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On the low- and high-frequency limit of quantum scattering by time-dependent potentials

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Abstract. Using time-dependent methods, we study the scattering of a quantum mechanical particle by short-range potentials with very slow or very fast periodic variations in time. The low- and high-frequency limits are derived as well as their first non-vanishing corrections, and their physical significance discussed.

1. Introduction

The scattering of a quantum mechanical particle by a time-dependent short-range potential $v(x, t)$ has been the subject of numerous investigations, at a general theoretical level and for specific systems. Typical examples are:

(i) tunnelling of a particle through a modulated barrier

$$v(x, t) = w_1(x) + \lambda(t) w_2(x) \tag{1.1}$$

where $w_1(x)$ and $w_2(x)$ are static potentials, and $\lambda(t)$ a time-dependent coupling strength (see for instance [1–6]);

(ii) scattering by a moving centre [5–10]

$$v(x, t) = v(x - a(t)) \tag{1.2}$$

where $a(t)$ is a prescribed classical trajectory. Such time-displaced potentials occur for instance in the study of the AC Stark effect and in the modelling of chemical reactions at surfaces.

In this paper, we consider time-dependent potentials of the form $v(x, \omega t)$, where ω^{-1} is either a very large or a very small parameter. We shall mainly be concerned with potentials that are periodic in time, $v(x, t) = v(x, t + 2\pi)$, so that $\omega \to 0$ is referred to as the low-frequency (or adiabatic) limit and $\omega \to \infty$ as the high-frequency limit. This includes, for instance, the system (1.1) with an oscillating barrier $\lambda(\omega t) = \lambda_0 \cos(\omega t)$, and the system (1.2) with an oscillating centre $a(\omega t) = a_0 \cos(\omega t)$.

Although the general formalism is well developed (existence of wave operators, unitarity of the scattering operator, see for instance [11–18]), it is notoriously difficult to perform explicit analytical calculations of scattering probabilities, even in the simplest models. It is therefore of interest to control the limit situations $\omega \to 0$ and $\omega \to \infty$, as well as the corresponding asymptotic expansions, and to discuss their physical significance.

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The result is that in both cases, for suitably smooth and short-range potentials, the scattering by the time-dependent potential effectively reduces to a static one. In the low-frequency limit, according to the discussion in section 2, we show that the scattering probabilities converge to the average probabilities associated with the family of time-independent potentials $v(x,\alpha)$, $0 \leqslant \alpha < 2\pi$. In the high-frequency limit, we obtain that the scattering quantities approach those associated with the static average potential $v_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, v(x,\alpha)$.

In section 2, we discuss the general aspects of scattering with time-dependent potentials, and, in particular, the implications of the fact that the dynamics is not invariant under time translations. One point is that particles entering the interaction region at different times do not experience the same configuration of the potential. We are thus led to distinguish the following two situations, the first one being the one more commonly realized.

(i) The arrival times of the particles in the interaction region have no relation with the characteristic time scale ω^{-1} of the variation in the potential. This happens when there is no control on the times at which the particles in the incident beam are prepared. One has the same effect with a single particle if the energy spreading ΔE of the incoming wavepacket is sufficiently narrow to give a time dispersion† $\Delta t \simeq \hbar/\Delta E$ larger than ω^{-1}. In all these situations, only averages of scattering probabilities over the time scale ω^{-1} can be observed.

(ii) The incoming beam consists of regular short pulses in phase with the variation in the potential so that all particles feel the same potential when they enter the interaction region.

Section 3 is devoted to the adiabatic limit of the scattering operator. The limit has been studied in [4, 11, 20, 21], and an asymptotic expansion of the S-operator is presented in [22]. Here we state and prove the result up to first order in ω‡. At the lowest order, the transition probabilities reduce to those associated with the static potential $v(x,\alpha)$, α fixed, in case (ii), or to their average in case (i).

The first-order correction in ω can be split into the sum of two terms. The first one involves again only static transition probabilities, but for the potential shifted in time, according to the time of incidence of the incoming particle. It thus makes explicit, at first order in ω, that incoming states prepared at different times (i.e. differing by energy-dependent phase factors) see different configurations of the potential. This term contributes non-trivially in case (ii), but its average vanishes so it does not contribute in case (i). The second term has a more complicated structure: it embodies, at order ω, the dynamical effects due to the effective time variation in the potential, and can be expressed in terms of energy derivatives of static quantities. Then, we discuss briefly the first correction to the adiabatic limit of scattering events when a fixed number n of energy quanta $\hbar\omega$ is exchanged with the external field (sidebands) as well as the corresponding energy transfer.

In section 4, we address the problem of the high-frequency limit of the scattering operator. The limit is of interest in the physics of atoms in intense laser fields [7–9] (it has also been investigated numerically in [5]), but we are unaware of a general proof of its existence and of its first non-vanishing correction.

In section 5 we specialize our results to the case of transmission and reflection probabilities in the one-dimensional scattering problem. In particular, the statistics of quanta will be obtained explicitly in the case of the time-displaced potential (1.2). Finally, in section 6, we present some concluding remarks.

† We recall that the proper sense of the time–energy uncertainty relation $\Delta E \Delta t \simeq \hbar$, for a free particle, is that one is unable to say when it will cross a given surface with an exactitude greater than $\Delta t \simeq \hbar/\Delta E$; see for instance [19].

‡ Reference [22] has appeared during the completion of this work, and the proof of proposition 1 is similar.

2. The scattering problem for time-dependent potentials: general setting

We are concerned with scattering systems with time-dependent potentials $V(\omega t)$, where $V(\omega t)$ is a multiplication operator by a sufficiently short-ranged function $v(x, \omega t)$ in configuration space $\mathbb{R}^d$, $d \geqslant 1$. The total Hamiltonian is

$$H(\omega t) = H_0 + V(\omega t) \tag{2.1}$$

where $H_0 = -\Delta/2m$ is the free Hamiltonian (m is the mass of the particle and we have set the Planck constant $\hbar = 1$) and ω is a parameter. By $U_\omega(t, t_0)$ we denote the corresponding evolution operator with initial condition $U_\omega(t_0, t_0) = I$. It is known that for a large class of potentials $v(x, t)$, that we do not need to specify here (see section 3), and any real τ, there exist wave operators

$$\Omega_\pm(\omega, \tau) = \operatorname*{s-lim}_{t\to\pm\infty} U_\omega^\dagger(t+\tau, \tau)\mathrm{e}^{-\mathrm{i}H_0 t} \tag{2.2}$$

and a unitary scattering operator

$$S(\omega, \tau) = \Omega_+^\dagger(\omega, \tau)\Omega_-(\omega, \tau). \tag{2.3}$$

The wave operators $\Omega_\pm(\omega) \equiv \Omega_\pm(\omega, 0)$ and $\Omega_\pm(\omega, \tau)$ are related by the generalized intertwining property

$$\Omega_\pm(\omega, \tau) = U_\omega(\tau, 0)\Omega_\pm(\omega)\mathrm{e}^{\mathrm{i}H_0\tau} \tag{2.4}$$

which, in turn, yields for the scattering operators $S(\omega) \equiv S(\omega, 0)$ and $S(\omega, \tau)$ the relation

$$S(\omega, \tau) = \mathrm{e}^{-\mathrm{i}H_0\tau} S(\omega)\mathrm{e}^{\mathrm{i}H_0\tau}. \tag{2.5}$$

Let F be an arbitrary projection operator in $\mathcal{H} = L^2(\mathbb{R}^d)$, commuting with H_0. We are interested in the quantity

$$\mathcal{P}_F(\omega, \tau, \varphi) = \|FS(\omega, \tau)\varphi\|_2^2 = \|FS(\omega)\mathrm{e}^{\mathrm{i}H_0\tau}\varphi\|_2^2 \tag{2.6}$$

which is the probability of finding asymptotically the scattering state in the subspace $F\mathcal{H}$, for an incoming wavepacket $\mathrm{e}^{\mathrm{i}H_0\tau}\varphi$. Clearly, φ and $\mathrm{e}^{\mathrm{i}H_0\tau}\varphi$ represent two identically prepared states, except for a time lag τ. Since the interaction is not invariant under time translations, all possible scattering events are not described by a single scattering operator, but by the whole family $S(\omega, \tau)$, $\tau \in \mathbb{R}$.

In the discussion hereabove, the origin of time was conventionally fixed so that the potential has amplitude $v(x, 0)$ at $t = 0$. We could equally as well have chosen the potential to equal $v(x, \alpha)$ at $t = 0$, for some $\alpha \neq 0$. In this case, we replace (2.1) by

$$H(\omega t + \alpha) = H_0 + V(\omega t + \alpha) \tag{2.7}$$

with the corresponding evolution $U_\omega^\alpha(t, t_0)$, wave operators $\Omega_\pm^\alpha(\omega, \tau)$ and scattering operator $S^\alpha(\omega, \tau)$. Now, we have the equality

$$U_\omega^{\omega\tau}(t, t_0) = U_\omega(t+\tau, t_0+\tau) \tag{2.8}$$

since both evolution operators in (2.8) obey the same differential equation with the same initial condition at $t = t_0$. Setting $S^\alpha(\omega) \equiv S^\alpha(\omega, 0)$, this implies, in view of the definitions (2.2), (2.3), that

$$S(\omega, \tau) = S^{\omega\tau}(\omega). \tag{2.9}$$

Combining (2.5) with (2.9) yields

$$S^\alpha(\omega) = \mathrm{e}^{-\mathrm{i}H_0\alpha/\omega} S(\omega) \mathrm{e}^{\mathrm{i}H_0\alpha/\omega} \tag{2.10}$$

or, in differential form,

$$\mathrm{i}\omega\partial_\alpha S^\alpha(\omega) = [H_0, S^\alpha(\omega)]. \tag{2.11}$$

Equations (2.9) or (2.10) may be seen as a precise version of the simple statement that two incoming packets with a time lag τ will feel the external potential with the time difference τ. Let us restrict our attention to potentials periodic in time $V(t) = V(t + 2\pi)$, so that $V(\omega t)$ has period $2\pi/\omega$. This implies immediately, by (2.10),

$$S^\alpha(\omega) = S^{\alpha+2\pi}(\omega) = \mathrm{e}^{-\mathrm{i}H_0 2\pi/\omega} S^\alpha(\omega) \mathrm{e}^{\mathrm{i}H_0 2\pi/\omega} \tag{2.12}$$

i.e. the scattering operator commutes with free evolution over one period. The commutation relation (2.12) is the precise law of quasi-energy conservation which states that, while H_0 may not be conserved by scattering, the energy can be changed only by discrete quanta $n\omega$, $n = 0, \pm 1, \pm 2, \ldots$ [14, 18]. In view of the periodicity $S^\alpha(\omega) = S^{\alpha+2\pi}(\omega)$, we can introduce the Fourier decomposition of the scattering operator

$$S^\alpha(\omega) = \sum_{n=-\infty}^{\infty} S^n(\omega) \mathrm{e}^{-\mathrm{i}n\alpha} \tag{2.13}$$

with

$$S^n(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\alpha \, S^\alpha(\omega) \mathrm{e}^{\mathrm{i}n\alpha}. \tag{2.14}$$

The coefficients $S^n(\omega)$, $n \neq 0$, are called the 'sideband' contributions to the total scattering operator. Relation (2.11) implies obviously

$$n\omega S^n(\omega) = [H_0, S^n(\omega)]. \tag{2.15}$$

Thus, every sideband $S^n(\omega)$, $n \neq 0$, describes scattering events with an energy shift equal to $n\omega$, or, in other words, with emission ($n > 0$) or absorption ($n < 0$) of exactly n quanta of energy ω. The probabilities of such events are

$$\mathcal{P}_F^n(\omega, \varphi) = \|F S^n(\omega)\varphi\|_2^2. \tag{2.16}$$

In particular, for $F = I$,

$$\mathcal{P}^n(\omega, \varphi) = \|S^n(\omega)\varphi\|_2^2 \tag{2.17}$$

is the probability of emitting ($n > 0$) or absorbing ($n < 0$) n quanta during the scattering process.

