Comment on ''Factorization of scattering matrices due to partitioning of potentials in one-dimensional Schrödinger-type equations'' [J. Math. Phys. 37, 5897 (1996)]

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In their recent paper,[1] Aktosun, Klaus, and van der Mee considered the problem of factorization of the scattering matrix for the one-dimensional Schrödinger equation and two of its generalizations (wave propagation in a nonhomogeneous medium and wave propagation in a nonconservative medium), and the related problem of decomposition of a potential into fragments.

Our first comment is to point out that in Sec. III of Ref. 2 some similar results were also obtained, using Levinson's theorem, for the special case of the two-potential system for the Schrödinger equation when each fragment is compactly supported. Moreover, the general description of bound-states and zero-energy resonances of finite periodic potentials presented in Sec. IV of Ref. 2 provides a nice example of a decomposition of a potential into fragments.

Our second comment concerns the factorization formula for the scattering matrix which was proven in Ref. 3 and reconsidered in Ref. 1 in order to allow for the inclusion of Dirac delta functions in the potential. It is worth mentioning, for completeness, the recent proof of the factorization formula given in Ref. 4, for the problem of a scattering particle with position-dependent mass. The proof uses an adaptation of the variable phase method which allows for the derivation of first-order differential equations for transmission and reflexion amplitudes. These equations remain consistent if the potential includes Dirac delta functions and their integration gives the factorization property of the scattering matrix. Since the results of Ref. 1 are based on the existence of a factorization formula, they remain also valid for the generalized Schrödinger equation with position-dependent mass (see also Ref. 5 for the analysis of the finite periodic potential for a particle having position-dependent mass).

We conclude our comments by presenting a third alternative proof of factorization of the scattering matrix using integral equations instead of the Schrödinger differential equation. We restrict ourselves to the factorization formula for the transmission amplitude [formula (2.18) of Ref. 1] and of a potential fragmented into only two pieces: $V(x) = V_{0,1}(x) + V_{1,2}(x)$ (for the notations we refer the reader to Ref. 1. The general case of a potential fragmented into N pieces follows by induction).

The transmission amplitude, $T(k)$, and the reflection amplitude for a particle incident from the right, $R(k)$, associated with the potential $V(x)$, admit the following well known integral representations:

$$T(k)=1-\frac{i}{2k}\int dx(V_{0,1}(x)+V_{1,2}(x))e^{ikx}\psi_r(k,x), \tag{1}$$

$$=T_{0,1}(k)-\frac{i}{2k}\int dx\ V_{1,2}(x)\psi_{l;0,1}(k,x)\psi_r(k,x), \tag{2}$$

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$$=T_{1,2}(k)-\frac{i}{2k}\int dx\ V_{0,1}(x)\psi_{l;1,2}(k,x)\psi_r(k,x), \tag{3}$$

and

$$R(k)=-\frac{i}{2k}\int dx(V_{0,1}(x)+V_{1,2}(x))e^{-ikx}\psi_r(k,x), \tag{4}$$

$$=R_{0,1}(k)-\frac{i}{2k}\int dx\ V_{1,2}(x)\psi_{r;0,1}(k,x)\psi_r(k,x), \tag{5}$$

$$=R_{1,2}(k)-\frac{i}{2k}\int dx\ V_{0,1}(x)\psi_{r;1,2}(k,x)\psi_r(k,x), \tag{6}$$

where $\psi_l(k,x)=T(k)f_l(k,x)$ and $\psi_r(k,x)=T(k)f_r(k,x)$ are the usual physical solutions for an incoming wave from the left and from the right, respectively. Since by definition $V_{0,1}(x)=0$ for x in the support of $V_{1,2}(x)$ and *vice versa*, one can replace in Eqs. (2) and (3) the solutions $\psi_{l;0,1}(k,x)$ and $\psi_{l;1,2}(k,x)$ by their asymptotic forms. Combining the obtained equations with Eq. (1), one finds the relation

$$T(k)=T_{0,1}(k)T_{1,2}(k)-T_{0,1}(k)L_{1,2}(k)\ \frac{i}{2k}\int dx\ V_{0,1}(x)e^{-ikx}\psi_r(k,x). \tag{7}$$

Similarly, one can replace in Eq. (5) $\psi_{r;0,1}(k,x)$ by its asymptotic form and combine it with Eqs. (4) and (7). The result is

$$T(k)=T_{0,1}(k)T_{1,2}(k)+T_{0,1}(k)L_{1,2}(k)R_{1,0}(k)\left[1-\frac{i}{2k}\int dx\ V_{1,2}(x)e^{ikx}\psi_r(k,x)\right]. \tag{8}$$

Finally, using Eq. (2) into Eq. (8) one obtains

$$T(k)=T_{0,1}(k)T_{1,2}(k)+L_{1,2}(k)R_{1,0}(k)T(k) \tag{9}$$

or equivalently

$$T(k)=\frac{T_{0,1}(k)T_{1,2}(k)}{1-L_{1,2}(k)R_{1,0}(k)}, \tag{10}$$

which is the desired factorization property for the transmission amplitude. Similar arguments lead to factorization formulas for reflection amplitudes.

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