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# Spin Precession Revisited

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*The passage of a spin-1/2 neutral particle through a region of uniform magnetic field and the corresponding precession mechanism is analyzed from the viewpoint of scattering theory, with particular consideration of the role of the field boundaries.*

For pedagogical purposes, in several good books on basic quantum mechanics, the Larmor precession of a spin-1/2 particle in a uniform magnetic field is presented as the simplest example of a quantum dynamical system.[1] For a magnetic field in the $z$-direction, the time-evolution operator is $U(t)=\exp(i\mu B\sigma_z t/h)$, with $B>0$ the field strength and $\mu$ the magnetic moment of the particle ($\mu$ is negative for a neutron). Using the commutation relations for the spin components $S_i=(h/2)\,\sigma_i$, $i=x, y, z$ ($\sigma_i$ are the Pauli matrices), one finds that the spin operators $S_\pm=S_x\pm iS_y$ evolve, in the Heisenberg picture, according to

$$S_\pm(t)=S_\pm e^{\pm i\omega t} \tag{1}$$

In other terms, the spin vector $\vec{S}(t)=(S_x(t), S_y(t))$ undergoes a rotation with constant angular velocity $\omega=-2\mu B/h$, the Larmor frequency. The motion (1) is formally the same as that of a classical magnetic moment in a uniform field.

The price to be paid for this beautiful simplicity is to ignore the translational degrees of freedom of the particle. We have of course to treat a more general dynamical problem (considering here a neutral particle with position $\vec{x}$ and momentum $\vec{p}$) governed by the Hamiltonian

$$H=\frac{|\vec{p}|^2}{2m}+V(\vec{x})-\mu B(\vec{x})\,\sigma_z \tag{2}$$

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where $B(\vec{x})$ is constant in a region $D$ of finite extension and $V(\vec{x})$ represents some additional spin-independent potential. If the particle remains localized in $D$, the picture (1) is correct. For instance, let the state of the particle be of the form $\varphi_E \otimes \chi$, where $\chi$ is any spin state and $\varphi_E$ is a bound state of $V(\vec{x})$ with energy $E$. If the spatial support of $\varphi_E(\vec{x})$ is well confined inside $D$, we may neglect the effects of the field boundaries, and treat $B(\vec{x})$ as constant in (2) when acting on $\varphi_E(\vec{x})$.

However, the situation is very different if the particle belongs to a beam traveling through the region where the magnetic field is acting: one has to take into account that the particle has to enter and leave the field region.

The problem of a traveling beam has been investigated in the context of neutron interferometry experiments, in relation with the question of the observability of the sign reversal of a spinor wave function subjected to a $2\pi$ rotation; see Refs. 2–5 and the literature quoted in these references. It is also discussed within the theory of the Larmor clock, a device proposed to measure tunneling times through potential barriers by means of the spin precession.[6–9] It has been recognized in many works that the spin motion of a neutral particle crossing a region of constant magnetic field shows a Larmor precession as in (1) under the following conditions:

(i) reflections of the wave packet on the field boundaries are disregarded;

(ii) the field strength is weak.

We do not claim originality in our forthcoming discussion of the spin precession mechanism, which is lucidly analyzed in Ref. 5, for instance. But in view of various controversies that have appeared in this large literature, we feel it is of interest to make the above-mentioned points (i) and (ii) mathematically precise in a simple situation, especially concerning the role played by the field boundaries. In this respect, we hope that this note will also serve a useful didactical purpose.

A neutral particle with spin 1/2 moves in one dimension along the $x$-direction through a magnetic field of strength $B > 0$ pointing in the $z$-direction. We characterize the spatial action of the field by $B(x) = Bw(x)$, where $w(x)$ is a dimensionless cut-off function given by

$$w(x) = \begin{cases} 1 & \text{if } |x| \leqslant L \\ g\left(\dfrac{|x| - L}{l}\right) & \text{otherwise} \end{cases} \qquad l, L > 0 \tag{3}$$

We assume that $g(x)$, $x \geqslant 0$, is a twice continuously differentiable function with compact support, $0 \leqslant g(x) \leqslant 1$ and $g(0) = 1$. The function $g(x)$ describes the switching on and off of the field in space. Since $dw(x)/dx = O(l^{-1})$ for

$|x| > L$, $l^{-1}$ is a measure of the size of the field gradient in the transition region.