The scattering probabilities that are relevant to situations (i) and (ii) described in the introduction are

$$\mathcal{P}_F^\alpha(\omega,\varphi) = \|FS^\alpha(\omega)\varphi\|_2^2 = \sum_{n,m=-\infty}^{\infty} (\varphi, S^{n\dagger}(\omega)FS^m(\omega)\varphi)\mathrm{e}^{\mathrm{i}(n-m)\alpha} \tag{2.18}$$

if the scattering is sensitive to the precise time at which the interaction occurs (case (ii)), and the corresponding average

$$\overline{\mathcal{P}}_F(\omega,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, \mathcal{P}_F^\alpha(\omega,\varphi) = \sum_{n=-\infty}^{\infty} \mathcal{P}_F^n(\omega,\varphi) \tag{2.19}$$

in the other situations (case (i)). Formula (2.19) applies, for instance, if the initial beam is constituted of a succession of incoming wavepackets with a small time lag $\mathrm{d}\tau$, that are scattered independently, and if the counters integrate all events during the period $2\pi/\omega$. Then, the observed probability will indeed be

$$\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} \mathrm{d}\tau\, \mathcal{P}_F(\omega,\tau,\varphi) = \overline{\mathcal{P}}_F(\omega,\varphi) \tag{2.20}$$

where the equality follows from (2.9) and (2.10). This applies also if the incoming state $\varphi(E)$, as a function of energy E, has support in an interval $\Delta E < \omega$, since then, as a consequence of (2.15), the off-diagonal contributions $(\varphi, S^{n\dagger}(\omega)FS^m(\omega)\varphi)$ vanish for $n \neq m$. Thus, for a small spreading in energy, i.e. for an uncertainty in time larger than $2\pi/\omega$, (2.18) also reduces to (2.19).

Notice that since the asymptotic observable $F^\alpha(\omega) = S^{\alpha\dagger}(\omega)FS^\alpha(\omega)$ also obeys (2.11) i.e.

$$\mathrm{i}\omega\partial_\alpha F^\alpha(\omega) = [H_0, F^\alpha(\omega)] \tag{2.21}$$

the average $\overline{F}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\, F^\alpha(\omega)$, as well as the sideband contributions $F^n(\omega) = S^{n\dagger}(\omega)FS^n(\omega)$ commute with the kinetic energy H_0. Thus, case (i) involves only the part of these observables on the energy shell, whereas in situation (ii), probabilities (2.18) depend, in particular, on the phase of the incident wavepacket and cannot simply be reduced to on-shell calculations.

Finally, let us consider the total energy variation in the particle during the scattering process

$$\Delta^\alpha(\omega) = H_0^\alpha(\omega) - H_0 \tag{2.22}$$

where $H_0^\alpha(\omega) = S^{\alpha\dagger}(\omega)H_0S^\alpha(\omega)$ is the asymptotic outgoing energy. According to (2.11), this quantity is related to the scattering operator by

$$\Delta^\alpha(\omega) = \mathrm{i}\omega S^{\alpha\dagger}(\omega)\partial_\alpha S^\alpha(\omega). \tag{2.23}$$

Introducing the sidebands (2.13) on the right-hand side of (2.23) and averaging over α gives the quasi-energy conservation law for an incoming state φ:

$$\sum_{n=-\infty}^{\infty} n\omega\mathcal{P}^n(\omega,\varphi) = (\varphi, \overline{\Delta}(\omega)\varphi) \tag{2.24}$$

where the $\mathcal{P}^n(\omega,\varphi)$ are defined in (2.17). Equation (2.24) is an energy balance: the loss or gain in energy of the scattered particles equals the average energy emitted or absorbed from the external field.

3. The low-frequency limit

3.1. The adiabatic theorem

Throughout the paper, we assume that the potential satisfies the bound

$$|v(x,t)| \leqslant b\,\langle x\rangle^{-\eta} \qquad \eta > 1 \tag{3.1}$$

where b is a constant independent of t and we have introduced the notation $\langle x\rangle = (1+|x|^2)^{1/2}$. Note that an equivalent formulation of (3.1) is to require that $\langle q\rangle^{\eta} V(t)$ is bounded uniformly with respect to t, where $q = (q_1, \ldots, q_d)$ is the multiplication operator by $x = (x_1, \ldots, x_d)$.

Then, one knows that the wave operators $\Omega^{\alpha}_{\pm}(\omega)$ associated with the Hamiltonian (2.7) exist as strong limits. In the periodic case, it is shown in [12] that they are complete†, and the scattering operator $S^{\alpha}(\omega)$ is unitary. Unitarity of the wave operators has also been proven for repulsive potentials, not necessarily time-periodic [15], and for potentials switched on and off in time [11]. We also quote the results from [16] for the case of moving potentials and for [17] for Hamiltonians asymptotically constant in time.

For the adiabatic as well as for the high-frequency limit (section 4), our proofs rely on the fact that, for suitably smooth incoming states, scattering states leave sufficiently rapidly a localized region in configuration space. This is expressed in the following lemma.

Lemma. Consider the scattering system (H, H_0), $H = H_0 + V$, where V is the multiplication by a static potential $v(x)$ such that both $v(x)$ and $x \cdot \nabla v(x)$ satisfy (3.1). *Let $\Omega_{\pm}$ be the corresponding wave operators and $\mathcal{D}$ the dense subset of $\mathcal{H}$ of vectors* $\varphi(x)$ such that their Fourier transform $\tilde{\varphi}(k)$ are infinitely differentiable functions of k with compact support and no support at the origin. Then, for any $\varphi \in \mathcal{D}$ and $\varepsilon > 0$, there exists a constant c, independent of t, such that

$$\|\langle q\rangle^{-\eta} \mathrm{e}^{-\mathrm{i}Ht}\Omega_{-}\varphi\|_2 \leqslant c(1+|t|)^{\varepsilon-\eta}. \tag{3.2}$$

The same estimate holds for Ω_{-} replaced by Ω_{+}.

The lemma is an immediate consequence of proposition 2 in [23]‡. It asserts that the scattering state $\mathrm{e}^{-\mathrm{i}Ht}\Omega_{-}\varphi$ propagates away sufficiently fast as $t \to \pm\infty$, provided that the incoming state is smooth and has non-vanishing kinetic energy, and η is large enough.

The above mentioned results hold for a wider class of potentials also allowing for local singularities (see [12, 23]). To avoid a technical development, we deal here only with bounded potentials.

To formulate the adiabatic limit, we introduce the scattering system $(H(\alpha), H_0)$, $H(\alpha) = H_0 + V(\alpha)$, determined by the static interaction $V(\alpha)$, α fixed, with corresponding wave operators $\Omega^{\alpha}_{\pm}$ and scattering operator $S^{\alpha} = \Omega^{\alpha\dagger}_{+}\Omega^{\alpha}_{-}$. Then, we expect that $S^{\alpha}(\omega)$ approaches S^{α} as $\omega \to 0$. More precisely, we have the following proposition.

† Complete means that the ranges $\mathrm{R}(\Omega^{\alpha}_{+}(\omega)) = \mathrm{R}(\Omega^{\alpha}_{-}(\omega))$ = absolutely continuous subspace of the monodromy operator $U^{\alpha}_{\omega}(2\pi/\omega, 0)$.

‡ Since $\tilde{\varphi}(k)$ is assumed to be infinitely differentiable, φ is in the domain of $|q|^{\rho}$, for all $\rho > 1$, and we have set the function $\phi(H)$ occurring in equation (30) of [23] equal to 1 on the support of $\tilde{\varphi}$, by intertwining.

Proposition 1. (i) If $V(t)$ satisfy (3.1) with $\eta > 1$ and is norm-continuous with respect to t, then

$$\operatorname*{s-lim}_{\omega\to 0} S^\alpha(\omega) = S^\alpha. \tag{3.3}$$

(ii) If in addition $V(t)$ is continuously differentiable in norm with respect to t, $v(x,t)$, $x \cdot \nabla v(x,t)$ and $\partial_t v(x,t)$ satisfy (3.1) with $\eta > 2$, then, for $\varphi \in \mathcal{D}$,

$$S^\alpha(\omega)\varphi = S^\alpha\varphi + \omega S_1^\alpha\varphi + \omega g^\alpha(\omega) \tag{3.4}$$

where

$$S_1^\alpha\varphi = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}t\, t\mathrm{e}^{\mathrm{i}H_0t}\Omega_+^{\alpha\dagger}V'(\alpha)\Omega_-^\alpha \mathrm{e}^{-\mathrm{i}H_0t}\varphi \tag{3.5}$$

and $V'(\alpha) = \partial_\alpha V(\alpha)$. The vector $g^\alpha(\omega)$ has a uniformly bounded norm and converges weakly to zero as $\omega \to 0$.

