.

$S_n - np \sqrt{npq}$ describes “number of standard deviations that $S_n$ is above or below its mean”.

Question: Does a similar statement hold if the $X_i$ are i.i.d. but have some other probability distribution?

Central limit theorem: Yes, if they have finite variance.
Recall: DeMoivre-Laplace limit theorem

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- Question: Does a similar statement hold if the $X_i$ are i.i.d. but have some other probability distribution?
- **Central limit theorem**: Yes, if they have finite variance.
Example

▶ Say we roll $10^6$ ordinary dice independently of each other.
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Let $X_i$ be the number on the $i$th die. Let $X = \sum_{i=1}^{10^6} X_i$ be the total of the numbers rolled.
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- Say we roll $10^6$ ordinary dice independently of each other.
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- What is $E[X]$?
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- What is the probability that $X$ is less than a standard deviations above its mean?

Central limit theorem: should be about $\sqrt{2/\pi} \int_{a - \infty} e^{-x^2/2} dx$. 
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  - Central limit theorem: should be about $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} \, dx$. 
Let $X_i$ be an i.i.d. sequence of random variables with finite mean $\mu$ and variance $\sigma^2$. 

Central limit theorem: 

$$\lim_{n \to \infty} P\{a \leq B_n \leq b\} \to \Phi(b) - \Phi(a).$$
Let $X_i$ be an i.i.d. sequence of random variables with finite mean $\mu$ and variance $\sigma^2$.

Write $S_n = \sum_{i=1}^n X_i$. So $E[S_n] = n\mu$ and $\text{Var}[S_n] = n\sigma^2$ and $\text{SD}[S_n] = \sigma\sqrt{n}$.
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Write $B_n = \frac{X_1+X_2+\ldots+X_n-n\mu}{\sigma\sqrt{n}}$. Then $B_n$ is the difference between $S_n$ and its expectation, measured in standard deviation units.
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$$\lim_{n \to \infty} P\{a \leq B_n \leq b\} \to \Phi(b) - \Phi(a).$$
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Let $X$ be a random variable.
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The **characteristic function** of $X$ is defined by

$$
\phi(t) = \phi_X(t) := E[e^{itX}].
$$

Like $M(t)$ except with $i$ thrown in.

Recall that $e^{it} = \cos(t) + i\sin(t)$.

Characteristic functions are similar to moment generating functions in some ways.

For example, $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$, just as $M_{X+Y}(t) = M_X(t)M_Y(t)$, if $X$ and $Y$ are independent.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.

And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$. 

Characteristic functions are well defined at all $t$ for all random variables $X$. 
Recall: characteristic functions

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► Characteristic functions are similar to moment generating functions in some ways.
► For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.
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- Characteristic functions are similar to moment generating functions in some ways.
- For example, $\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.
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### Review

- **Characteristics Function**
- $\phi(t) = \phi_X(t) := E[e^{itX}]$
- **Similar to Moment Generating Functions**
- $\phi_{X+Y} = \phi_X \phi_Y$
- $M_{X+Y} = M_X M_Y$
- $\phi_{aX}(t) = \phi_X(at)$
- $M_{aX}(t) = M_X(at)$
- $E[X^m] = i^m\phi_X^{(m)}(0)$
Let $X$ be a random variable and $X_n$, a sequence of random variables.
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Say $X_n$ converge in distribution or converge in law to $X$ if
\[
\lim_{n \to \infty} F_{X_n}(x) = F_X(x)
\]
at all $x \in \mathbb{R}$ at which $F_X$ is continuous.
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\]

Recall: the weak law of large numbers can be rephrased as the statement that
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A_n = \frac{X_1 + X_2 + \ldots + X_n}{n}
\text{ converges in law to } \mu \text{ (i.e., to the random variable that is equal to } \mu \text{ with probability one) as } n \to \infty.
\]
Let $X$ be a random variable and $X_n$ a sequence of random variables.

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Recall: the weak law of large numbers can be rephrased as the statement that $A_n = \frac{X_1 + X_2 + \ldots + X_n}{n}$ converges in law to $\mu$ (i.e., to the random variable that is equal to $\mu$ with probability one) as $n \to \infty$.

The central limit theorem can be rephrased as the statement that $B_n = \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma \sqrt{n}}$ converges in law to a standard normal random variable as $n \to \infty$. 
Lévy’s continuity theorem (see Wikipedia): if

\[ \lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t) \]

for all \( t \), then \( X_n \) converge in law to \( X \).
Lévy’s continuity theorem (see Wikipedia): if

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for all \( t \), then \( X_n \) converge in law to \( X \).

By this theorem, we can prove the central limit theorem by showing \( \lim_{n \to \infty} \phi_{B_n}(t) = e^{-t^2/2} \) for all \( t \).
Lévy’s continuity theorem (see Wikipedia): if

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Moment generating function continuity theorem: if moment generating functions \( M_{X_n}(t) \) are defined for all \( t \) and \( n \) and \( \lim_{n \to \infty} M_{X_n}(t) = M_X(t) \) for all \( t \), then \( X_n \) converge in law to \( X \).
Lévy’s continuity theorem (see Wikipedia): if

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Moment generating function continuity theorem: if moment generating functions $M_{X_n}(t)$ are defined for all $t$ and $n$ and $\lim_{n \to \infty} M_{X_n}(t) = M_X(t)$ for all $t$, then $X_n$ converge in law to $X$.

By this theorem, we can prove the central limit theorem by showing $\lim_{n \to \infty} M_{B_n}(t) = e^{t^2/2}$ for all $t$. 
Proof of central limit theorem with moment generating functions

- Write $Y = \frac{X - \mu}{\sigma}$. Then $Y$ has mean zero and variance 1.
Proof of central limit theorem with moment generating functions

- Write $Y = \frac{X-\mu}{\sigma}$. Then $Y$ has mean zero and variance 1.
- Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$. 

