18.600: Lecture 11

Binomial random variables and repeated trials

Scott Sheffield

MIT
Outline

Bernoulli random variables

Properties: expectation and variance

More problems
Outline

Bernoulli random variables

Properties: expectation and variance

More problems
Bernoulli random variables

- Toss fair coin \( n \) times. (Tosses are independent.) What is the probability of \( k \) heads?

\[
\binom{n}{k} \frac{1}{2^n}.
\]

What if coin has \( p \) probability to be heads?

\[
\binom{n}{k} p^k (1-p)^{n-k}.
\]

Writing \( q = 1-p \), we can write this as

\[
\binom{n}{k} p^k q^{n-k}.
\]

Can use binomial theorem to show probabilities sum to one:

\[
1 = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.
\]

Number of heads is binomial random variable with parameters \((n, p)\).
Bernoulli random variables

- Toss fair coin \( n \) times. (Tosses are independent.) What is the probability of \( k \) heads?
- Answer: \( \binom{n}{k}/2^n \).
Bernoulli random variables

- Toss fair coin $n$ times. (Tosses are independent.) What is the probability of $k$ heads?
- Answer: $\binom{n}{k}/2^n$.
- What if coin has $p$ probability to be heads?

Writing $q = 1 - p$, we can write this as $\binom{n}{k}p^kq^{n-k}$.

Can use binomial theorem to show probabilities sum to one:

$1 = 1^n = (p+q)^n = \sum_{k=0}^{n} \binom{n}{k}p^kq^{n-k}$.

Number of heads is binomial random variable with parameters $(n, p)$. 
Bernoulli random variables

- Toss fair coin \( n \) times. (Tosses are independent.) What is the probability of \( k \) heads?
- Answer: \( \binom{n}{k} / 2^n \).
- What if coin has \( p \) probability to be heads?
- Answer: \( \binom{n}{k} p^k (1 - p)^{n-k} \).
Bernoulli random variables

- Toss fair coin \( n \) times. (Tosses are independent.) What is the probability of \( k \) heads?
- Answer: \( \binom{n}{k}/2^n \).
- What if coin has \( p \) probability to be heads?
- Answer: \( \binom{n}{k} p^k (1 - p)^{n-k} \).
- Writing \( q = 1 - p \), we can write this as \( \binom{n}{k} p^k q^{n-k} \).
Toss fair coin $n$ times. (Tosses are independent.) What is the probability of $k$ heads?

Answer: $\binom{n}{k}/2^n$.

What if coin has $p$ probability to be heads?

Answer: $\binom{n}{k}p^k(1-p)^{n-k}$.

Writing $q = 1 - p$, we can write this as $\binom{n}{k}p^kq^{n-k}$.

Can use binomial theorem to show probabilities sum to one:
Toss fair coin \( n \) times. (Tosses are independent.) What is the probability of \( k \) heads?

Answer: \( \binom{n}{k}/2^n \).

What if coin has \( p \) probability to be heads?

Answer: \( \binom{n}{k}p^k(1-p)^{n-k} \).

Writing \( q = 1 - p \), we can write this as \( \binom{n}{k}p^k q^{n-k} \).

Can use binomial theorem to show probabilities sum to one:

\[
1 = 1^n = (p + q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.
\]
Bernoulli random variables

- Toss fair coin \( n \) times. (Tosses are independent.) What is the probability of \( k \) heads?
  - Answer: \( \binom{n}{k}/2^n \).
- What if coin has \( p \) probability to be heads?
  - Answer: \( \binom{n}{k} p^k (1 - p)^{n-k} \).
- Writing \( q = 1 - p \), we can write this as \( \binom{n}{k} p^k q^{n-k} \).
- Can use binomial theorem to show probabilities sum to one:
  - \( 1 = 1^n = (p + q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \).
- Number of heads is **binomial random variable with parameters** \( (n, p) \).
Examples

- Toss 6 fair coins. Let $X$ be number of heads you see. Then $X$ is binomial with parameters $(n, p)$ given by $(6, 1/2)$. 

- Probability mass function for $X$ can be computed using the 6th row of Pascal's triangle.

- If coin is biased (comes up heads with probability $p \neq 1/2$), we can still use the 6th row of Pascal's triangle, but the probability that $X = i$ gets multiplied by $p^i (1-p)^{n-i}$. 

Toss 6 fair coins. Let $X$ be number of heads you see. Then $X$ is binomial with parameters $(n, p)$ given by $(6, 1/2)$.