Proof. (i) We set

$$\Omega_t^\alpha = \mathrm{e}^{\mathrm{i}H(\alpha)t}\mathrm{e}^{-\mathrm{i}H_0t} \qquad \Omega_t^\alpha(\omega) = U_\omega^{\alpha\dagger}(t,0)\mathrm{e}^{-\mathrm{i}H_0t}. \tag{3.6}$$

Then,

$$\|(\Omega_+^\alpha(\omega) - \Omega_+^\alpha)\varphi\|_2 \leqslant \|(\Omega_+^\alpha(\omega) - \Omega_t^\alpha(\omega))\varphi\|_2 + \|(\Omega_+^\alpha - \Omega_t^\alpha)\varphi\|_2 + \|(\Omega_t^\alpha(\omega) - \Omega_t^\alpha)\varphi\|_2. \tag{3.7}$$

We have the standard estimate for $\varphi \in \mathcal{D}$,

$$\begin{aligned}\|(\Omega_+^\alpha(\omega) - \Omega_t^\alpha)\varphi\|_2 &\leqslant \int_t^\infty \mathrm{d}s\, \|V(\alpha+\omega s)\mathrm{e}^{-\mathrm{i}H_0s}\varphi\|_2 \\ &\leqslant \sup_s \|V(\alpha+\omega s)\langle q\rangle^\eta\| \int_t^\infty \mathrm{d}s\, \|\langle q\rangle^{-\eta}\mathrm{e}^{-\mathrm{i}H_0s}\varphi\|_2.\end{aligned} \tag{3.8}$$

Using (3.1) and applying (3.2) in the free case with $\eta > 1$ and ε sufficiently small, we see that (3.8) tends to zero as $t \to \infty$, uniformly with respect to ω. The same is obviously true for the second term on the right-hand side of (3.7). To estimate the last term in (3.7) we write

$$\Omega_t^\alpha(\omega) = U_\omega^{\alpha\dagger}(t,0)\mathrm{e}^{-\mathrm{i}H(\alpha)t}\Omega_t^\alpha = \Omega_t^\alpha - \mathrm{i}\int_0^t \mathrm{d}s\, U_\omega^{\alpha\dagger}(s,0)(V(\omega s+\alpha) - V(\alpha))\mathrm{e}^{-\mathrm{i}H(\alpha)s}\Omega_t^\alpha \tag{3.9}$$

leading to

$$\|(\Omega_t^\alpha(\omega) - \Omega_t^\alpha)\varphi\|_2 \leqslant \int_0^t \mathrm{d}s\, \|V(\omega s+\alpha) - V(\alpha)\| \tag{3.10}$$

which tends to zero as $\omega \to 0$, for any fixed t, because of the norm-continuity of $V(t)$. Thus, letting first $\omega \to 0$, and then $t \to \infty$ in (3.7), we have shown that $\lim_{\omega\to 0}\Omega_+^\alpha(\omega)\varphi = \Omega_+^\alpha\varphi$ for any $\varphi \in \mathcal{D}$, and the same proof clearly holds for $\Omega_-^\alpha(\omega)$. Since the $\Omega_\pm^\alpha(\omega)$ are uniformly bounded with respect to ω, the result is true for all $\varphi \in \mathcal{H}$, implying that $S^\alpha(\omega)$ converges weakly to S^α as $\omega \to 0$. But, since the $S^\alpha(\omega)$ are isometric, $S^\alpha(\omega) \to S^\alpha$ strongly.

(ii) Consider the scattering system defined by the pair $(H(\omega t+\alpha), H(\alpha))$. It is easy to see that the corresponding wave operators $\tilde{\Omega}^\alpha_\pm(\omega)$ exist on the range $\mathrm{R}(\Omega^\alpha_-) = \mathrm{R}(\Omega^\alpha_+)$. For this, it suffices to note, following the estimate (3.8), that for $\varphi \in \mathcal{D}$,

$$\begin{aligned}\|(V(\omega t+\alpha)-V(\alpha))\mathrm{e}^{-\mathrm{i}H(\alpha)t}\Omega^\alpha_-\varphi\|_2 &\leqslant (\|V(\omega t+\alpha)\langle q\rangle^\eta\|+\|V(\alpha)\langle q\rangle^\eta\|)\|\langle q\rangle^{-\eta}\mathrm{e}^{-\mathrm{i}H(\alpha)t}\Omega^\alpha_-\varphi\|_2 \\ &\leqslant 2bc(1+|t|)^{\varepsilon-\eta}\end{aligned} \qquad (3.11)$$

which is integrable for ε sufficiently small. This enables us to represent the full scattering operator $S^\alpha(\omega)$ by the chain rule [18], as

$$S^\alpha(\omega) = \Omega^{\alpha\dagger}_+ \tilde{S}^\alpha(\omega)\Omega^\alpha_- \qquad (3.12)$$

where $\Omega^\alpha_\pm$ are the wave operators belonging to $(H(\alpha), H_0)$ and $\tilde{S}^\alpha(\omega) = \tilde{\Omega}^{\alpha\dagger}_+(\omega)\tilde{\Omega}^\alpha_-(\omega)$ is the scattering operator of the system $(H(\omega t+\alpha), H(\alpha))$, defined on $\mathrm{R}(\Omega^\alpha_-)$. Let $\psi = \Omega^\alpha_-\varphi$, $\varphi \in \mathcal{D}$. One can represent $\tilde{S}^\alpha(\omega)\psi$ as a weak limit in $\mathrm{R}(\Omega^\alpha_-)$ by

$$(\chi, \tilde{S}^\alpha(\omega)\psi) = \lim_{\substack{t\to\infty \\ t_0\to-\infty}} (\chi, \tilde{S}^\alpha(\omega; t, t_0)\psi) \qquad \chi \in \mathrm{R}(\Omega^\alpha_-) \qquad (3.13)$$

with

$$\begin{aligned}\tilde{S}^\alpha(\omega; t, t_0) &= \mathrm{e}^{\mathrm{i}H(\alpha)t}U^\alpha_\omega(t, t_0)\mathrm{e}^{-\mathrm{i}H(\alpha)t_0} \\ &= I - \mathrm{i}\mathrm{e}^{\mathrm{i}H(\alpha)t}U^\alpha_\omega(t, 0)\int_{t_0}^t \mathrm{d}s\, U^{\alpha\dagger}_\omega(s, 0)(V(\omega s+\alpha) - V(\alpha))\mathrm{e}^{-\mathrm{i}H(\alpha)s}.\end{aligned} \qquad (3.14)$$

It is clear that the formal limit $t_0 \to -\infty$, $t \to \infty$ and $\omega \to 0$ of the second term of (3.14), when inserted in (3.12), gives the expression (3.5) of the first-order correction. The relevant estimates to prove (3.4) are as follows. From (3.14), we have (omiting from now on the index and argument α)

$$\|(\tilde{S}(\omega; t, t_0) - I)\psi\|_2 \leqslant \int_{-\infty}^{\infty} \mathrm{d}s\, \|(V(\omega s) - V)\mathrm{e}^{-\mathrm{i}Hs}\psi\|_2. \qquad (3.15)$$

As seen in (3.11), the integrand in (3.15) is majorized uniformly in ω by an integrable function of s, and it tends pointwise to zero by the continuity of $V(t)$. Thus, dominated convergence implies

$$\lim_{\omega\to 0}\sup_{t,t_0}\|(\tilde{S}(\omega; t, t_0) - I)\psi\|_2 = 0 \qquad (3.16)$$

and, in view of the unitarity of $\tilde{S}(\omega; t, t_0)$, the limit (3.16) can be extended to all $\psi \in \mathrm{R}(\Omega_-)$. Then, we write the equality (3.14) in the form

$$\begin{aligned}(\chi, \tilde{S}(\omega; t, t_0)\psi) = (\chi, \psi) &- \mathrm{i}\omega \int_{t_0}^t \mathrm{d}s\, s(\chi, \mathrm{e}^{\mathrm{i}Hs}V'\mathrm{e}^{-\mathrm{i}Hs}\psi) \\ &+ \omega(\chi, g_1(\omega; t, t_0)) + \omega(\chi, g_2(\omega; t, t_0))\end{aligned} \qquad (3.17)$$

with

$$(\chi, g_1(\omega; t, t_0)) = -\mathrm{i}\left(U_\omega^\dagger(t,0)\mathrm{e}^{-\mathrm{i}Ht}\chi, \int_{t_0}^{t} \mathrm{d}s\, s U_\omega^\dagger(s,0)\left(\frac{V(\omega s) - V}{\omega s} - V'\right)\mathrm{e}^{-\mathrm{i}Hs}\psi\right) \tag{3.18}$$

$$(\chi, g_2(\omega; t, t_0)) = -\mathrm{i}\int_{t_0}^{t} \mathrm{d}s\, s((\tilde{S}(\omega; t, s) - I)\chi, \mathrm{e}^{\mathrm{i}Hs}V'\mathrm{e}^{-\mathrm{i}Hs}\psi). \tag{3.19}$$

By the Schwartz inequality we have, for $g_1(\omega; t, t_0)$,

$$|(\chi, g_1(\omega; t, t_0))| \leqslant \|\chi\|_2 \int_{-\infty}^{\infty} \mathrm{d}s\, |s| \left\| \left(\frac{V(\omega s) - V}{\omega s} - V'\right)\mathrm{e}^{-\mathrm{i}Hs}\psi \right\|_2. \tag{3.20}$$

By assumption, $\frac{1}{\omega s}(V(\omega s) - V) - V'$ satisfies the bound (3.1) and tends in the norm to zero as $\omega \to 0$. By (3.2) with $\eta > 2$, the integrand of (3.20) is majorized uniformly in ω by an integrable function and tends pointwise to zero as $\omega \to 0$. We conclude by dominated convergence that $\lim_{\omega\to 0}(\chi, g_1(\omega; t, t_0)) = 0$, uniformly with respect to t_0 and t. For $g_2(\omega; t, t_0)$ we have

$$\begin{aligned} |(\chi, g_2(\omega; t, t_0))| &\leqslant \int_{-\infty}^{\infty} \mathrm{d}s\, |s| \|(\tilde{S}(\omega; t, s) - I)\chi\|_2 \|V'\mathrm{e}^{-\mathrm{i}Hs}\psi\|_2 \\ &\leqslant \sup_{t,s} \|(\tilde{S}(\omega; t, s) - I)\chi\|_2 \int_{-\infty}^{\infty} \mathrm{d}s\, |s| \|V'\mathrm{e}^{-\mathrm{i}Hs}\psi\|_2. \end{aligned} \tag{3.21}$$

We conclude from (3.16) that $\lim_{\omega\to 0}(\chi, g_2(\omega; t, t_0)) = 0$, uniformly with respect to t_0 and t. Finally, using once more the decay property (3.2), it is easy to see that, as $t_0 \to -\infty$, $t \to \infty$, the limits of all terms in (3.17) exist so that, for $\psi = \Omega_-\varphi$, $\varphi \in \mathcal{D}$,

$$\tilde{S}(\omega)\psi = \psi - \mathrm{i}\omega \int_{-\infty}^{\infty} \mathrm{d}s\, s\mathrm{e}^{\mathrm{i}Hs}V'\mathrm{e}^{-\mathrm{i}Hs}\psi + \omega(g_1(\omega) + g_2(\omega)) \tag{3.22}$$

where $g_1(\omega)$ and $g_2(\omega)$ obviously still converge weakly to zero, as $\omega \to 0$, in $\mathrm{R}(\Omega_-)$. When (3.22) is inserted into (3.12), we obtain the result (3.4), (3.5), by setting $g(\omega) = \Omega_+^\dagger(g_1(\omega) + g_2(\omega))$ and using the intertwining relations $H(\alpha)\Omega_\pm^\alpha = \Omega_\pm^\alpha H_0$. □

3.2. Discussion of the adiabatic limit

For a time-periodic potential, assuming that we are in situation (i) described in the introduction, we conclude from (2.19) and (3.3) that

$$\lim_{\omega\to 0} \overline{\mathcal{P}}_F(\omega, \varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, \mathcal{P}_F^\alpha(\varphi) \tag{3.23}$$

where

$$\mathcal{P}_F^\alpha(\varphi) = \|FS^\alpha\varphi\|_2^2 \tag{3.24}$$

is the transition probability for the static potential $V(\alpha)$. Thus, in the adiabatic limit, the average scattering probability (2.19), for the time-dependent problem, converges to the average of the static quantities associated with the family of static potentials $V(\alpha)$, $0 \leqslant \alpha \leqslant 2\pi$. According to (2.14), we also conclude from proposition 1 that

$$\operatorname*{s-lim}_{\omega\to 0} S^n(\omega) = S^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha\, S^\alpha e^{-in\alpha} \tag{3.25}$$

so that all the sidebands contribute to the low-frequency limit of (2.19). In particular, the statistical distribution $\mathcal{P}^n(\omega, \varphi)$ of emitted and absorbed quanta (2.17) has a non-trivial limit as $\omega \to 0$, and an explicit example is given in section 5.

