Exponential random variables

Minimum of independent exponentials

Memoryless property

Relationship to Poisson random variables
Outline

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Say $X$ is an exponential random variable of parameter $\lambda$ when its probability distribution function is

$$f(x) = \begin{cases} 
\lambda e^{-\lambda x} & x \geq 0 \\
0 & x < 0 
\end{cases}.$$
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For \( a > 0 \) have

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F_X(a) = \int_0^a f(x)\,dx = \int_0^a \lambda e^{-\lambda x} \,dx = -e^{-\lambda x}\bigg|_0^a = 1 - e^{-\lambda a}
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Thus $P\{X < a\} = 1 - e^{-\lambda a}$ and $P\{X > a\} = e^{-\lambda a}$.

Formula $P\{X > a\} = e^{-\lambda a}$ is very important in practice.
Suppose $X$ is exponential with parameter $\lambda$, so

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{when} \quad x \geq 0.$$
Suppose $X$ is exponential with parameter $\lambda$, so $f_X(x) = \lambda e^{-\lambda x}$ when $x \geq 0$.

What is $E[X^n]$? (Say $n \geq 1$.)
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What is $E[X^n]$? (Say $n \geq 1$.)

Write $E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} \, dx$. 

If $\lambda = 1$, then $E[X^n] = n!$. Could take this as definition of $n!$. It makes sense for $n = 0$ and for non-integer $n$. 

Variance: $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/\lambda^2$. 
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$$E[X^n] = -\int_0^\infty nx^{n-1} \frac{e^{-\lambda x}}{-\lambda} \, dx + x^n \lambda \frac{e^{-\lambda x}}{-\lambda} \bigg|_0^\infty.$$
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We get $E[X^n] = \frac{n}{\lambda} E[X^{n-1}]$. 

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 CLAIM: If $X_1$ and $X_2$ are independent and exponential with parameters $\lambda_1$ and $\lambda_2$ then $X = \min\{X_1, X_2\}$ is exponential with parameter $\lambda = \lambda_1 + \lambda_2$. 
CLAIM: If $X_1$ and $X_2$ are independent and exponential with parameters $\lambda_1$ and $\lambda_2$ then $X = \min\{X_1, X_2\}$ is exponential with parameter $\lambda = \lambda_1 + \lambda_2$.

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How could we prove this?

Have various ways to describe random variable $Y$: via density function $f_Y(x)$, or cumulative distribution function $F_Y(a) = P\{Y \leq a\}$, or function $P\{Y > a\} = 1 - F_Y(a)$. 
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Last one has simple form for exponential random variables. We have \( P\{Y > a\} = e^{-\lambda a} \) for \( a \in [0, \infty) \).
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> If $X_1, \ldots, X_n$ are independent exponential with $\lambda_1, \ldots, \lambda_n$, then $\min\{X_1, \ldots, X_n\}$ is exponential with $\lambda = \lambda_1 + \ldots + \lambda_n$. 

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- Suppose $X$ is exponential with parameter $\lambda$. 
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- To make this precise, we ask what is the probability distribution of $Y = X - b$ *conditioned on* $X > b$?
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$$P\{X > b + a\} / P\{X > b\} = e^{-\lambda(b+a)} / e^{-\lambda b} = e^{-\lambda a}.$$

Thus, conditional law of $X - b$ given that $X > b$ is same as the original law of $X$. 
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If we plan to toss a coin until the first heads comes up, then we have a .5 chance to get a heads in one step, a .25 chance in two steps, etc.
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Given that the first 5 tosses are all tails, there is conditionally a .5 chance we get our first heads on the 6th toss, a .25 chance on the 7th toss, etc.
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Despite our having had five tails in a row, our expectation of the amount of time remaining until we see a heads is the same as it originally was.
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Bob: No, that’s your mistake. You should never assume that, because maybe somebody tampered with the coin.
Alice: Yeah, yeah, I get it. I can’t win here.
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Bob: No, I don’t think you get it yet. It’s a subtle point in statistics. It’s very important.
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Exchange continued for duration of shuttle ride (Alice increasingly irritated, Bob increasingly patronizing).
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Suppose the duration of a couple’s relationship is exponential with $\lambda^{-1}$ equal to two weeks.
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Given that it has lasted for 10 weeks so far, what is the conditional probability that it will last an additional week?
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How about an additional four weeks? Ten weeks?
Alice assumes Bob means “independent tosses of a fair coin.” Under this assumption, all $2^{11}$ outcomes of eleven-coin-toss sequence are equally likely. Bob considers HHHHHHHHHHH more likely than HHHHHHHHHHHT, since former could result from a faulty coin.
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Alice sees Bob’s point but considers it annoying and churlish to ask about coin toss sequence and criticize listener for assuming this means “independent tosses of fair coin”.

Without that assumption, Alice has no idea what context Bob has in mind. (An environment where two-headed novelty coins are common? Among coin-tossing cheaters with particular agendas?...)

Alice: you need assumptions to convert stories into math.

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Radioactive decay: maximum of independent exponentials

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- Let \( T_1 \) be the amount of time you wait until the first particle decays, \( T_2 \) the amount of *additional* time until the second particle decays, etc., so that \( T = T_1 + T_2 + \ldots + T_n \).
Suppose you start at time zero with $n$ radioactive particles. Suppose that each one (independently of the others) will decay at a random time, which is an exponential random variable with parameter $\lambda$.

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Let $T_1$ be the amount of time you wait until the first particle decays, $T_2$ the amount of additional time until the second particle decays, etc., so that $T = T_1 + T_2 + \ldots + T_n$.

Claim: $T_1$ is exponential with parameter $n\lambda$. 

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Let \( T_1 \) be the amount of time you wait until the first particle decays, \( T_2 \) the amount of additional time until the second particle decays, etc., so that \( T = T_1 + T_2 + \ldots + T_n \).

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Claim: \( T_2 \) is exponential with parameter \( (n - 1)\lambda \).

And so forth. \( E[T] = \sum_{i=1}^{n} E[T_i] = \lambda^{-1} \sum_{j=1}^{n} \frac{1}{j} \) and (by independence) \( \text{Var}[T] = \sum_{i=1}^{n} \text{Var}[T_i] = \lambda^{-2} \sum_{j=1}^{n} \frac{1}{j^2} \).
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Let $T_1, T_2, \ldots$ be independent exponential random variables with parameter $\lambda$. 

We can view them as waiting times between “events”. How do you show that the number of events in the first $t$ units of time is Poisson with parameter $\lambda t$?

We actually did this already in the lecture on Poisson point processes. You can break the interval $[0, t]$ into $n$ equal pieces (for very large $n$), let $X_k$ be number of events in $k$th piece, use memoryless property to argue that the $X_k$ are independent.

When $n$ is large enough, it becomes unlikely that any interval has more than one event. Roughly speaking: each interval has one event with probability $\frac{\lambda t}{n}$, zero otherwise.

Take $n \to \infty$ limit. Number of events is Poisson $\lambda t$. 

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We actually did this already in the lecture on Poisson point processes. You can break the interval $[0, t]$ into $n$ equal pieces (for very large $n$), let $X_k$ be number of events in $k$th piece, use memoryless property to argue that the $X_k$ are independent.

When $n$ is large enough, it becomes unlikely that any interval has more than one event. Roughly speaking: each interval has one event with probability $\lambda t/n$, zero otherwise.
Let $T_1, T_2, \ldots$ be independent exponential random variables with parameter $\lambda$.

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Take $n \to \infty$ limit. Number of events is Poisson $\lambda t$. 