Probability mass function for $X$ can be computed using the 6th row of Pascal’s triangle.
Examples

- Toss 6 fair coins. Let $X$ be number of heads you see. Then $X$ is binomial with parameters $(n, p)$ given by $(6, 1/2)$.
- Probability mass function for $X$ can be computed using the 6th row of Pascal’s triangle.
- If coin is biased (comes up heads with probability $p \neq 1/2$), we can still use the 6th row of Pascal’s triangle, but the probability that $X = i$ gets multiplied by $p^i(1 - p)^{n-i}$.
Room contains \( n \) people. What is the probability that exactly \( i \) of them were born on a Tuesday?
Room contains $n$ people. What is the probability that exactly $i$ of them were born on a Tuesday?

Answer: use binomial formula $\binom{n}{i} p^i q^{n-i}$ with $p = \frac{1}{7}$ and $q = 1 - p = \frac{6}{7}$. 
Room contains \( n \) people. What is the probability that exactly \( i \) of them were born on a Tuesday?

Answer: use binomial formula \( \binom{n}{i} p^i q^{n-i} \) with \( p = 1/7 \) and \( q = 1 - p = 6/7 \).

Let \( n = 100 \). Compute the probability that nobody was born on a Tuesday.
Other examples

- Room contains $n$ people. What is the probability that exactly $i$ of them were born on a Tuesday?
- Answer: use binomial formula $\binom{n}{i} p^i q^{n-i}$ with $p = 1/7$ and $q = 1 - p = 6/7$.
- Let $n = 100$. Compute the probability that nobody was born on a Tuesday.
- What is the probability that exactly 15 people were born on a Tuesday?
Bernoulli random variables

Properties: expectation and variance

More problems
Outline

Bernoulli random variables

Properties: expectation and variance

More problems
Let $X$ be a binomial random variable with parameters $(n, p)$. 
Let $X$ be a binomial random variable with parameters $(n, p)$.

What is $E[X]$?
Let $X$ be a binomial random variable with parameters $(n, p)$.

What is $E[X]$?

Direct approach: by definition of expectation,

$$E[X] = \sum_{i=0}^{n} P\{X = i\}i.$$
Let $X$ be a binomial random variable with parameters $(n, p)$.

What is $E[X]$?

Direct approach: by definition of expectation,

$$E[X] = \sum_{i=0}^{n} P\{X = i\} i.$$ 

What happens if we modify the $n$th row of Pascal’s triangle by multiplying the $i$ term by $i$?
Let $X$ be a binomial random variable with parameters $(n, p)$.

What is $E[X]$?

Direct approach: by definition of expectation,

$$E[X] = \sum_{i=0}^{n} P\{X = i\}i.$$  

What happens if we modify the $n$th row of Pascal’s triangle by multiplying the $i$ term by $i$?

For example, replace the 5th row $(1, 5, 10, 10, 5, 1)$ by $(0, 5, 20, 30, 20, 5)$. Does this remind us of an earlier row in the triangle?
Let $X$ be a binomial random variable with parameters $(n, p)$.

What is $E[X]$?

Direct approach: by definition of expectation,
$$E[X] = \sum_{i=0}^{n} P\{X = i\} i.$$ 

What happens if we modify the $n$th row of Pascal’s triangle by multiplying the $i$ term by $i$?

For example, replace the 5th row $(1, 5, 10, 10, 5, 1)$ by $(0, 5, 20, 30, 20, 5)$. Does this remind us of an earlier row in the triangle?

Perhaps the prior row $(1, 4, 6, 4, 1)$?
Recall that \( \binom{n}{i} = \frac{n \times (n-1) \times \ldots \times (n-i+1)}{i \times (i-1) \times \ldots \times 1} \). This implies a simple but important identity: \( i \binom{n}{i} = n \binom{n-1}{i-1} \).
Recall that \( \binom{n}{i} = \frac{n \times (n-1) \times \ldots \times (n-i+1)}{i \times (i-1) \times \ldots \times (1)} \). This implies a simple but important identity: \( i \binom{n}{i} = n \binom{n-1}{i-1} \).

Using this identity (and \( q = 1 - p \)), we can write

\[
E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i} = \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i q^{n-i}.
\]
Recall that \( \binom{n}{i} = \frac{n \times (n-1) \times \ldots \times (n-i+1)}{i \times (i-1) \times \ldots \times 1} \). This implies a simple but important identity: \( i \binom{n}{i} = n \binom{n-1}{i-1} \).

