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LETTERS AND COMMENTS

A simple semiclassical derivation of Hartman's effect

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Abstract. We present a very simple semiclassical derivation of Hartman's effect valid for potential barriers of general shape. The derivation also gives some insight into the tunnelling time phenomenon.

Following the standard stationary phase argument on quasi-monochromatic wave packets, it is well known that transmission and reflection (group) time delays are given in one-dimensional quantum mechanics by the energy derivative of the phases of the transmission and reflection amplitudes, respectively (see [1] and the references cited therein).

The explicit calculation of time delay in the problem of a particle tunnelling through a rectangular barrier of width d, yields the so-called Hartman's effect [2, 3]: as $d \to \infty$, the transmission time delay $\tau_{\mathrm{tr}}(E)$ is given by

$$\tau_{\mathrm{tr}}(E) \to -\frac{d}{v}\left[1 + \mathrm{O}\left(\frac{1}{d}\right)\right] \tag{1}$$

where $v = \sqrt{2E/m}$ is the group velocity, m the mass of the particle and E its incoming kinetic energy. According to (1), as the width of the barrier d increases, the so-called tunnelling time‡

$$t_{\mathrm{tr}}(E) = \frac{d}{v} + \tau_{\mathrm{tr}}(E) \tag{2}$$

which can be defined as the sum of the free reference traversal time d/v and transmission delay time, becomes independent of the barrier width. This implies, in particular, that for sufficiently large barriers the effective group velocity of the particle inside the barrier can become arbitrarily large, allowing for traversal mean group velocities even larger than the light speed in vacuum§. Such a surprising observation is at the heart of what has been called the tunnelling time problem, which has attracted increasing interest in the last two decades, mainly motivated by the prospect of making ultrafast electronic devices based on tunnelling.

How can a tunnelling particle travel in a barrier faster than an identical free reference particle? Possible interpretations of Hartman's effect are still under debate. However, it has been generally recognized that one of the basic ingredients of the explanation is to understand the effect in terms of a reshaping of the wave packet, the observed time advance of the tunnelling particle being then a consequence of the fact that the barrier acts like a filter transmitting preferentially the leading edge of the incoming wave packet instead of the trailing one. As a consequence, the transmitted peak appears to be shifted to earlier times in comparison with the peak of an identical free reference incoming particle.

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‡ The tunnelling time is the time spent, on average, by the tunnelling particle inside the barrier region. The notion of tunnelling time does not, however, have a unique definition in quantum mechanics since it corresponds to the joint measurement of two incompatible observables. Reviews of the long debate on this problem can be found, for instance, in [1] and in [3–7].

§ This does not mean, however, that information can be conveyed faster than light. See, for instance, the experiment with single photons of Steinberg *et al* [8], and the references cited therein.

The usual procedure for deriving Hartman's effect (1) involves solving the stationary Schrödinger equation for the rectangular barrier, finding the explicit expression for the phase of the transmission amplitude, differentiating it with respect to energy, and finally studying its asymptotic behaviour as $d \to \infty$. The disadvantage of such a procedure is that it represents a rather lengthy calculation limited to the very special case of the rectangular barrier. The purpose of the present note is to present an alternative very simple semiclassical derivation of Hartman's effect (1), and to show that the derivation allows for a simple interpretation of the effect in terms of two ingredients: classical reflection from a barrier and unitarity of the scattering matrix. The derivation requires practically no calculations and has the advantage of being general since it does not depend on the specification of the form of the transition region of the barrier. All that is needed is to assume that the barrier is smoothly switched on and off on a larger scale than the wavelength of the incoming particle, so that the semiclassical approximation applies†.

For simplicity, we start by assuming that the barrier is symmetric. Let

$$V(x) = \begin{cases} V_0 & \text{if } |x| \leqslant \frac{d}{2} \\ V_0 g(|x| - \frac{d}{2}) & \text{otherwise} \end{cases} \qquad (3)$$

$V_0 > 0$

be the potential function describing the barrier, with $g(r)$, $r \geqslant 0$, being a smooth function describing the switching on and off of the barrier in space, with support in the interval $[0, \ell/2]$, $\ell > 0$, $g(0) = 1$. Note that $V(x)$ has its support inside the interval $[-(d+\ell)/2, (d+\ell)/2]$.

We consider a classical particle with incident energy $E < V_0$, coming, say, from the left. Since the incoming energy is below the barrier height, the particle will reach the extreme point $x_0 < -d/2$, the smallest x such that $E + V(x) = 0$, and will be reflected back to the left. By definition, the classical time delay, $\tau_{\rm re}^{\rm cl}(E)$, experienced by the reflected particle is given by the difference between the time it spends inside the barrier region and the time spent in the same region by a free reference particle which is reflected back at the origin [9].