Let $\sigma_z \chi^\sigma = \sigma \chi^\sigma$, $\sigma = \pm 1$. Then, for this model, the stationary solution $\psi_E^\sigma(x)$ of the Schrödinger equation associated with the Hamiltonian (2) (setting now $V(x) = 0$) that describes a particle coming from the left with energy $E$ and spin state $\chi^\sigma$ obeys

$$\left( -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \sigma\mu B w(x) - E \right) \psi_E^\sigma(x) = 0 \tag{4}$$

and behaves asymptotically as

$$\begin{aligned} \psi_E^\sigma(x) &= e^{ikx} + R_E^\sigma e^{-ikx}, \qquad & x \to -\infty \\ \psi_E^\sigma(x) &= T_E^\sigma e^{ikx}, & x \to \infty \end{aligned} \tag{5}$$

where $\hbar k = \sqrt{2mE}$, and $R_E^\sigma$, $T_E^\sigma$ are the corresponding reflection and transmission coefficients. These coefficients are given by the following proposition:

**Proposition.** Assume that $w(x)$ is defined as in (3) and that $E - |\mu| B \geqslant E_0 > 0$. Then

$$\begin{aligned} R_E^\sigma &= O\left(\frac{|\mu| B}{E}\frac{1}{kl}\right) \\ T_E^\sigma &= \exp\left( i \int_{-\infty}^{\infty} dx\, (k^\sigma(x) - k) \right) + O\left(\frac{|\mu| B}{E}\frac{1}{kl}\right) \end{aligned} \tag{6}$$

where we have set $\hbar k^\sigma(x) = \sqrt{2m(E + \sigma\mu B w(x))}$.

*Proof.* Define $f_{E,\pm}^\sigma(x) = k^\sigma(x)^{-1/2} \exp(\pm i \int_0^x dy\, k^\sigma(y))$ and write $\psi_E^\sigma(x)$ in the form (dropping now the indices $\sigma$, $E$)

$$\psi(x) = \alpha_+(x) f_+(x) + \alpha_-(x) f_-(x) \tag{7}$$

Since $w(x)$ has compact support, we have $k(x) = k$ for $x$ large enough, and the comparison of (5) and (7) as $x \to \pm\infty$ leads to

$$\begin{aligned} \alpha_+(-\infty) &= \sqrt{k} \exp\left( i \int_{-\infty}^{0} dy\, (k(y) - k) \right) \\ \alpha_+(\infty) &= \sqrt{k}\, T \exp\left( -i \int_{0}^{\infty} dy\, (k(y) - k) \right) \\ \alpha_-(-\infty) &= \sqrt{k}\, R \exp\left( -i \int_{-\infty}^{0} dy\, (k(y) - k) \right) \\ \alpha_-(\infty) &= 0 \end{aligned} \tag{8}$$

To determine the functions $\alpha_{\pm}(x)$, we impose the additional relation

$$\frac{d\psi}{dx}(x) \equiv \psi'(x) = ik(x)(\alpha_{+}(x)\, f_{+}(x) - \alpha_{-}(x)\, f_{-}(x)) \tag{9}$$

One checks that $\psi(x)$ verifies (4) if and only if

$$\alpha'(x) = A(x)\, \alpha(x) \tag{10}$$

with

$$A(x) = \begin{pmatrix} 0 & A_{+}(x) \\ A_{-}(x) & 0 \end{pmatrix}, \qquad \alpha(x) = \begin{pmatrix} \alpha_{+}(x) \\ \alpha_{-}(x) \end{pmatrix} \tag{11}$$

and

$$A_{\pm}(x) = \frac{1}{2}\frac{k'(x)}{k(x)} \exp\left( \mp 2i \int_0^x dy\, k(y) \right) \tag{12}$$

The integral equation corresponding to (10)

$$\alpha(x) = \alpha(0) + \int_0^x dy\, A(y)\, \alpha(y) \tag{13}$$

can be solved by iteration, leading to the estimate

$$\|\alpha(\pm\infty) - \alpha(0)\| \leqslant \|\alpha(0)\| \sup_x \left\| \int_0^x dy\, A(y) \right\| \exp\left( \int_0^{\pm\infty} dy\, \|A(y)\| \right) \tag{14}$$

From (12) and the form (3) of the cut-off $w(x)$ we find

$$\int_0^x dy\, A_{\pm}(y) = \begin{cases} 0 & \text{if} \quad |x| \leqslant L \\ \dfrac{\mu B}{4E} \displaystyle\int_0^{(|x|-L)/l} dy\, \frac{g'(y)}{h(y)^2} \exp\left( \mp 2ik \left( Lh(0) + l \int_0^y dz\, h(z) \right) \right) & \text{otherwise} \end{cases} \tag{15}$$

where we have introduced the dimensionless quantity

$$h(x) = \frac{\sqrt{2m(E + \sigma\mu Bg(x))}}{\hbar k} = \sqrt{1 + \sigma \frac{\mu B}{E} g(x)} \tag{16}$$