Using this identity (and \( q = 1 - p \)), we can write

\[
E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i} = \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i q^{n-i}.
\]

Rewrite this as \( E[X] = np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{(i-1)} q^{(n-1)-(i-1)} \).
Recall that \( \binom{n}{i} = \frac{n \times (n-1) \times \ldots \times (n-i+1)}{i \times (i-1) \times \ldots \times (1)} \). This implies a simple but important identity: \( i \binom{n}{i} = n \binom{n-1}{i-1} \).

Using this identity (and \( q = 1 - p \)), we can write

\[
E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i} = \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i q^{n-i}.
\]

Rewrite this as \( E[X] = np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{(i-1)} q^{(n-1)-(i-1)} \).

Substitute \( j = i - 1 \) to get

\[
E[X] = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j} = np(p + q)^{n-1} = np.
\]
Let $X$ be a binomial random variable with parameters $(n, p)$. Here is another way to compute $E[X]$. 

- Write $X = \sum_{j=1}^{n} X_j$, where $X_j$ is 1 if the $j$th coin is heads, 0 otherwise.
- In other words, $X_j$ is the number of heads (zero or one) on the $j$th toss.
- Note that $E[X_j] = p \cdot 1 + (1 - p) \cdot 0 = p$ for each $j$.
- Conclude by additivity of expectation that $E[X] = n \sum_{j=1}^{n} E[X_j] = np$. 
Let $X$ be a binomial random variable with parameters $(n, p)$. Here is another way to compute $E[X]$.

Think of $X$ as representing number of heads in $n$ tosses of coin that is heads with probability $p$. 

Note that $E[X_j] = p \cdot 1 + (1 - p) \cdot 0 = p$ for each $j$. 

Conclude by additivity of expectation that $E[X] = \sum_{j=1}^{n} E[X_j] = np$. 


Let $X$ be a binomial random variable with parameters $(n, p)$. Here is another way to compute $E[X]$.

Think of $X$ as representing number of heads in $n$ tosses of a coin that is heads with probability $p$.

Write $X = \sum_{j=1}^{n} X_j$, where $X_j$ is 1 if the $j$th coin is heads, 0 otherwise.
Let $X$ be a binomial random variable with parameters $(n, p)$. Here is another way to compute $E[X]$.

Think of $X$ as representing number of heads in $n$ tosses of coin that is heads with probability $p$.

Write $X = \sum_{j=1}^{n} X_j$, where $X_j$ is 1 if the $j$th coin is heads, 0 otherwise.

In other words, $X_j$ is the number of heads (zero or one) on the $j$th toss.
Decomposition approach to computing expectation

- Let $X$ be a binomial random variable with parameters $(n, p)$. Here is another way to compute $E[X]$.
- Think of $X$ as representing number of heads in $n$ tosses of a coin that is heads with probability $p$.
- Write $X = \sum_{j=1}^{n} X_j$, where $X_j$ is 1 if the $j$th coin is heads, 0 otherwise.
- In other words, $X_j$ is the number of heads (zero or one) on the $j$th toss.
- Note that $E[X_j] = p \cdot 1 + (1 - p) \cdot 0 = p$ for each $j$. 
Let $X$ be a binomial random variable with parameters $(n, p)$. Here is another way to compute $E[X]$.

Think of $X$ as representing number of heads in $n$ tosses of coin that is heads with probability $p$.

Write $X = \sum_{j=1}^{n} X_j$, where $X_j$ is 1 if the $j$th coin is heads, 0 otherwise.

In other words, $X_j$ is the number of heads (zero or one) on the $j$th toss.

Note that $E[X_j] = p \cdot 1 + (1 - p) \cdot 0 = p$ for each $j$.

Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} p = np.$$
Let $X$ be binomial $(n, p)$ and fix $k \geq 1$. What is $E[X^k]$?
Let $X$ be binomial $(n, p)$ and fix $k \geq 1$. What is $E[X^k]$?

Recall identity: $i(n) = n(i-1)$. 

Thus $E[X^k] = npE[Y^k]$ where $Y$ is binomial with parameters $(n-1, p)$. 

$E[X^k] = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$.
Let $X$ be binomial $(n, p)$ and fix $k \geq 1$. What is $E[X^k]$?

Recall identity: $i(\binom{n}{i}) = n \binom{n-1}{i-1}$.