Let us discuss now the general structure of the first-order correction. According to (3.4), the linear correction to an asymptotic observable $F^\alpha(\omega) = S^{\alpha\dagger}(\omega) F S^\alpha(\omega)$ is, for $\varphi \in \mathcal{D}$,

$$(\varphi, F^\alpha(\omega)\varphi) = (S^\alpha\varphi, FS^\alpha\varphi) + \omega\, \mathrm{Re}(S^\alpha\varphi, FS^\alpha_1\varphi) + o(\omega). \tag{3.26}$$

From now on, we analyse the structure of this linear correction in formal terms, writing simply

$$F^\alpha(\omega) = F^\alpha + \omega F^\alpha_1 + o(\omega) \tag{3.27}$$

where

$$F^\alpha = S^{\alpha\dagger} F S^\alpha \qquad F^\alpha_1 = S^{\alpha\dagger}_1 F S^\alpha + S^{\alpha\dagger} F S^\alpha_1. \tag{3.28}$$

Equations (3.27) and (3.28) must be understood in the sense of the quadratic form (3.26), with $\varphi \in \mathcal{D}$. Introducing the expression (3.5) into (3.27), (3.28), leads to

$$F^\alpha_1 = i\int_{-\infty}^{\infty} dt\, t e^{iH_0 t}[\Xi^\alpha, F^\alpha] e^{-iH_0 t} \qquad \Xi^\alpha = \Omega^{\alpha\dagger}_- V'(\alpha)\Omega^\alpha_-. \tag{3.29}$$

At this point, it is useful to introduce the symmetric operator (the formal 'time operator')

$$T_0 = \frac{1}{4}\left(\frac{1}{H_0} p\cdot q + q\cdot p \frac{1}{H_0}\right) \tag{3.30}$$

which represents the derivation with respect to energy in the spectral representation of H_0 [24], i.e.

$$[T_0, f(H_0)] = i\frac{\partial f(H_0)}{\partial H_0} \tag{3.31}$$

on $\mathcal{D}$, for differentiable functions of H_0. From the relation

$$\partial_\alpha S^\alpha = -i\int_{-\infty}^{\infty} dt\, e^{iH_0 t} S^\alpha \Xi^\alpha e^{-iH_0 t} \tag{3.32}$$

we deduce

$$\partial_\alpha F^\alpha = i\int_{-\infty}^{\infty} dt\, e^{iH_0 t}[\Xi^\alpha, F^\alpha] e^{-iH_0 t} \tag{3.33}$$

and then, using $[T_0, e^{\pm iH_0 t}] = \mp t e^{\pm iH_0 t}$, we find that F^α_1 can be transformed into the sum of the two contributions

$$F^\alpha_1 = K^\alpha_1 + D^\alpha_1 \tag{3.34}$$

where

$$K_1^\alpha = -\tfrac{1}{2}\{T_0, \partial_\alpha F^\alpha\} \qquad D_1^\alpha = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}H_0 t}\{T_0, [\Xi^\alpha, F^\alpha]\}\mathrm{e}^{-\mathrm{i}H_0 t} \tag{3.35}$$

and $\{A, B\} = AB + BA$ denotes the anticommutator. Notice that $[H_0, D_1^\alpha] = 0$, whereas $[H_0, K_1^\alpha] = \mathrm{i}\partial_\alpha F^\alpha$, implying that F_1^α does not commute in general with H_0. However, since K_1^α is a derivative of a periodic function of α, it has zero average so that

$$\overline{F}_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, F_1^\alpha = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, D_1^\alpha = \overline{D}_1 \tag{3.36}$$

is indeed an on-shell quantity, in accordance with (2.21).

The term K_1^α may be called 'kinematical'. It involves only static quantities and it is a manifestation, at first order in ω, of the fact (already discussed in section 2) that the incoming state will find the potential in a configuration which depends on its time of arrival in the interaction region (we recall that, classically, the observable $-T_0$ has the meaning of an arrival time at the origin). This is illustrated in the one-dimensional example in section 5.

On the other hand, the contribution D_1^α may be called 'dynamical' since it incorporates the genuine new dynamical effects of the potential on the scattering. The structure of D_1^α is also briefly analysed in section 5. Here we conclude from the following corollary that potentials of the form $v(x, \mu(t))$, depending on t only through a single real scalar periodic function $\mu(t)$, have no first-order dynamical corrections to the averaged probabilities (2.19). This includes, for instance, potentials of the form (1.1), setting $\lambda(t) = \mu(t)$, and potentials of the form (1.2) with $a(t) = a(\mu(t))$.

Corollary. Let $\mu(t)$ be a 2π-periodic real function of t, and $v(x, \mu(t))$ a potential satisfying the assumptions of proposition 1, part (ii). Then $\overline{F}_1 = \overline{D}_1 = 0$.

Proof. One has $v'(x, \mu(t)) = \mu'(t)w(x, \mu(t))$ with $w(x, \mu(t)) = \partial_\mu v(x, \mu)|_{\mu=\mu(t)}$. According to (3.29), F_1^α is of the form $F_1^\alpha = \mu'(\alpha)G_1^\alpha$ where G_1^α is defined as (3.29) with $v'(x, \mu(\alpha))$ replaced by $w(x, \mu(\alpha))$. Clearly, for $\varphi \in \mathcal{D}$, the function $(\varphi, G_1^\alpha\varphi) = g(\mu(\alpha))$ depends on α only through $\mu(\alpha)$, and $g(\mu)$ is finite for each μ (μ in compact sets) by the estimate of proposition 1, part (ii). Thus, the first-order correction can be written as $(\varphi, F_1^\alpha\varphi) = \mu'(\alpha)g(\mu(\alpha))$, and its average

$$(\varphi, \overline{F}_1\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, \mu'(\alpha)g(\mu(\alpha)) = \frac{1}{2\pi}\int^{\mu(\alpha)} \mathrm{d}\mu\, g(\mu)\bigg|_{-\pi}^{\pi} = 0 \tag{3.37}$$

vanishes as a consequence of the periodicity of $\mu(\alpha)$. □

The linear order correction to the adiabatic limit of sidebands is

$$S^n(\omega) = S^n + \omega S_1^n + \mathrm{o}(\omega) \tag{3.38}$$

with

$$S_1^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, S_1^\alpha \mathrm{e}^{\mathrm{i}n\alpha}. \tag{3.39}$$

For the asymptotic observables $F^n(\omega) = S^{n\dagger}(\omega)FS^n(\omega)$, we obtain

$$F^n(\omega) = F^n + \omega F_1^n + \mathrm{o}(\omega) \tag{3.40}$$

where

$$F^n = S^{n\dagger}FS^n \qquad F_1^n = S_1^{n\dagger}FS^n + S^{n\dagger}FS_1^n. \tag{3.41}$$

Introducing $[T_0, \mathrm{e}^{-\mathrm{i}H_0t}] = t\mathrm{e}^{-\mathrm{i}H_0t}$ into (3.5), we can write S_1^α as the sum

$$S_1^\alpha = S_{1,A}^\alpha + S_{1,B}^\alpha \tag{3.42}$$

where $S_{1,A}^\alpha = -\partial_\alpha S^\alpha T_0$ (use (3.32)) and $S_{1,B}^\alpha$, which commutes with H_0, is given by

$$S_{1,B}^\alpha = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}H_0t} S^\alpha \Xi^\alpha T_0 \mathrm{e}^{-\mathrm{i}H_0t}. \tag{3.43}$$

According to this decomposition, we find that

$$F_1^n = F_{1,A}^n + F_{1,B}^n \tag{3.44}$$

where

$$F_{1,A}^n = S^{n\dagger}FS_{1,A}^n + S_{1,A}^{n\dagger}FS^n = -\mathrm{i}n[T_0, F^n] = n\frac{\partial F^n}{\partial H_0} \tag{3.45}$$

and

$$F_{1,B}^n = S^{n\dagger}FS_{1,B}^n + S_{1,B}^{n\dagger}FS^n. \tag{3.46}$$

For the last equality of (3.45) we have used (3.31), and $S_{1,A}^n$, $S_{1,B}^n$ are the Fourier components of $S_{1,A}^\alpha$ and $S_{1,B}^\alpha$ respectively. The first-order contributions to sidebands sum up to the total first-order correction to scattering (3.36), i.e.

$$\sum_{n=-\infty}^{\infty} F_1^n = \overline{F}_1 = \overline{D}_1. \tag{3.47}$$

In general, individual sidebands have non-vanishing first-order corrections. However, for the class of potentials considered in the corollary, these corrections have to sum to zero. Examples are given in section 5.

3.3. Energy transfer in the adiabatic limit

Since the energy of the scattered particle is not conserved, it is of interest to investigate in more detail the energy transfer (2.23) between the particle and the external field. The expression of $\Delta^\alpha(\omega)$ up to second order in ω is readily obtained by inserting the expansion (3.4) into (2.23). One finds

$$\Delta^\alpha(\omega) = \omega\Delta_1^\alpha + \omega^2\Delta_2^\alpha + \mathrm{o}(\omega^2) \tag{3.48}$$

with

$$\Delta_1^\alpha = \mathrm{i}S^{\alpha\dagger}\partial_\alpha S^\alpha \qquad \Delta_2^\alpha = \mathrm{i}(S_1^{\alpha\dagger}\partial_\alpha S^\alpha + S^{\alpha\dagger}\partial_\alpha S_1^\alpha). \tag{3.49}$$

We first comment on the first-order term, which is given by (see (3.32))

$$\Delta_1^\alpha = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}H_0t}\Xi^\alpha \mathrm{e}^{-\mathrm{i}H_0t}. \tag{3.50}$$

For a time-dependent perturbation of the type (1.1) with $w_2(x) \geqslant 0$ (resp. $w_2(x) \leqslant 0$) we clearly have $\Xi^\alpha = \lambda'(\alpha)\Omega_-^{\alpha\dagger}W_2\Omega_-^\alpha$, where $\Omega_-^{\alpha\dagger}W_2\Omega_-^\alpha$ is a positive (resp. negative) operator. Hence, the sign of the energy transfer is determined by that of $\lambda'(\alpha)$.