The time spent by the particle inside the infinitesimal interval between x and $x + \mathrm{d}x$ is $2\mathrm{d}x/v(x)$, where $v(x) = \sqrt{2(E - V(x))/m}$ is the speed at point x and the factor 2 takes into account the fact that, since the particle is reflected, it passes the same point twice. By integration, it follows that the classical reflection time delay is simply given by the difference [9]

$$\tau_{\rm re}^{\rm cl}(E) = 2\left[\int_{-(d+\ell)/2}^{x_0} \mathrm{d}x\, \frac{1}{v(x)} - \int_{-(d+\ell)/2}^{0} \mathrm{d}x\, \frac{1}{v}\right]. \qquad (4)$$

Performing in (4) the change of variables $y = x + d/2$, and defining the point $y_0 < 0$ by $y_0 = x_0 + d/2$, one easily obtains

$$\tau_{\rm re}^{\rm cl}(E) = \left[\int_{-\ell/2}^{y_0} \mathrm{d}y \sqrt{\frac{2m}{2(E - V_0 g(|y|))}} - \frac{\ell}{v}\right] - \frac{d}{v}. \qquad (5)$$

Since the term in the square brackets in (5) is independent of d, one finds for the classical reflection time delay the following asymptotic behaviour, as $d \to \infty$:

$$\tau_{\rm re}^{\rm cl}(E) \to -\frac{d}{v}\left[1 + \mathrm{O}\left(\frac{1}{d}\right)\right]. \qquad (6)$$

One can then conclude the derivation with the following two simple remarks:

(a) classical reflection time delay (4)–(5) also corresponds to the semiclassical approximation of the quantum mechanical reflection time delay [9];

(b) for a parity-invariant potential the difference between the phases of the transmission and reflection amplitudes is (modulo π) a constant equal to $\pi/2$, implying that the quantum mechanical transmission and reflection time delays (given by the energy derivatives of these phases) must coincide (see also the appendix):

$$\tau_{\rm tr}(E) = \tau_{\rm re}(E). \qquad (7)$$

† For a sufficiently smooth potential function, the semiclassical approximation holds when $k\ell \gg 1$, i.e. when the scale ℓ at which the switching on and off of the barrier in space occurs is on a much larger scale than the de Broglie wavelength $2\pi/k$ of the particle. This occurs, for instance, if the potential is assumed to be produced by a macroscopic device.

In view of the above two remarks, it follows that equations (4)–(5) also correspond to the semiclassical approximation of the quantum mechanical transmission time delay and that, in particular, (6) gives the correct asymptotic behaviour for the semiclassical transmission time delay, which is the desired expression (1) for the (here semiclassical) Hartman's effect.

The same is also true for a non-symmetric barrier. Indeed, in that case one has simply to replace (7) by the more general equality (see appendix):

$$\tau_{\mathrm{tr}}(E) = \frac{1}{2}\left(\tau_{\mathrm{re}}^{+}(E) + \tau_{\mathrm{re}}^{-}(E)\right) \tag{8}$$

where $\tau_{\mathrm{re}}^{+}(E)$ and $\tau_{\mathrm{re}}^{-}(E)$ are the reflection time delays for a particle coming from the left and from the right, respectively. Thus, the same arguments as for the symmetric case apply.

According to the simple semiclassical derivation presented in this note, one can understand Hartman's effect in terms of two simple ingredients: classical reflection from a barrier and unitarity of the scattering matrix. Since a classical reflected particle does not explore the entire width of the barrier, the time it spends inside it cannot depend on the barrier width d, as d increases. On the other hand, unitarity of the scattering matrix (a pure quantum mechanical property) forces transmission and reflection time delays to coincide (in the sense of (7) for a parity-invariant potential and in the sense of (8) in the general case), implying that the same must be true (at least on a semiclassical level) for the time spent inside the barrier by the tunnelling particle.

Appendix

The coincidence of the quantum mechanical transmission and reflection time delays is a simple consequence of the unitarity of the one-dimensional scattering matrix

$$S(E) = \begin{pmatrix} T(E) & R(E) \\ L(E) & T(E) \end{pmatrix}$$

where $T(E)$ is the transmission amplitude, $R(E)$ and $L(E)$ the reflection amplitudes for a particle coming from the right and from the left, respectively. Indeed, the unitarity relation $S^{\dagger}(E)S(E) = I$ implies probability conservation $|T(E)|^2+|R(E)|^2 = |T(E)|^2 + |L(E)|^2 = 1$, as well as the relation $T^*(E)R(E) + L^*(E)T(E) = 0$ or, equivalently, using $|R(E)| = |L(E)|$,

$$\alpha_{\mathrm{T}}(E)+\frac{\pi}{2} = \frac{1}{2}\left(\alpha_{\mathrm{R}}(E)+\alpha_{\mathrm{L}}(E)\right) \pmod{\pi}$$

for the corresponding phases of the transmission and reflection amplitudes. Deriving the above equation with respect to energy, one obtains (8). In the special case of a symmetric potential, $\alpha_{\mathrm{R}}(E) = \alpha_{\mathrm{L}}(E)$, and one finds (7). Note that the same result can also be obtained by exploiting the properties of the Wronskian; see, for instance, [10].

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