Generally, $E[X^k]$ can be written as

$$
\sum_{i=0}^{n} i \binom{n}{i} p^i (1 - p)^{n-i} i^{k-1}.
$$
Interesting moment computation

Let $X$ be binomial $(n, p)$ and fix $k \geq 1$. What is $E[X^k]$?

Recall identity: $i\binom{n}{i} = n\binom{n-1}{i-1}$.

Generally, $E[X^k]$ can be written as

$$
\sum_{i=0}^{n} i \binom{n}{i} p^i (1 - p)^{n-i} i^{k-1}.
$$

Identity gives

$$
E[X^k] = np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1 - p)^{n-i} i^{k-1} =
$$

$$
np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1 - p)^{n-1-j} (j + 1)^{k-1}.
$$
Interesting moment computation

- Let $X$ be binomial $(n, p)$ and fix $k \geq 1$. What is $E[X^k]$?
- Recall identity: $i^n = n^{i_{i-1}}$.
- Generally, $E[X^k]$ can be written as
  \[
  \sum_{i=0}^{n} i \binom{n}{i} p^i (1 - p)^{n-i} i^{k-1}.
  \]
- Identity gives
  \[
  E[X^k] = np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1 - p)^{n-i} i^{k-1} =
  \]
  \[
  np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1 - p)^{n-1-j} (j+1)^{k-1}.
  \]
- Thus $E[X^k] = npE[(Y + 1)^{k-1}]$ where $Y$ is binomial with parameters $(n - 1, p)$. 
Let $X$ be binomial $(n, p)$. What is $E[X]$?

We know $E[X] = np$.

We computed identity $E[X^k] = npE[(Y+1)^k - 1]$ where $Y$ is binomial with parameters $(n-1, p)$.


So $\text{Var}[X] = E[X^2] - E[X]^2 = np(n-1)p + np - (np)^2 = npq$, where $q = 1 - p$.

Commit to memory: variance of binomial $(n, p)$ random variable is $npq$.

This is $n$ times the variance you'd get with a single coin. Coincidence?
Computing the variance

Let $X$ be binomial $(n, p)$. What is $E[X]$?

We know $E[X] = np$. 
Computing the variance

- Let \( X \) be binomial \((n, p)\). What is \( E[X] \)?
- We know \( E[X] = np \).
- We computed identity \( E[X^k] = npE[(Y + 1)^{k-1}] \) where \( Y \) is binomial with parameters \((n - 1, p)\).
Let $X$ be binomial $(n, p)$. What is $E[X]$?

We know $E[X] = np$.

We computed identity $E[X^k] = npE[(Y + 1)^{k-1}]$ where $Y$ is binomial with parameters $(n - 1, p)$.

Computing the variance

- Let $X$ be binomial $(n, p)$. What is $E[X]$?
- We know $E[X] = np$.
- We computed identity $E[X^k] = npE[(Y + 1)^{k-1}]$ where $Y$ is binomial with parameters $(n - 1, p)$.
- So $\text{Var}[X] = E[X^2] - E[X]^2 = np(n - 1)p + np - (np)^2 = np(1 - p) = npq$, where $q = 1 - p$. 

Commit to memory: variance of binomial $(n, p)$ random variable is $npq$. This is $n$ times the variance you’d get with a single coin. Coincidence?
Let $X$ be binomial $(n, p)$. What is $E[X]$?

We know $E[X] = np$.

We computed identity $E[X^k] = npE[(Y + 1)^{k-1}]$ where $Y$ is binomial with parameters $(n - 1, p)$.


So $\text{Var}[X] = E[X^2] - E[X]^2 = np(n - 1)p + np - (np)^2 = np(1 - p) = npq$, where $q = 1 - p$.

Commit to memory: variance of binomial $(n, p)$ random variable is $npq$. 

This is $n$ times the variance you'd get with a single coin. Coincidence?
Let $X$ be binomial $(n, p)$. What is $E[X]$?

We know $E[X] = np$.

We computed identity $E[X^k] = npE[(Y + 1)^{k-1}]$ where $Y$ is binomial with parameters $(n - 1, p)$.


So $\text{Var}[X] = E[X^2] - E[X]^2 = np(n - 1)p + np - (np)^2 = np(1 - p) = npq$, where $q = 1 - p$.

Commit to memory: variance of binomial $(n, p)$ random variable is $npq$.