However, for the class of potentials considered in the corollary, the averaged first-order energy transfer $\overline{\Delta}_1$ vanishes by the same arguments and one has to examine the averaged second-order term $\overline{\Delta}_2$. After an integration by parts, using (3.5) and $[T_0, \mathrm{e}^{\pm\mathrm{i}H_0t}] = \mp t\mathrm{e}^{\pm\mathrm{i}H_0t}$, one can write $\overline{\Delta}_2$ in the form

$$\begin{aligned}\overline{\Delta}_2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, (S_1^{\alpha\dagger}S^\alpha\Delta_1^\alpha + \Delta_1^\alpha S^{\alpha\dagger}S_1^\alpha) \\ &= \frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha \int_{-\infty}^{\infty}\mathrm{d}t\, \mathrm{e}^{\mathrm{i}H_0t}(\Xi^\alpha T_0\Delta_1^\alpha - \Delta_1^\alpha T_0\Xi^\alpha)\mathrm{e}^{-\mathrm{i}H_0t}.\end{aligned} \tag{3.51}$$

This formula can be further reduced if the spectrum of H_0 is simple (see also section 5). For instance, for potentials of the form (1.1) invariant under rotations, we can restrict the formula to a subspace with fixed angular momentum. In such subspaces, Δ_1^α reduces to a function of energy only, given by

$$\Delta_1^\alpha(E) = 2\pi\langle E|\Xi^\alpha|E\rangle \tag{3.52}$$

where, in the spherically symmetric case, $|E\rangle$ stands for the improper eigenvector of H_0 with fixed angular momentum. Then,

$$\begin{aligned}\overline{\Delta}_2(E) &= \mathrm{i}\int_{-\pi}^{\pi}\mathrm{d}\alpha\, \Delta_1^\alpha(E)(\langle E|\Xi^\alpha T_0|E\rangle - \langle E|T_0\Xi^\alpha|E\rangle) \\ &= \int_{-\pi}^{\pi}\mathrm{d}\alpha\, \Delta_1^\alpha(E)\partial_E\langle E|\Xi^\alpha|E\rangle = \frac{1}{4\pi}\partial_E\int_{-\pi}^{\pi}\mathrm{d}\alpha\, (\Delta_1^\alpha(E))^2\end{aligned} \tag{3.53}$$

where for the second equality we have used $\langle E|T_0 = \mathrm{i}\partial_E\langle E|$. Thus, at given energy E, there is a positive or negative energy transfer according to the sign of the quantity (3.53). One can check on examples that both signs can occur, according to the value of the incoming energy E.

4. The high-frequency limit

4.1. The high-frequency theorem

As mentioned in the introduction, the high-frequency limit is also of interest in various physical situations, in particular in the theory of ionization of atoms in strong laser fields. The point here is that the fast oscillating Fourier components of the potential do not contribute to the scattering as $\omega \to \infty$. We introduce explicitly the Fourier series of the periodic interaction $V(t) = V(t + 2\pi)$,

$$V(t) = \sum_{n=-\infty}^{\infty} V_n \mathrm{e}^{\mathrm{i}nt} \tag{4.1}$$

where V_n is the multiplication operator by

$$v_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t\, v(x,t) \mathrm{e}^{-\mathrm{i}nt} = v_{-n}^*(x). \tag{4.2}$$

In this section we make the hypothesis that the potential is sufficiently differentiable with respect to its time and space variables. More precisely, we shall assume that $v(x, t)$ satisfy the following conditions:

(1) $v(x, t)$ is twice continuously differentiable with respect to t, and both $v(x, t)$ and its time derivatives satisfy (3.1);

(2) the average potential $v_0(x)$ defines a static scattering system (H, H_0), $H = H_0 + V_0$, with complete wave operators $\Omega_\pm$ and scattering operator S, for which the propagation property (3.2) is true;

(3) $v(x, t)$ is n-times differentiable with respect to x and the partial derivatives $\partial_x^m v(x,t)$ and $\partial_t \partial_x^m v(x,t)$ satisfy (3.1), $m = 1, \ldots, n$, $n \geqslant 1$, where $\partial_x^m = \partial_{x_1}^{m_1} \cdots \partial_{x_d}^{m_d}$, $m = m_1 + \cdots + m_d$.

According to (1), by partial integration, one has

$$|v_n(x)| \leqslant \frac{c}{n^2} \langle x \rangle^{-\eta} \qquad \eta > 1 \tag{4.3}$$

so that the series (4.1) are norm convergent. We decompose the time-dependent Hamiltonian (2.7) as (setting here $\alpha = 0$)

$$H(\omega t) = H + \tilde{V}(\omega t) \qquad H = H_0 + V_0 \tag{4.4}$$

where

$$\tilde{V}(t) = V(t) - V_0 = \sum_{n \neq 0}^{\infty} V_n \mathrm{e}^{\mathrm{i}nt} \tag{4.5}$$

has the same properties as $V(t)$. By condition (2), we can apply the chain rule to the pair of scattering systems $(H(\omega t), H)$ and (H, H_0). The arguments are identical to those leading to (3.12) in the previous section. This enables us to represent the scattering operator $S(\omega)$ of the full time-dependent problem as

$$S(\omega) = \Omega_+^\dagger \tilde{S}(\omega) \Omega_- \qquad \tilde{S}(\omega) = \tilde{\Omega}_+^\dagger(\omega) \tilde{\Omega}_-(\omega). \tag{4.6}$$

Proposition 2. (i) Assume that conditions (1)–(3) hold for $n \geqslant 2$. Then,

$$\operatorname*{s-lim}_{\omega \to \infty} S^\alpha(\omega) = S \tag{4.7}$$

where $S = \Omega_+^\dagger \Omega_-$ is the scattering operator for the static average potential $v_0(x)$.

(ii) If in addition (3) *holds for* $n \geqslant 6$, *then for* $\varphi \in \mathcal{D}$

$$S^\alpha(\omega)\varphi = S\varphi + \omega^{-2} S_2 \varphi + \omega^{-2} f^\alpha(\omega) \tag{4.8}$$

where

$$S_2 \varphi = \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}H_0 t} \Omega_+^\dagger M \Omega_- \mathrm{e}^{-\mathrm{i}H_0 t} \varphi \tag{4.9}$$

$$M = \sum_{n \neq 0} \frac{1}{n^2} \left(\tfrac{1}{2}(V_n V_n^\dagger H + H V_n V_n^\dagger) - V_n H V_n^\dagger \right) \tag{4.10}$$

and the vector $f^\alpha(\omega)$ converges strongly to zero as $\omega \to \infty$.

Proof. (i) We consider the case $\alpha = 0$, dropping the index α. In view of (4.6), it suffices to show that $\text{s-lim}_{\omega\to\infty} \tilde{S}(\omega)\psi = \psi$, for $\psi = \Omega_-\varphi$, $\varphi \in \mathcal{D}$. We can represent $\tilde{S}(\omega)\psi$ by the formula

$$\tilde{S}(\omega)\psi = \psi - \mathrm{i} \operatorname*{s-lim}_{\substack{t\to\infty \\ t_0\to-\infty}} \int_{t_0}^{t} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Ht} U_\omega(t,s)\tilde{V}(\omega s)\mathrm{e}^{-\mathrm{i}Hs}\psi. \tag{4.11}$$

Introducing

$$\tilde{V}(t) = \frac{\mathrm{d}\tilde{V}_1(t)}{\mathrm{d}t} \qquad \tilde{V}_1(t) = -\sum_{n\neq 0} \frac{\mathrm{i}}{n} V_n \mathrm{e}^{\mathrm{i}nt} \tag{4.12}$$

and performing an integration by parts in the last term of (4.11), one obtains

$$\begin{aligned} -\mathrm{i}\int_{t_0}^{t} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Ht} U_\omega(t,s)\tilde{V}(\omega s)\mathrm{e}^{-\mathrm{i}Hs}\psi &= -\frac{\mathrm{i}}{\omega}\mathrm{e}^{\mathrm{i}Ht}U_\omega(t,s)\tilde{V}_1(\omega s)\mathrm{e}^{-\mathrm{i}Hs}\psi\Big|_{t_0}^{t} \\ &\quad - \frac{1}{\omega}\int_{t_0}^{t} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Ht}U_\omega(t,s)(\tilde{V}(\omega s)\tilde{V}_1(\omega s) + [H, \tilde{V}_1(\omega s)])\mathrm{e}^{-\mathrm{i}Hs}\psi. \end{aligned} \tag{4.13}$$

Using that $\tilde{V}_1(\omega s)$ is bounded uniformly with respect to ω, the limit $t_0 \to -\infty$, $t \to \infty$, of the integrated term vanishes by the estimate (3.2) ($\psi = \Omega_-\varphi$). On the other hand, the last term in (4.13) can be estimated by

$$\frac{1}{\omega}\int_{-\infty}^{\infty} \mathrm{d}s\, \|\tilde{V}(\omega s)\tilde{V}_1(\omega s)\mathrm{e}^{-\mathrm{i}Hs}\psi\|_2 + \frac{1}{\omega}\int_{-\infty}^{\infty} \mathrm{d}s\, \|[H, \tilde{V}_1(\omega s)]\mathrm{e}^{-\mathrm{i}Hs}\psi\|_2. \tag{4.14}$$

By (3.2), the first integral in (4.14) is convergent. To control the second one, we observe that

$$[H, \tilde{V}_1(\omega s)] = [H_0, \tilde{V}_1(\omega s)] = \frac{1}{2m}\bigg(-2\mathrm{i}\sum_{j=1}^{d} \partial_j \tilde{V}_1(\omega s)p_j - \sum_{j=1}^{d} \partial_j^2 \tilde{V}_1(\omega s)\bigg) \tag{4.15}$$

where ∂_j are the partial space derivatives, and p_j are the components of the momentum operator. Thus, the second term in (4.14) is majorized by a sum of contributions of the form

$$\frac{1}{\omega}\int_{-\infty}^{\infty} \mathrm{d}s\, \|\partial_j \tilde{V}_1(\omega s)p_j \mathrm{e}^{-\mathrm{i}Hs}\psi\|_2 \tag{4.16}$$

and

$$\frac{1}{\omega}\int_{-\infty}^{\infty} \mathrm{d}s\, \|\partial_j^2 \tilde{V}_1(\omega s)\mathrm{e}^{-\mathrm{i}Hs}\psi\|_2. \tag{4.17}$$

Our hypothesis on the potential implies that $\partial_j \tilde{V}_1(\omega s)$ as well as $\partial_j^2 \tilde{V}_1(\omega s)$ verify (3.1) so that the s-integral in (4.17) is finite and independent of ω, by (3.2). For (4.16), using the intertwining relation, we write

$$\begin{aligned} \|\partial_j \tilde{V}_1(\omega s)p_j \mathrm{e}^{-\mathrm{i}Hs}\psi\|_2 &= \|\partial_j \tilde{V}_1(\omega s)p_j(H+\mu)^{-1}\mathrm{e}^{-\mathrm{i}Hs}\chi\|_2 \\ &\leqslant \|\partial_j \tilde{V}_1(\omega s)\langle q\rangle^{\eta}\| \|\langle q\rangle^{-\eta} p_j (H+\mu)^{-1}\langle q\rangle^{\eta}\| \|\langle q\rangle^{-\eta}\mathrm{e}^{-\mathrm{i}Hs}\chi\|_2. \end{aligned} \tag{4.18}$$

In (4.18), $(H+\mu)^{-1}$ is the resolvant of H evaluated at a sufficiently large positive number μ (H is bounded below) and $\chi = \Omega_-(H_0+\mu)\varphi$, with $(H_0+\mu)\varphi$ still belonging to $\mathcal{D}$. It is known that the second factor on the right-hand side of the inequality is finite (lemma 3 of [23], see also appendix A), so the decay (3.2) also applies to (4.18). We conclude that the quantity (4.14) is $O(\omega^{-1})$. Thus, $S(\omega) \to S$ strongly on $\mathcal{D}$, as $\omega \to 0$, and since $S(\omega)$ is uniformly bounded with respect to ω, the convergence holds on the whole of $\mathcal{H}$. The proof is the same for $S^\alpha(\omega)$, $\alpha \neq 0$, since it amounts simply to replacing V_n everywhere by $V_n e^{in\alpha}$.