Noticing that $h(x) \geqslant \sqrt{E_0/E} > 0$, we clearly have the bound

$$\int_0^{\pm\infty} dy\, |A_{\pm}(y)| = \frac{|\mu|\, B}{4E} \int_0^{\infty} dy\, \frac{|g'(y)|}{h(y)^2} < \infty \tag{17}$$

which is independent of $l$. Moreover, an integration by parts on (15), together with the properties of $g(x)$, shows that

$$\sup_x \left| \int_0^x dy\, A_{\pm}(y) \right| = O\left(\frac{|\mu| B}{E} \frac{1}{kl}\right) \tag{18}$$

We conclude from (14), (17), and (18) that

$$\alpha(\pm\infty) = \alpha(0) + O\left(\frac{|\mu| B}{E} \frac{1}{kl}\right) \tag{19}$$

Combining (8) with (19) leads to the result of the proposition.

The methods used in the proposition are familiar in the semiclassical treatment of reflection and transmission coefficients[10,11] (see also Proposition 2 in Ref. 9). If $w(x)$ is $n+1$ times differentiable, the estimate (6) for the reflection coefficient could be improved to $O((kl)^{-n})$. However, the estimate for the transmission coefficient appears to be optimal.

Let $\chi^{in} = c^+\chi^+ + c^-\chi^-$, $|c^+|^2 + |c^-|^2 = 1$, be the incoming spin state and $\langle S_\pm \rangle^{in} = (\chi^{in}, S_\pm \chi^{in}) = c^{\pm *} c^{\mp}$. Then, when the incoming energy is $E$, the transmitted part of this spin state through the magnetic field region is

$$\chi_E^{tr} = T_E^+ c^+ \chi^+ + T_E^- c^- \chi^- \tag{20}$$

Hence, according to the proposition, $\langle S_\pm \rangle_E^{tr} = (\chi_E^{tr}, S_\pm \chi_E^{tr})$ is equal to

$$\langle S_\pm \rangle_E^{tr} = \langle S_\pm \rangle^{in}\, T_E^{\pm *} T_E^{\mp} = \langle S_\pm \rangle^{in} \left( e^{\pm i\Theta_E(B)} + O\left(\frac{|\mu| B}{E} \frac{1}{kl}\right)\right) \tag{21}$$

with

$$\begin{aligned} \Theta_E(B) &= \int_{-\infty}^{\infty} dx\, (k^-(x) - k^+(x)) \\ &= \omega \frac{2L}{v} \left( \frac{\sqrt{1 + \mu B/E} - \sqrt{1 - \mu B/E}}{\mu B/E} \right. \\ &\quad \left. + \frac{l}{L} \int_0^{\infty} dx \frac{\sqrt{1 + (\mu B/E)\, g(x)} - \sqrt{1 - (\mu B/E)\, g(x)}}{\mu B/E} \right) \end{aligned} \tag{22}$$

where $v = \hbar k/m$ is the velocity and $\omega = -2\mu B/\hbar$ is the Larmor frequency. Except for the condition $E > |\mu|\, B$ (the energy is above the potential barriers), the expression (21) is general: the field is not necessarily weak nor did we specify its form in the transition region.

From now on, we assume that the switching on and off of the field occurs on a much larger scale than the wavelength of the particle i.e.,

$l \gg k^{-1}$. This assumption is natural if the field is produced by an apparatus of macroscopic size. Then, according to (6), reflections at the boundaries can be neglected and the spin operator, after transmission, has undergone a pure rotation.[2] Can we interpret the spin rotation angle $\Theta_E(B)$ as a Larmor precession? Performing a limited Taylor expansion of $\Theta_E(B)$ with respect to the field strength $B$ gives

$$\Theta_E(B) = \omega \frac{2L}{v}\left(1 + O\left(\frac{l}{L}\right)\right)\left(1 + O\left(\frac{|\mu| B}{E}\right)^2\right) \tag{23}$$

Since $2L/v$ is the time of flight of a free classical particle of velocity $v$ in a field region of extension $2L$, we can interpret $\Theta_E(B)$ as a classical Larmor precession if the conditions

$$k^{-1} \ll l \ll L \tag{24}$$

$$|\mu|\, B \ll E \tag{25}$$

are fulfilled. The two inequalities (24) ensure that the effects of the field boundaries can be neglected and (25) is the weak field condition, making thus assumptions (i) and (ii) precise.