This is $n$ times the variance you’d get with a single coin. Coincidence?
Compute variance with decomposition trick

- \( X = \sum_{j=1}^{n} X_j \), so
  
  \[
  E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]
  \]
Compute variance with decomposition trick

- $X = \sum_{j=1}^{n} X_j$, so
  
  $E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]$

- $E[X_i X_j]$ is $p$ if $i = j$, $p^2$ otherwise.
Compute variance with decomposition trick

- $X = \sum_{j=1}^{n} X_j$, so
  \[ E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j] \]
- $E[X_i X_j]$ is $p$ if $i = j$, $p^2$ otherwise.
- $\sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]$ has $n$ terms equal to $p$ and $(n - 1)n$ terms equal to $p^2$. 

\[ E[X^2] = np + (n - 1)np^2 = np + np^2 - np^2 = np \]

Thus

\[ \text{Var}[X] = E[X^2] - E[X]^2 = np - npq = npq \]
Compute variance with decomposition trick

- \( X = \sum_{j=1}^{n} X_j \), so
  \[ E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j] \]
- \( E[X_i X_j] \) is \( p \) if \( i = j \), \( p^2 \) otherwise.
- \( \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j] \) has \( n \) terms equal to \( p \) and \( (n - 1)n \) terms equal to \( p^2 \).
- So
  \[ E[X^2] = np + (n - 1)np^2 = np + (np)^2 - np^2. \]
Compute variance with decomposition trick

- $X = \sum_{j=1}^{n} X_j$, so
  
  $E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]$

- $E[X_i X_j]$ is $p$ if $i = j$, $p^2$ otherwise.

- $\sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]$ has $n$ terms equal to $p$ and $(n - 1)n$ terms equal to $p^2$.

- So $E[X^2] = np + (n - 1)np^2 = np + (np)^2 - np^2$.

- Thus
  

Outline

Bernoulli random variables

Properties: expectation and variance

More problems
Bernoulli random variables

Properties: expectation and variance

More problems
More examples

- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a .05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?
More examples

- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a .05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?

\[ \sum_{j=201}^{205} \binom{205}{j} .95^j .05^{205-j} \]

- In a 100 person senate, forty people always vote for the Republicans' position, forty people always for the Democrats' position and 20 people just toss a coin to decide which way to vote. What is the probability that a given vote is tied?

\[ \left( \frac{20}{10} \right) / 2 \]

- You invite 50 friends to a party. Each one, independently, has a 1/3 chance of showing up. What is the probability that more than 25 people will show up?

\[ \sum_{j=26}^{50} \binom{50}{j} \left( \frac{1}{3} \right)^j \left( \frac{2}{3} \right)^{50-j} \]
More examples

- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a .05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?

\[ \sum_{j=201}^{205} \binom{205}{j} \cdot 0.95^j \cdot 0.05^{205-j} \]

- In a 100 person senate, forty people always vote for the Republicans’ position, forty people always for the Democrats’ position and 20 people just toss a coin to decide which way to vote. What is the probability that a given vote is tied?

\[ \binom{20}{10} / 2^{20} \]
More examples

- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a 0.05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?
  \[ \sum_{j=201}^{205} \binom{205}{j} \cdot 0.95^j \cdot 0.05^{205-j} \]

- In a 100 person senate, forty people always vote for the Republicans’ position, forty people always for the Democrats’ position and 20 people just toss a coin to decide which way to vote. What is the probability that a given vote is tied?
  \[ \binom{20}{10} / 2^{20} \]
More examples

- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a .05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?

\[ \sum_{j=201}^{205} \binom{205}{j} \cdot 0.95^j \cdot 0.05^{205-j} \]

- In a 100 person senate, forty people always vote for the Republicans’ position, forty people always for the Democrats’ position and 20 people just toss a coin to decide which way to vote. What is the probability that a given vote is tied?

\[ \binom{20}{10} / 2^{20} \]

- You invite 50 friends to a party. Each one, independently, has a 1/3 chance of showing up. What is the probability that more than 25 people will show up?
More examples

- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a .05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?

\[ \sum_{j=201}^{205} \binom{205}{j} .95^j .05^{205-j} \]

- In a 100 person senate, forty people always vote for the Republicans’ position, forty people always for the Democrats’ position and 20 people just toss a coin to decide which way to vote. What is the probability that a given vote is tied?

\[ \binom{20}{10} / 2^{20} \]

- You invite 50 friends to a party. Each one, independently, has a 1/3 chance of showing up. What is the probability that more than 25 people will show up?

\[ \sum_{j=26}^{50} \binom{50}{j} (1/3)^j (2/3)^{50-j} \]