(ii) To find the first non-vanishing correction to the high-frequency limit (4.7), we carry on the integration by parts on the last term of (4.13). Introducing

$$\tilde{V}_1(t) = \frac{d\tilde{V}_2(t)}{dt} \qquad \tilde{V}_2(t) = -\sum_{n\neq 0}\frac{1}{n^2}V_n e^{int} \tag{4.19}$$

and observing that $\tilde{V}(t)\tilde{V}_1(t) = (1/2)\,d\tilde{V}_1^2(t)/dt$, one finds the following contributions:

$$\begin{aligned}
&-\frac{1}{\omega^2}e^{iHt}U_\omega(t,s)([H,\tilde{V}_2(\omega s)]+\tfrac{1}{2}\tilde{V}_1^2(\omega s))e^{-iHs}\psi\Big|_{t_0}^{t}\\
&+\frac{i}{\omega^2}\int_{t_0}^{t}ds\, e^{iHt}U_\omega(t,s)[H,[H,\tilde{V}_2(\omega s)]]e^{-iHs}\psi\\
&+\frac{i}{\omega^2}\int_{t_0}^{t}ds\, e^{iHt}U_\omega(t,s)\tilde{V}(\omega s)[H,\tilde{V}_2(\omega s)]e^{-iHs}\psi\\
&+\frac{i}{\omega^2}\int_{t_0}^{t}ds\, e^{iHt}U_\omega(t,s)\tfrac{1}{2}[H,\tilde{V}_1^2(\omega s)]e^{-iHs}\psi\\
&+\frac{i}{2\omega^2}\int_{t_0}^{t}ds\, e^{iHt}U_\omega(t,s)\tfrac{1}{2}\tilde{V}(\omega s)\tilde{V}_1^2(\omega s)e^{-iHs}\psi.
\end{aligned} \tag{4.20}$$

Using the same arguments as for (4.13), the integrated term goes to zero as $t_0 \to -\infty$, $t \to \infty$. Introducing

$$\tilde{V}_2(t) = \frac{d\tilde{V}_3(t)}{dt} \qquad \tilde{V}_3(t) = \sum_{n\neq 0}\frac{i}{n^3}V_n e^{int} \tag{4.21}$$

and $\tilde{V}(t)\tilde{V}_1^2(t) = (\frac{1}{3})d\tilde{V}_1^3(t)/dt$, the first and last integral of (4.20) can be integrated by parts one step further and seen to be $O(\omega^{-3})$ by an immediate generalization of the treatement of the terms (4.13). The multiple commutator of local potentials with H can be rearranged as in (4.15) as the sum of monomials of derivatives of these potentials times powers of the momentum, provided that assumption (ii) holds. Then, bounds can be obtained as in (4.18) using the result in appendix A. For the second and third integrals, we write

$$\begin{aligned}
[H,\tilde{V}_1^2(t)] &= -\sum_{n,m\neq 0}\frac{1}{nm}[H,V_nV_m]e^{i(n+m)t}\\
&= \sum_{n\neq 0}\frac{1}{n^2}[H,V_nV_{-n}] - \sum_{\substack{n,m\neq 0\\ n\neq m}}\frac{1}{nm}[H,V_nV_m]e^{i(n+m)t}
\end{aligned} \tag{4.22}$$

and

$$\tilde{V}(t)[H, \tilde{V}_2(t)] = -\sum_{n,m\neq 0} \frac{1}{m^2} V_n[H, V_m] \mathrm{e}^{\mathrm{i}(n+m)t}$$
$$= -\sum_{n\neq 0} \frac{1}{n^2} V_n[H, V_{-n}] - \sum_{\substack{n,m\neq 0 \\ n\neq m}} \frac{1}{m^2} V_n[H, V_m] \mathrm{e}^{\mathrm{i}(n+m)t}. \tag{4.23}$$

The time-dependent contributions of (4.22) and (4.23) can be integrated by parts (integrating the phase factors $\mathrm{e}^{\mathrm{i}(n+m)t}$), and controlled by the same arguments as hereabove. Finally, combining together the time-independent contributions of (4.22) and (4.23), we remain with the term

$$\frac{\mathrm{i}}{\omega^2} \int_{t_0}^{t} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Ht} U_\omega(t, s) M \mathrm{e}^{-\mathrm{i}Hs} \psi \tag{4.24}$$

up to a correction of order ω^{-3}. As a consequence of the same estimates as in part (i) of the proposition, one sees easily that $\mathrm{e}^{\mathrm{i}Ht} U_\omega(t, s)$ tends pointwise strongly to $\mathrm{e}^{-\mathrm{i}Hs}$, as $\omega \to \infty$, uniformly with respect to t (note that for any s, $\mathrm{e}^{-\mathrm{i}Hs}\psi = \Omega_- \mathrm{e}^{-\mathrm{i}H_0 s}\varphi$ with $\mathrm{e}^{-\mathrm{i}H_0 s}\varphi \in \mathcal{D}$). Thus, using dominated convergence, the strong limit $t_0 \to -\infty$, $t \to \infty$ and $\omega \to \infty$ of the integral (4.24) exists and, by intertwining and (4.6), is given by the result of part (ii) of the proposition. □

4.2. Discussion of the high-frequency limit

According to proposition 2, as $\omega \to \infty$, the scattering operator $S^\alpha(\omega)$ approaches the static S associated with the time-independent average potential $v_0(x)$. Since the latter is independent of α, we also find that, in this limit, the scattering process becomes independent of the initial phase α of the potential or, equivalently, of the time of incidence of the incoming particle in the interaction region.

According to (2.14), we also conclude from part (i) of proposition 2 that

$$\operatorname*{s-lim}_{\omega\to\infty} S^n(\omega) = 0 \tag{4.25}$$

for $n \neq 0$. In contrast to the adiabatic case, the sidebands do not contribute to the high-frequency limit of (2.19). In fact, if the potential satisfies the assumptions of part (ii) of proposition 2, we have the stronger result

$$S^n(\omega) = \mathrm{o}(\omega^{-2}) \tag{4.26}$$

for $n \neq 0$, since the correction (4.9) is independent of α.

The first non-vanishing correction to the asymptotic observable $F^\alpha(\omega) = S^{\alpha\dagger}(\omega) F S^\alpha(\omega)$ is, for $\varphi \in \mathcal{D}$,

$$(\varphi, F^\alpha(\omega)\varphi) = (S\varphi, FS\varphi) + \omega^{-2}\,\mathrm{Re}(S\varphi, FS_2\varphi) + \mathrm{o}(\omega^{-2}). \tag{4.27}$$

On a more formal level, we can also write

$$F^\alpha(\omega) = F_0 + \omega^{-2} F_2 + \mathrm{o}(\omega^{-2}) \tag{4.28}$$

where $F_0 = S^\dagger F S$ and

$$F_2 = S^\dagger F S_2 + S_2^\dagger F S = \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}H_0 t}[F_0, \Omega_-^\dagger M \Omega_-]\mathrm{e}^{-\mathrm{i}H_0 t}. \tag{4.29}$$

For a potential of the form (1.1), the operator M is given by

$$M = \sum_{n=-\infty}^{\infty} \frac{|\lambda_n|^2}{2n^2}(W_2^2 H_0 + H_0 W_2^2 - 2W_2 H_0 W_2) = \sum_{n=-\infty}^{\infty} \frac{|\lambda_n|^2}{2n^2}[[H_0, W_2], W_2]$$
$$= -\sum_{n=-\infty}^{\infty} \frac{|\lambda_n|^2}{2n^2}\frac{1}{m}\sum_{j=1}^{d}(\partial_j W_2)^2 \tag{4.30}$$

where for the last equality we have used (4.15), and W_2 denotes the multiplication operator by the function $w_2(x)$.

5. Application to one-dimensional scattering

5.1. The adiabatic limit for transmission and reflection probabilities

We introduce the improper eigenvectors $|E, \pm\rangle$ of H_0, corresponding to positive and negative momentum p, satisfying the orthogonality relation

$$\langle E, \sigma | E', \sigma'\rangle = \delta_{\sigma,\sigma'}\delta(E - E') \tag{5.1}$$

and the projection operators

$$\int_0^{\infty} \mathrm{d}E\, |E, \pm\rangle\langle E, \pm| = F_\pm \qquad F_+ + F_- = I \tag{5.2}$$

onto the set of states with positive and negative momentum. Then, if φ is an incoming state describing a particle approaching the potential from the left, i.e. $F_+\varphi = \varphi$, the corresponding average transmission and reflection probabilities are given by

$$\overline{\mathcal{P}}_\pm(\omega, \varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, \|F_\pm S^\alpha(\omega)\varphi\|_2^2 = \int_0^{\infty} \mathrm{d}E\, \overline{\mathcal{P}}_\pm(\omega, E)|\varphi(E)|^2. \tag{5.3}$$

According to proposition 1, one obtains

$$\lim_{\omega\to 0} \overline{\mathcal{P}}_\pm(\omega, \varphi) = \int_0^{\infty} \mathrm{d}E \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, |A_\pm^\alpha(E)|^2\right)|\varphi(E)|^2 \tag{5.4}$$

where $A_\pm^\alpha(E) = \langle \pm|S^\alpha(E)|+\rangle$ are, respectively, the transmission and reflection amplitudes for an incident particle with momentum $k = \sqrt{2mE}$ and static potential $v(x, \alpha)$.

One can use (5.4) to investigate the effect, on the static transmission and reflection probabilities, of a slow time-dependent perturbation of the potential. For the case of a displaced potential of the form (1.2) the answer is clear since the amplitudes $A_\pm^\alpha(E)$ for different α differ only by a phase factor and thus the average probability (5.4) is equal to that determined by $v(x)$. However, for an additive time-dependent perturbation, either increase or decrease in the transmission and reflection probabilities can occur. More specifically, consider an interaction of the form (1.1) with $\lambda(\omega t) = \lambda_0 \cos(\omega t)$, $\lambda_0 > 0$, and denote by $A_\pm(E, \lambda)$ the transmission and reflection amplitudes for the static potential $w_1(x) + \lambda w_2(x)$. If, for a given energy E, $|A_\pm(E, \lambda)|^2$ is a convex (concave) function of λ, $|\lambda| \leqslant \lambda_0$, then the perturbation of the coupling will increase (decrease) the probability in the adiabatic limit. To show this, we have to compare the average (5.4) with the static ($\lambda_0 = 0$) probability $|A_\pm(E, 0)|^2$. If $|A_\pm(E, \lambda)|^2$ is convex, we have

$$|A^\alpha_\pm(E)|^2 = |A_\pm(E, \lambda_0 \cos\alpha)|^2 \geqslant |A_\pm(E, 0)|^2 + \lambda_0 \cos\alpha\, \partial_\lambda |A_\pm(E, \lambda)|^2_{\lambda=0} \tag{5.5}$$

yielding

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\alpha\, |A^\alpha_\pm(E)|^2 \geqslant |A_\pm(E, 0)|^2 \tag{5.6}$$

and conversely for the concave case (see also the related discussion in [25]).