Still assuming (24), is it possible to interpret the spin rotation (21) as a Larmor precession beyond the weak field assumption (25)? For this, it would be natural to ask, according to (1), if $\Theta_E(B)$ can be written in the form $\omega t_E(B)$, where $t_E(B)$ is the quantum mechanical mean sojourn time spent by the incoming particle with energy $E$ and spin state $\chi^{\text{in}}$ in the field region $[-L, L]$. According to the theory of sojourn times[12] (and using the results of the proposition), one finds that $t_E(B)$ is given by

$$\begin{aligned} t_E(B) &= \frac{2L}{v^+}\,|c^+|^2 + \frac{2L}{v^-}\,|c^-|^2 + \frac{2L}{v}\, O\left(\frac{|\mu| B}{E}\frac{1}{kl}\right) \\ &= \frac{2L}{v}\left(\frac{|c^+|^2}{\sqrt{1+\mu B/E}} + \frac{|c^-|^2}{\sqrt{1-\mu B/E}} + O\left(\frac{|\mu| B}{E}\frac{1}{kl}\right)\right) \end{aligned} \tag{26}$$

where $v^\pm = \hbar k^\pm/m$ and $\hbar k^\pm = \sqrt{2m(E \pm \mu B)}$. Clearly, since $|c^+|^2 + |c^-|^2 = 1$, $\omega t_E(B)$ agrees with $\Theta_E(B)$ to first order in $B$, but not in general [except to first order in $B$, $\omega t_E(B)$ depends on the spin state $\chi^{\text{in}}$, which is not the case for the rotation angle $\Theta_E(B)$]. Thus, the concept of sojourn time of the particle in the field region does not enable one to give a consis-

[2] To our knowledge the existing literature has only discussed the abrupt switching on of the field in space, i.e., a step function. In this case, the above conclusion is no longer valid.

tent interpretation of the spin rotation angle (22) as a Larmor precession beyond the weak field regime (25).

Consider, however, the motion of the transmitted waves for the two spin components $\sigma = \pm 1$ corresponding to an initial packet having some smooth energy dispersion $\varphi(E)$ with support $E - |\mu| B > E_0$

$$\varphi^{\sigma}_{\mathrm{tr}}(x, t) = \int_{E_0 + |\mu| B}^{\infty} dE\, T^{\sigma}_{E} \varphi(E) \exp\left(\frac{i}{\hbar}(\hbar kx - Et)\right) \tag{27}$$

The stationary phase argument together with the result (6) of the proposition (assuming $l \gg k^{-1}$ and the initial packet sharply peaked at about $E$) show that these wave packets have experienced the time delays[12]

$$\tau^{\sigma}_{E} = \frac{d}{dE} \arg T^{\sigma}_{E} = \int_{-\infty}^{\infty} dx \left(\frac{m}{\hbar k^{\sigma}(x)} - \frac{m}{\hbar k}\right) \tag{28}$$

The expression (28) is nothing but the time delay corresponding to the scattering of a classical particle of energy $E$ by the potential $-\sigma\mu B w(x)$.

According to the present analysis, when we let a spin-1/2 particle go through a homogeneous magnetic field (with weak gradients as its boundaries) classical features do not show up primarly in the spin evolution, but rather in the particle translational motion. Indeed, the scattering effect due to the field on both spin component waves $\varphi^{\sigma}_{\mathrm{tr}}(x, t)$ is just the classical time delay $\tau^{\sigma}_{E}$. For $\mu < 0$, we have obviously $\tau^{+}_{E} > \tau^{-}_{E}$, as a consequence of the fact that the "up" and "down" components of the incoming packet move with different velocities $v^{+} > v^{-}$ in the field region. On the other hand, according to formula (22), the spin rotation angle $\Theta_E(B)$ results in a genuine quantum mechanical interference phenomenon between the two wave components. Then it turns out that in sufficiently weak fields $\Theta_E(B)$ reduces to the usual classical Larmor rotation angle.

This interpretation is in complete accordance with the view expressed in Ref. 5 (see also Ref. 9) that the basic effect is the Stern–Gerlach longitudinal splitting of the wave function (see Ref. 13 and references therein) and that Larmor precession results in an interference between these two Stern–Gerlach states. The consideration of the effects of the field boundaries is important: the presence there of the field gradient is necessary to generate the spatial splitting, but gradients must be weak enough to ensure full transmission of the particle through the field region.

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