Write $Y = \frac{X - \mu}{\sigma}$. Then $Y$ has mean zero and variance 1.

Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$.

We know $g(0) = 0$. Also $M'_Y(0) = E[Y] = 0$ and $M''_Y(0) = E[Y^2] = \text{Var}[Y] = 1$. 
Proof of central limit theorem with moment generating functions

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We know $g(0) = 0$. Also $M'_Y(0) = E[Y] = 0$ and $M''_Y(0) = E[Y^2] = \text{Var}[Y] = 1$.

Chain rule: $M'_Y(0) = g'(0)e^{g(0)} = g'(0) = 0$ and $M''_Y(0) = g''(0)e^{g(0)} + g'(0)^2e^{g(0)} = g''(0) = 1$. 

Proof of central limit theorem with moment generating functions

- Write $Y = \frac{X - \mu}{\sigma}$. Then $Y$ has mean zero and variance 1.
- Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$.
- We know $g(0) = 0$. Also $M_Y'(0) = E[Y] = 0$ and $M_Y''(0) = E[Y^2] = \text{Var}[Y] = 1$.
- Chain rule: $M_Y'(0) = g'(0)e^{g(0)} = g'(0) = 0$ and $M_Y''(0) = g''(0)e^{g(0)} + g'(0)^2e^{g(0)} = g''(0) = 1$.
- So $g$ is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = 1$.
  Taylor expansion: $g(t) = t^2/2 + o(t^2)$ for $t$ near zero.
Proof of central limit theorem with moment generating functions

- Write \( Y = \frac{X - \mu}{\sigma} \). Then \( Y \) has mean zero and variance 1.
- Write \( M_Y(t) = E[e^{tY}] \) and \( g(t) = \log M_Y(t) \). So \( M_Y(t) = e^{g(t)} \).
- We know \( g(0) = 0 \). Also \( M_Y'(0) = E[Y] = 0 \) and \( M_Y''(0) = E[Y^2] = \text{Var}[Y] = 1 \).
- Chain rule: \( M_Y'(0) = g'(0)e^{g(0)} = g'(0) = 0 \) and \( M_Y''(0) = g''(0)e^{g(0)} + g'(0)^2e^{g(0)} = g''(0) = 1 \).
- So \( g \) is a nice function with \( g(0) = g'(0) = 0 \) and \( g''(0) = 1 \). Taylor expansion: \( g(t) = \frac{t^2}{2} + o(t^2) \) for \( t \) near zero.
- Now \( B_n \) is \( \frac{1}{\sqrt{n}} \) times the sum of \( n \) independent copies of \( Y \).
Proof of central limit theorem with moment generating functions

- Write $Y = \frac{X-\mu}{\sigma}$. Then $Y$ has mean zero and variance 1.
- Write $M_Y(t) = E[e^{tY}]$ and $g(t) = \log M_Y(t)$. So $M_Y(t) = e^{g(t)}$.
- We know $g(0) = 0$. Also $M_Y'(0) = E[Y] = 0$ and $M_Y''(0) = E[Y^2] = \text{Var}[Y] = 1$.
- Chain rule: $M_Y'(0) = g'(0)e^{g(0)} = g'(0) = 0$ and $M_Y''(0) = g''(0)e^{g(0)} + g'(0)^2e^{g(0)} = g''(0) = 1$.
- So $g$ is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = 1$.
- Taylor expansion: $g(t) = t^2/2 + o(t^2)$ for $t$ near zero.
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- So \( M_{B_n}(t) = (M_Y(t/\sqrt{n}))^n = e^{ng(t/\sqrt{n})} \).
- But \( e^{ng(t/\sqrt{n})} \approx e^{n(t/\sqrt{n})^2/2} = e^{t^2/2} \), in sense that LHS tends to \( e^{t^2/2} \) as \( n \) tends to infinity.
Proof of central limit theorem with characteristic functions

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- But the proof can be repeated almost verbatim using characteristic functions instead of moment generating functions.
- Then it applies for any $X$ with finite variance.
Almost verbatim: replace $M_Y(t)$ with $\phi_Y(t)$

\[ \phi(Y(t)) = E[e^{itY}] \]

\[ g(t) = \log \phi_Y(t) \]

\[ \phi_Y(t) = e^{g(t)} \]

We know $g(0) = 0$. Also $\phi_Y'(0) = iE[Y] = 0$ and $\phi_Y''(0) = i^2 E[Y^2] = -1$.

Chain rule:

\[ \phi_Y'(0) = g'(0) e^{g(0)} = g'(0) \]

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So $g$ is a nice function with $g(0) = g'(0) = 0$ and $g''(0) = -1$.

Taylor expansion:

\[ g(t) = -\frac{t^2}{2} + o(t^2) \text{ for } t \text{ near zero}. \]

Now $B_n$ is $1/\sqrt{n}$ times the sum of $n$ independent copies of $Y$.

\[ \phi(B_n(t)) = \left( \phi(Y(t/\sqrt{n})) \right)^n = e^{ng(t/\sqrt{n})}. \]

But $e^{ng(t/\sqrt{n})} \approx e^{-n(t\sqrt{n})/2} = e^{-t^2/2}$, in sense that LHS tends to $e^{-t^2/2}$ as $n$ tends to infinity.
Write $\phi_Y(t) = E[e^{itY}]$ and $g(t) = \log \phi_Y(t)$. So $\phi_Y(t) = e^{g(t)}$. 

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*Kind of* true for homogenous population, ignoring outliers.