Let us now analyse the form taken by the first-order 'kinematical' and 'dynamical' corrections (3.35) to transmission and reflection probabilities. For K^α_1 we have

$$\begin{aligned}(\varphi, K^\alpha_1 \varphi) &= -\tfrac{1}{2} \left(\varphi, \{T_0, \partial_\alpha |A^\alpha_\pm|^2\} \varphi\right) = -\operatorname{Re}(T_0 \varphi, \partial_\alpha |A^\alpha_\pm|^2 \varphi) \\ &= \int_0^\infty \mathrm{d}E\, \beta'(E) \partial_\alpha |A^\alpha_\pm(E)|^2 |\varphi(E)|^2 \end{aligned} \tag{5.7}$$

where for the last equality we have used (3.31), and $\beta'(E) = \partial_E \arg\varphi(E)$. If $\varphi(E)$ is real, this correction vanishes. In contrast, if $\varphi(E)$ has a non-vanishing phase $\beta(E)$, we observe that the mean position of the incoming state φ is expressed by

$$(\varphi, q\varphi) = \int_0^\infty \mathrm{d}E\, x_E |\varphi(E)|^2 \tag{5.8}$$

where $x_E = -v\beta'(E)$ and $v = k/m$ is the velocity. Thus, we may write, to first order in ω,

$$\begin{aligned}(\varphi, (F^\alpha + \omega K^\alpha_1)\varphi) &= \int_0^\infty \mathrm{d}E\, \left(|A^\alpha_\pm(E)|^2 - \omega \frac{x_E}{v} \partial_\alpha |A^\alpha_\pm(E)|^2\right) |\varphi(E)|^2 \\ &= \int_0^\infty \mathrm{d}E\, |A^{\alpha - \omega(x_E/v)}_\pm(E)|^2 |\varphi(E)|^2 + \mathrm{o}(\omega). \end{aligned} \tag{5.9}$$

For $\varphi(E)$ real and well peaked at about E, $-x_E/v$ represents the time difference between the preparation of the two initial states $\mathrm{e}^{\mathrm{i}\beta(E)}\varphi(E)$ and $\varphi(E)$, so that (5.9) is a manifestation, at first order in ω, of the fact that the two states experience the action of the potential with a time lag $-x_E/v$.

To analyse the structure of the 'dynamical' contribution D^α_1, we note that, formally, $\langle E, \pm | T_0 = \mathrm{i}\partial_E \langle E, \pm|$, and find

$$(\varphi, D^\alpha_1 \varphi) = \int_0^\infty \mathrm{d}E\, D^\alpha_1(E) |\varphi(E)|^2 \tag{5.10}$$

where

$$D^\alpha_1(E) = \mathrm{i}\pi \langle E, + | \{T_0, [\Xi^\alpha, F^\alpha]\} | E, + \rangle = 2\pi \operatorname{Re} \partial_{E'} \langle E, + | [\Xi^\alpha, F^\alpha] | E', + \rangle |_{E'=E}. \tag{5.11}$$

Working out (5.11) one step further in the case of transmission, $D_{\uparrow}^{\alpha}(E)$ can be rewritten as the sum

$$D_{\uparrow}^{\alpha}(E) = 2\pi\langle E,+|\Xi^{\alpha}|E,+\rangle\partial_E|A_{+}^{\alpha}(E)|^2 + I^{\alpha}(E) \tag{5.12}$$

where the interference contribution $I^{\alpha}(E)$ is given by

$$I^{\alpha}(E) = 2\pi\,\mathrm{Re}\,\partial_{E'}[A_{+}^{\alpha*}(E)A_{-}^{\alpha}(E)\langle E',+|\Xi^{\alpha}|E,-\rangle \\ - A_{+}^{\alpha}(E')A_{-}^{\alpha*}(E')\langle E',-|\Xi^{\alpha}|E,+\rangle]_{E'=E}. \tag{5.13}$$

We shall not take the calculation of the expressions for (5.12), (5.13) any further as they are rather complicated. Notice, however, that in the special case of a barrier (or well), the matrix elements $\langle E',\sigma'|\Xi^{\alpha}|E,\sigma\rangle$ are entirely expressible in terms of the transmission and reflection coefficients, by simple integration (see [26] for the method).

5.2. *The adiabatic limit for sidebands*

The adiabatic limit of transmitted and reflected sidebands

$$\mathcal{P}_{\pm}^{n}(\omega,\varphi) = \|F_{\pm}S^{n}(\omega)\varphi\|_2^2 = \int_0^{\infty} \mathrm{d}E\,\mathcal{P}_{\pm}^{n}(\omega,E)|\varphi(E)|^2. \tag{5.14}$$

is

$$\lim_{\omega\to 0}\mathcal{P}_{\pm}^{n}(\omega,\varphi) = \int_0^{\infty} \mathrm{d}E\,\mathcal{P}_{\pm}^{n}(E)|\varphi(E)|^2 \tag{5.15}$$

where

$$\mathcal{P}_{\pm}^{n}(E) = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\,A_{\pm}^{\alpha}(E)\mathrm{e}^{in\alpha}\right|^2. \tag{5.16}$$

For a modulated potential of the form (1.1), the adiabatic sidebands probabilities (5.16) cannot, in general, be evaluated explicitly (an exception is the modulated δ-function barrier which was considered in [4]). On the other hand, for a displaced potential of the form (1.2), one has $S^{\alpha} = \mathrm{e}^{ipa(\alpha)}S\mathrm{e}^{-ipa(\alpha)}$, yielding

$$\mathcal{P}_{+}^{n}(E) = \delta_{n,0}|A_{+}(E)|^2 \qquad \mathcal{P}_{-}^{n}(E) = |c_n(E)|^2|A_{-}(E)|^2 \tag{5.17}$$

where the $A_{\pm}(E)$ are, respectively, the transmission and reflection amplitudes for the potential $v(x)$, and

$$c_n(E) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\,\mathrm{e}^{\mathrm{i}(n\alpha-2ka(\alpha))}. \tag{5.18}$$

Thus, the statistics of quanta (2.17), i.e.

$$\mathcal{P}^{n}(E) = \mathcal{P}_{+}^{n}(E) + \mathcal{P}_{-}^{n}(E) \tag{5.19}$$

is fully determined by the coefficients $c_n(E)$ in the adiabatic limit. In particular, for $a(\alpha) = a_0\cos\alpha$, we have $c_n(E) = \mathrm{i}^n J_n(-2ka_0)$, where the J_n are Bessel functions of the first kind.

For the displaced potential we can even find explicitly the n-dependence of the first-order corrections. Indeed, one has

$$S_{1,B}^{\alpha}=a'(\alpha)\mathrm{e}^{\mathrm{i}a(\alpha)p}S_{1,B}\mathrm{e}^{-\mathrm{i}a(\alpha)p} \tag{5.20}$$

with

$$S_{1,B}=-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\;\mathrm{e}^{\mathrm{i}H_0t}S\Xi T_0\mathrm{e}^{-\mathrm{i}H_0t} \qquad \Xi=\Omega_-^{\dagger}(\partial_x V)\Omega_- \tag{5.21}$$

independent of α. Thus, using

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\, a'(\alpha)\mathrm{e}^{\mathrm{i}(n\alpha-2ka(\alpha))}=\frac{n}{2k}c_n(E) \tag{5.22}$$

we immediately find for the reflected sidebands

$$\mathcal{P}_-^n(\omega,E)=\mathcal{P}_-^n(E)+\omega\mathcal{P}_{-,1}^n(E)+\mathrm{o}(\omega) \tag{5.23}$$

where $\mathcal{P}_-^n(E)$ is given by (5.17) and

$$\mathcal{P}_{-,1}^n(E)=n\partial_E\mathcal{P}_-^n(E)+\frac{n}{2k}|c_n(E)|^2B(E). \tag{5.24}$$

The first term of (5.24) arises from the contribution (3.45) and $B(E)$ is the on-shell part of the operator

$$B=F_+(S^{\dagger}F_-S_{1,B}+S_{1,B}^{\dagger}F_-S)F_+. \tag{5.25}$$

In the same way, one finds that the correction to the transmitted sidebands vanishes for all n, i.e.

$$\mathcal{P}_+^n(\omega,E)=\mathcal{P}_+^n(E)+\mathrm{o}(\omega). \tag{5.26}$$

One sees that the statistics of sidebands at first order is again determined by knowledge of the coefficients $c_n(E)$, given by (5.18). At this order, only the reflection of the particle is accompanied by a process of emission or absorption.

5.3. Energy transfer

We can also immediately deduce the form of the second-order energy transfer from (2.24), (5.23) and (5.26),

$$\overline{\Delta}_2(E)=\sum_{-\infty}^{\infty}n\left(\mathcal{P}_{-,1}^n(E)+\mathcal{P}_{+,1}^n(E)\right)=\frac{C(E)}{4\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\,(a'(\alpha))^2. \tag{5.27}$$

The last equality follows from (5.24) and the fact that

$$\sum_{-\infty}^{\infty}n^2|c_n(E)|^2=\frac{4mE}{\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\,(a'(\alpha))^2. \tag{5.28}$$

The function $C(E)$ determines the sign of the energy transfer and can be found from (5.24), (5.25). We can give a more explicit expression for $C(E)$ in the case of a parity invariant potential $v(x) = v(-x)$. For a particle coming from the left, we have to compute the restriction of (3.51) to the subspace $F_+\mathcal{H}$. According to

$$\Delta_1^\alpha = a'(\alpha)\mathrm{e}^{\mathrm{i}pa(\alpha)}\Delta_1\mathrm{e}^{-\mathrm{i}pa(\alpha)} \tag{5.29}$$

where

$$\Delta_1 = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}H_0t}\,\Xi\,\mathrm{e}^{-\mathrm{i}H_0t} \qquad \Xi = \Omega_-^\dagger(\partial_x V)\Omega_- \tag{5.30}$$

and to

$$\mathrm{e}^{\mathrm{i}pa(\alpha)}T_0\mathrm{e}^{-\mathrm{i}pa(\alpha)} = T_0 + \sqrt{\frac{m}{2H_0}}\,\hat{p}a(\alpha) \qquad \hat{p} = p/|p| \tag{5.31}$$

one obtains

$$\begin{aligned} F_+\overline{\Delta}_2F_+ = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\,(a'(\alpha))^2\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}H_0t}\mathrm{i}F_+(\Xi T_0\Delta_1 - \Delta_1 T_0\Xi)F_+\mathrm{e}^{-\mathrm{i}H_0t} \\ + (a'(\alpha))^2a(\alpha)\sqrt{\frac{m}{2H_0}}\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}H_0t}\mathrm{i}F_+(\Xi\hat{p}\Delta_1 - \Delta_1\hat{p}\Xi)F_+\mathrm{e}^{-\mathrm{i}H_0t}. \end{aligned} \tag{5.32}$$

Using $\hat{p} = F_+ - F_-$, the second contribution in (5.32) vanishes. For the first one we insert the identity $I = F_+ + F_-$ between Ξ and Δ_1, and use the fact that $F_+\Xi F_- = -F_-\Xi F_+$ (since $\partial_x v(x) = -\partial_x v(-x)$). This yields

$$F_+\overline{\Delta}_2F_+ = \tfrac{1}{2}\overline{(a'(\alpha))^2}F_+\frac{\partial\Delta_1^2}{\partial H_0}F_+ \tag{5.33}$$

and the same result remains clearly true for a particle coming from the right (i.e. replacing F_+ by F_-). Finally, introducing the improper eigenvectors (5.1), we immediately find that the on-shell part $C(E)$ of (5.33) takes the form of an energy derivative, given by (Ξ defined in (5.21))

$$C(E) = 4\pi^2\partial_E(\langle E,+|\Xi|E,+\rangle^2 + |\langle E,-|\Xi|E,+\rangle|^2). \tag{5.34}$$

This quantity can be positive or negative, depending on the value of the incoming energy E.

