Log-Odds

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1 Why we need log-odds

Our conventional way of expressing probabilities has always frustrated me. For example, it is very easy to say nonsensical statements like, "110% chance of working". Or, it is not obvious that the difference between 50% and 50.01% is trivial compared to the difference between 99.98% and 99.99%. It also fails to accommodate the math correctly when we want to say things like, "five times more likely", because 50% * 5 overflows 100%.

Jacob and I have (re)discovered a mapping from probabilities to log-odds which addresses all of these issues. To boot, it accommodates Bayes' theorem beautifully. For something so simple and fundamental, it certainly took a great deal of google searching/wikipedia surfing to discover that they are actually called "log-odds", and that they were "discovered" in 1944, instead of the 1600s. Also, nobody seems to use log-odds, even though they are conceptually powerful. Thus, this primer serves to explain why we need log-odds, what they are, how to use them, and when to use them.

Log-odds are not a panacea. There are some things for which regular probabilities are the only way to go. For example, you cannot do simple additions with log-odds. On the other hand, log-odds work beautifully with Bayesian statistical updates.

While e would be the natural choice of base, I've written this primer in base 10 logarithms. To quote E.T. Jaynes's Probability Theory, "Our minds are thoroughly conditioned to the base 10 number system, and base 10 logarithms have an immediate, clear intuitive meaning to all of us. However, we just don't know what to make of a conclusion stated in terms of natural logarithms, until it is translated back into base 10 terms." Therefore, we may as well skip this intermediate step by writing everything as a base 10 logarithm. All logarithms in this primer are base 10 logarithms.

2 The Formula

Let's say we have some probability $0\% < x < 100\%$. Then, our fix is to say that our log-odds(shorthand: dB) is equal to

$$10 * \log_{10}\left(\frac{x}{100\% - x}\right)$$

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This scale works like the decibel scale for sound intensity - each increase of 10 dB corresponds to a tenfold increase in likelihood.

We can reverse this formula - if our log-probability is L, then our normal probability is equal to

$$\left(1 - \frac{1}{1 + 10^{\frac{L}{10}}}\right) * 100\%$$

3 Why this formula is ideal

Log-odds have several properties which make them nice.

- Zero: a probability of 50% has a log-odd of 0 dB.
- Range: 0% and 100% probabilities are mapped to $-\infty$ dB and ∞ dB (thus clarifying that no probability is ever really 0% or 100%)
- Symmetry: 99% is roughly 20 dB, and 1% is roughly -20 dB. In other words, two probabilities that used to be complements will become negatives of each other.
- Easy interpretation of 0s and 9s: A 99.999999% probability becomes 80 dB. For extreme probabilities, the log-odds is simply the number of nines, times ten. (For probabilities close to 0%, it's simplest to invert the probability and count the number of nines, as before.)

Additionally, using log-odds fixes several problems.

- It is now harder to say nonsensical things like "110% chance", since it would be analogous to saying "$\infty + 500$ dB".
- Differences are amplified where they matter. 50.00% and 50.01% become 0 dB and .0017 dB, whereas 99.98% and 99.99% become 37 dB and 40 dB.

- We can express concepts like “five times more likely” as additions. Simply add $\log 5$ to your log-odd, and your probability will converge neatly to 100%, no mess, guaranteed.

4 How to use log-odds

Here is a chart of probabilities and log-odds. It’s useful to train yourself to think in terms of log-odds. The human mind doesn’t understand the difference between 99.9% and 99.99%, but the difference between 3 dB and 4 dB is obvious. Log-odds make it clear that increasing from 99.9% to 99.99% is just as hard as increasing from 50% to 91%, or that the difference between 60% and 70% is roughly the same as the difference between 3% and 2%.

probability	log-odd
50%	0 dB
60%	1.8 dB
70%	3.7 dB
80%	6.0 dB
91%	10 dB
95%	13 dB
98%	17 db
99%	20 dB
99.5%	23 dB
99.7%	25 dB
99.9%	30 dB
99.99%	40 dB
99.999%	50 dB

probability	log-odd
50%	0 dB
40%	-1.8 dB
30%	-3.7 dB
20%	-6.0 dB
9%	-10 dB
5%	-13 dB
2%	-17 dB
1%	-20 dB
.5%	-23 dB
.3%	-25 dB
.1%	-30 dB
.01%	-40 dB
.001%	-50 dB

Poker hand (5-card)	Odds	log-odds
Pair	1.36:1	-1.3 dB
Two Pair	20:1	-13 dB
Three of a Kind	46:1	-17 dB
Straight	250:1	- 24 dB
Flush	500:1	-27 dB
Full House	700:1	-28 dB
Four of a Kind	4000:1	-36 dB
Straight Flush	70,000:1	-48 dB
Royal Flush	650,000:1	-58 dB

Most calculations involving probabilities do not translate well to log-odds, which is probably why they haven't taken hold in the math world. There is, however, one application where log-odds shine. Bayesian updates are tedious to do manually, but they become trivial when using log-odds.

