18.600: Lecture 3
What is probability?

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Outline

- Formalizing probability
- Sample space
- DeMorgan’s laws
- Axioms of probability
Outline

Formalizing probability

Sample space

DeMorgan’s laws

Axioms of probability
What does “I’d say there’s a thirty percent chance it will rain tomorrow” mean?

Neurological: When I think “it will rain tomorrow” the “truth-sensing” part of my brain exhibits 30 percent of its maximum electrical activity.

Frequentist: Of the last 1000 days that meteorological measurements looked this way, rain occurred on the subsequent day 300 times.

Market preference (“risk neutral probability”): The market price of a contract that pays 100 if it rains tomorrow agrees with the price of a contract that pays 30 tomorrow no matter what.

Personal belief: If you offered me a choice of these contracts, I’d be indifferent. (If need for money is different in two scenarios, I can replace dollars with “units of utility.”)
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Even more fundamental question: defining a set of possible outcomes.

- Roll a die $n$ times. Define a sample space to be $\{1, 2, 3, 4, 5, 6\}^n$, i.e., the set of $a_1, \ldots, a_n$ with each $a_j \in \{1, 2, 3, 4, 5, 6\}$.

- Shuffle a standard deck of cards. Sample space is the set of $52!$ permutations.

- Will it rain tomorrow? Sample space is $\{R, N\}$, which stand for “rain” and “no rain.”

- Randomly throw a dart at a board. Sample space is the set of points on the board.
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Event: subset of the sample space

If a set \( A \) is comprised of some of the elements of \( B \), say \( A \) is a subset of \( B \) and write \( A \subset B \).

Similarly, \( B \supset A \) means \( A \) is a subset of \( B \) (or \( B \) is a superset of \( A \)).

If \( S \) is a finite sample space with \( n \) elements, then there are \( 2^n \) subsets of \( S \).

Denote by \( \emptyset \) the set with no elements.
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Intersections, unions, complements

- $A \cup B$ means the union of $A$ and $B$, the set of elements contained in at least one of $A$ and $B$. 

- $A \cap B$ means the intersection of $A$ and $B$, the set of elements contained on both $A$ and $B$. 

- $A^c$ means complement of $A$, set of points in whole sample space $S$ but not in $A$. 

- $A \setminus B$ means "$A$ minus $B$" which means the set of points in $A$ but not in $B$. In symbols, $A \setminus B = A \cap (B^c)$. 

- $\cup$ is associative. So $(A \cup B) \cup C = A \cup (B \cup C)$ and can be written $A \cup B \cup C$. 

- $\cap$ is also associative. So $(A \cap B) \cap C = A \cap (B \cap C)$ and can be written $A \cap B \cap C$. 

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Venn diagrams

\[ \begin{array}{c}
A \\
B \\
\end{array} \]
Venn diagrams

$A \cap B$

$A^c \cap B$

$A \cap B^c$

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— “It will not snow or rain” means “It will not snow and it will not rain.”

— \( S \cup R \subseteq S^c \cap R^c \)

— More generally: \( \bigcup_{i=1}^{n} E_i \subseteq \bigcap_{i=1}^{n} (E_i)^c \)

— “It will not both snow and rain” means “Either it will not snow or it will not rain.”

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DeMorgan’s laws

- “It will not snow or rain” means “It will not snow and it will not rain.”
- If $S$ is event that it snows, $R$ is event that it rains, then $(S \cup R)^c = S^c \cap R^c$
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- $P(A) \in [0, 1]$ for all $A \subset S$. 
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- Finite additivity: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.
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- $P(A) \in [0, 1]$ for all $A \subseteq S$.  
- $P(S) = 1$.  
- Finite additivity: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.  
- Countable additivity: $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \emptyset$ for each pair $i$ and $j$.  

Neurological: When I think “it will rain tomorrow” the “truth-sensing” part of my brain exhibits 30 percent of its maximum electrical activity. Should have $P(A) \in [0,1]$ and presumably $P(S) = 1$ but not necessarily $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$.
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Personal belief: $P(A)$ is amount such that I’d be indifferent between contract paying 1 if $A$ occurs and contract paying $P(A)$ no matter what. Seems to satisfy axioms with some notion of utility units, strong assumption of “rationality”...