5.4. The high-frequency limit

We conclude our study of one-dimensional examples by asking what is the effect, on the transmission probability, of introducing a high-frequency time-dependent perturbation. For modulated barriers of the form (1.1), with $\lambda(\omega t) = \lambda_0\cos(\omega t)$, the average potential is equal to the static one ($\lambda_0 = 0$). Thus, the addition of a time-periodic perturbation of zero average has no effect on the scattering process in the limit $\omega \to \infty$. The situation is different for an oscillating position potential of the form (1.2), with say $a(\omega t) = a_0\cos(\omega t)$, since the effective static potential

$$v_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\, v(x - a_0\cos\alpha) \tag{5.35}$$

may, in general, differ appreciably from $v(x)$. For instance, if $v(x)$ is a barrier and a_0 is sufficiently large, $v_0(x)$ is considerably broader and reduced in height with respect to $v(x)$. It can also develop a bimodal structure which may give rise to interesting resonance effects at some specified energies. The effective potential (5.35) is known as the 'dressed' potential associated with $v(x)$, in the theory of atoms in intense, high-frequency laser fields [7–10].

Finally, we specialize the correction (4.29) to the case of an oscillating potential of the form (1.1) with $\lambda(t) = \lambda_0 \cos t$. According to (4.30), we find for the transmission probability the formula ($A_+(E)$ here denotes the static amplitude for $\lambda_0 = 0$)

$$\mathcal{P}_+(\omega, E) = |A_+(E)|^2 + \left(\frac{\lambda_0}{\omega}\right)^2 \operatorname{Im}\left(A_+^*(E)A_-(E)\int_{-\infty}^{\infty} \mathrm{d}x\, (\partial_x w_2(x))^2 \psi_-^*(E,x)\psi_+(E,x)\right) + \mathrm{o}(\omega^{-2}) \tag{5.36}$$

where the kernels $\psi_\pm(E,x) = \sqrt{2\pi k/m}\,\langle x|\Omega_-|E,\pm\rangle$ are the usual solutions of the stationary Schrödinger equation for the averaged potential $v_0(x) = w_1(x)$. When $w_1(x) = 0$, the ω^{-2} term vanishes in (5.36) (since $A_-(E) = 0$). But, by unitarity, $\mathcal{P}_+(\omega, E) = 1 - \mathcal{P}_-(\omega, E)$, and after a straightfoward calculation of the reflection coefficient $\mathcal{P}_-(\omega, E)$, one finds

$$\mathcal{P}_+(\omega, E) = 1 - \left(\frac{\lambda_0}{\omega}\right)^4 \left|\int_{-\infty}^{\infty} \mathrm{d}x\, (\partial_x w_2(x))^2 \mathrm{e}^{2ikx}\right|^2 + \mathrm{o}(\omega^{-4}). \tag{5.37}$$

6. Concluding remarks

We have proven that the scattering by a local time-periodic potential has well defined low- and high-frequency limits, and we have studied the first terms of the corresponding asymptotic expansions of scattering probabilities for suitably smooth potentials. All our considerations are non-perturbative with respect to the potential strength. Inspection of the proofs of both limits shows that they will also apply to potentials with other types of time variations, not necessarily periodic, for instance potentials switched on and off in time.

The status of both expansions is not the same. The validity of the low-frequency expansion requires differentiability of the potential in its time variable and a fast spatial decay (to have the nth-order term defined one needs to apply the lemma in section 3 with η sufficiently large, see [22]). On the other hand, as seen from proposition 2, the high-frequency limit will necessitate strong local differentiability properties for the potential with respect to its spatial variables.

Already at the lowest order, the corrections to the adiabatic limit have a complicated structure. Their explicit calculation requires, in principle, the full solutions of the static scattering problems for the potentials $v(x, \alpha)$, and this complexity increases considerably for higher-order terms. One can, however, emphasize an interesting point: for smooth periodic potentials depending on time only via a single scalar function (see the corollary in section 3), the first non-vanishing correction to scattering probabilities at low and high frequency are of order ω^2 and ω^{-2} respectively. Thus, scattering with these types of interaction has a certain stability with respect to dynamical perturbations. In particular, a potential with oscillating coupling and zero average is essentially transparent in the high-frequency regime (see (5.37)).

A number of questions deserve further investigations. We mention some of them. The first correction terms can be worked out in more detail in specific cases: for instance if the static potential has a resonance, how stable is the resonance under slow oscillations of the potential and what is the nature of the energy transfer with the external field? We have not established here the high-frequency expansion beyond the ω^{-2} term of proposition 2(ii). In particular, one could expect the sideband probabilities to be very small for sufficiently smooth potentials. They have been shown to be $o(\omega^{-2})$, but what is their asymptotic behaviour? Interesting comparisons with the corresponding classical scattering problems could also be made in a semi-classical regime. In the case when the potential has local singularities (square potential barrier, Coulomb potential), what is the behaviour of the correction to the high-frequency limit? For instance, we have checked that for a perturbation by a modulated δ-function, the corrections are $O(1/\sqrt{\omega})$ only.

Finally, in this paper, we have only used the time-dependent formulation of scattering theory. How to recover the results in the quasistationary formalism [14] is an open problem. We plan to come back to these questions in future work.

Appendix

According to lemma 3 of [23],

$$\|\langle q\rangle^{-\eta}(H+\mu)^{-1}\langle q\rangle^{\eta}\| < \infty \qquad \|\langle q\rangle^{-\eta}p_j(H+\mu)^{-1}\langle q\rangle^{\eta}\| < \infty. \tag{A.1}$$

To prove part (ii) of proposition 2, we also need the result

$$\|\langle q\rangle^{-\eta}p^n(H+\mu)^{-n}\langle q\rangle^{\eta}\| < \infty \tag{A.2}$$

where $p^n = p_{j_1}\cdots p_{j_n}$ and n is a given integer. We shall prove (A.2) by a recursion procedure. We assume it is true up to n and have to show it for $n+1$. To simplify the notation we write $p \equiv p_j$ for all $j = 1, \ldots, d$ without distinction between the different components of p. We have

$$p^{n+1}(H+\mu)^{-(n+1)} = p(H+\mu)^{-1}p^n(H+\mu)^{-n} + p[p^n,(H+\mu)^{-1}](H+\mu)^{-n}. \tag{A.3}$$

The first term of (A.3) is bounded by assumption. For the second term we observe that

$$[p^n,(H+\mu)^{-1}] = (H+\mu)^{-1}[H,p^n](H+\mu)^{-1} = (H+\mu)^{-1}[V,p^n](H+\mu)^{-1}. \tag{A.4}$$

Inserting

$$[p^n, V] = \sum_{j=1}^{n}\binom{n}{j}(-1)^j\partial_x^j V p^{n-j} \tag{A.5}$$

in (A.4), then using (A.1), one finds that the second term of (A.3) is also bounded, provided that all the partial derivatives $\partial_x^j V$, $j = 1, \ldots, n$, satisfy (3.1). Thus, (A.2) is true for $n+1$.

References

[1] Kazanskii A K, Ostrovskii V N and Solov'ev E A 1976 *Sov. Phys.–JETP* **43** 254
[2] Büttiker M and Landauer R 1982 *Phys. Rev. Lett.* **49** 1739
[3] Douglas Stone A, Azbel M Ya and Lee P A 1985 *Phys. Rev.* B **31** 1707
[4] Hauge E H and Støvneng J A 1989 *J. Stat. Phys.* **57** 841
[5] Pimpale A, Holloway S and Smith R J 1991 *J. Phys. A: Math. Gen.* **24** 3533
[6] Pimpale A and Razavy M 1991 *Fortschr. Phys.* **39** 885
[7] Henneberger W C 1968 *Phys. Rev. Lett.* **21** 838
[8] Gavrila M (ed) 1992 *Atoms in Intense Laser Fields* (New York: Academic)
[9] Gavrila M and Kaminski J Z 1984 *Phys. Rev. Lett.* **52** 6139
[10] Holloway S 1990 *Interactions of Atoms and Molecules with Solid Surfaces* ed V Bortolani, N H March and M P Tosi (New York: Plenum) p 567
[11] Dollard J D 1966 *J. Math. Phys.* **7** 802
[12] Yajima B K 1977 *J. Math. Soc. Japan* **29** 729
[13] Howland J S 1979 *Indiana Univ. Math. J.* **28** 471
[14] Yafaev D R 1991 *Recent Developments in Quantum Mechanics* ed A Boutet de Monvel *et al* (Dordrecht: Kluwer) p 367
[15] Yafaev D R 1984 *Proc. Steklov Inst. Math.* **2 (V.159)** 219
[16] Neidhart H 1992 *Integr. Eq. Oper. Th.* **15** 100
[17] Gesztesy F, Mitter H and Perush M 1985 *J. Math. Anal. Appl.* **110** 265
[18] Reed M and Simon B 1979 *Scattering Theory* (New York: Academic)
[19] Rayski J and Rayski J M Jr 1977 *The Uncertainty Principle and Foundations of Quantum Mechanics* ed W C Price and S S Chissick (Chichester: Wiley)
[20] Narnhofer H and Thirring W 1982 *Phys. Rev.* A **26** 3646
[21] Sokolovski D and Hänggi P 1988 *Europhys. Lett.* **1** 7
[22] Martinez A and Nakamura S 1994 *C. R. Acad. Sci., Paris* **318** 1153
[23] Amrein W, Cibils M B and Sinha K B 1987 *Ann. Inst. H. Poincaré* **47** 367
[24] Amreim W 1981 *Non-Relativistic Quantum Dynamics (Math. Phys. Studies 2)* (Dordrecht: Reidel)
[25] Levitan J and Suchanecki Z 1994 *Phys. Rev.* A **50** 2761
[26] van Tiggelen B A, Tip A and Lagendijk A 1993 *J. Phys. A: Math. Gen.* **26** 1731