What is a Bayesian update? As humans, we have a very poor grasp of what "confidence" means. In general, we are either confident in something, or we are not. If we want to be more precise about things, we should assign probabilities to our confidence numbers. Additionally, we would like to know how our confidence in something should increase or decrease in response to new pieces of evidence. Bayes' Theorem tells you precisely how to update your confidence in light of new evidence. The mathematical form of Bayes' Theorem is somewhat daunting, and the notation is confusing for beginners. Therefore I will refrain from ever explicitly stating Bayes' theorem, and intuitively demonstrate how to perform a Bayesian update.

4.1 Bayes' Theorem

Here is an example of a Bayesian update.

Let's take a disease, say phenylketonuria (PKU). PKU has an incidence rate of about 1 in 10,000. If we were to pick some random person off the street, our confidence that he has PKU is very slim: .01%, or -40 dB. Additionally, there is a simple blood test that will diagnose PKU. Let's say this test will correctly diagnose PKU in a sick person 99% of the time, and it will falsely diagnose PKU in a healthy person 3% of the time. (These numbers are made up, but the point is that no test is perfect - all tests occasionally have false positives and false negatives).

Let's take a test population of 10,001 people, 1 of whom has PKU, and the other 10,000 of whom are healthy. Then, give everybody the PKU blood test. The one person with PKU will have a positive test result with 99% probability, and of the 10,000 healthy people, about 300 will return a false positive. The problem is that we don't know who's who. But we have narrowed our search space significantly. If we were to pick a random person off the street, his chance of having PKU would be 1 in 10,000. But now that we've got some positive test results, we've narrowed our probability to 1 in 300. (If we were to repeat this process, the chances improve to 1 in 10, since the likelihood of getting two false positives in a row is low.) With a low-sensitivity test, you have to administer the test multiple times to be able to detect a very rare test.

This process of updating our probability of PKU from .01% to .33% to 10% after being given new information (i.e. a positive test result) is essentially Bayes' theorem.

4.2 Bayes Factors

Is there an easy way to get these numbers without going through an admittedly long calculation? There is, if you use log-odds.

Let's take our correct diagnosis rate and divide it by our false positive rate to give what we call a "Bayes Factor" (also known as a likelihood ratio). The Bayes Factor is a ratio - In this case, 99/3 = 33. It's not a coincidence that 33 is roughly the ratio of .01% to .33% to 10%. But the ratios aren't exactly 33. $\frac{10\%}{.33\%} = 30$, not 33. They differ very slightly, and it would be nice if we could "multiply" by 33 each time without having to worry about subtle errors building up. (These errors are not due to rounding)

In our previous example, a person off the street has PKU with -40 dB. Our Bayes Factor is 33, and $10\log(33) = 15.19$. We can say that our PKU test provides 15 decibels of evidence.

- -40.00 dB = .01%
- -40.00 dB + 15.19 dB = -24.81 dB. (Checkpoint: -24.8 dB = .329%)
- -24.81 dB + 15.19 dB = -9.62 dB. (Checkpoint: -9.62 dB = 9.84%)
- -9.62 dB + 15.19 dB = 5.57 dB. This is equal to 78.3%.

This procedure is simple, gives the right answers, and demonstrates what happens as your probability crosses the 50% mark.

4.3 Other uses for log-odds

Any procedure where you have two groups, each decaying at their own rate, will benefit from the usage of log-odds. For example, you can use log-odds to simply radioactive decay of a U238/U235 mixture, which has different half-lives for the two isotopes. Or you can use log-odds to calculate ethanol distillations, since water and ethanol behave differently at each liquid-vapor equilibrium.

5 Conclusion

Log-odds have been thought of independently by many people. However, most people treat them as a vague notion, and continue to use regular probabilities in their mental representations. After reading this primer, you should be equipped to think and make actual calculations using log-odds.

Appendix: Proof that Bayes factor works with log-odds

Let a be the number of sick people within the group under consideration. Let b be the number of healthy people. Then, within this group, the chance that you are a sick person is $\frac{a}{a+b}$. Equivalently, the log-odd that you are a sick person is $\ln \frac{\frac{a}{a+b}}{\frac{b}{a+b}} = \ln \frac{a}{b}$.

Let ϵ be the background rate of sick people. Let Q be the correct diagnosis rate of our test, and let F be the false diagnosis rate of our test. We arbitrarily start with a population of 1, to make the math easier.

Let's call $\ln \frac{a}{b} = L_0$.

stage	# sick	# healthy	% sick	dB sick
start	ϵ	$1-\epsilon$	ϵ	L_0
test #1	ϵQ	$(1-\epsilon)F$	$\frac{\epsilon Q}{\epsilon Q+(1-\epsilon)F}$	$L_0 + \ln(\frac{Q}{F})$
test #2	ϵQ^2	$(1-\epsilon)F^2$	$\frac{\epsilon Q^2}{\epsilon Q^2+(1-\epsilon)F^2}$	$L_0 + 2\ln(\frac{Q}{F})$
test #n	ϵQ^n	$(1-\epsilon)F^n$	$\frac{\epsilon Q^n}{\epsilon Q^n+(1-\epsilon)F^n}$	$L_0 + n\ln(\frac{Q}{F})$

After each round of testing, the number of sick people under consideration shrinks by a factor of Q, and the number of healthy people under consideration shrinks by a factor of F. Thus, repeatedly adding $\ln(\frac{Q}{F})$ is valid.