Expressing the (almost) Infinite

Don't say "infinitely" when you mean "very"; otherwise you'll have no word left when you want to talk about something really infinite. ~C. S. Lewis

Problems

1. The sum $1 + 2 + 3 + \ldots + m$ can be concisely written in sigma notation as $\sum_{n=1}^{m} n$.[1]
   a. Using python, find $\sum_{n=1}^{1000} n$ (remember that `range(n)` gives `[0,1,...,n-1]`). Verify that your method is correct by performing the calculation manually as shown below.

      $1 + 2 + \ldots + 500 + 501 + \ldots + 999 + 100$

      $= (1 + 1000) + (2 + 999) + \ldots + (500 + 501)$

      $1001 + 1001 + \ldots + 1001$

   b. Modify your program to find $\sum_{n=1}^{1000} n^2$.
2. Sometimes you can approximate a complicated answer by doing a only small part (albeit the most important part) of the work. For instance, in this problem we'll calculate an approximation for an infinite series[2] and make a (ultimately correct) guess at it's value.
   a. Modify you program above to find $\sum_{n=1}^{10} \frac{1}{n^2}$. Remember to use `1.0 / n ** 2` since `1 / n ** 2` will evaluate to `0`.
   b. Find $\sum_{n=1}^{100} \frac{1}{n^2}$, $\sum_{n=1}^{1000} \frac{1}{n^2}$, $\sum_{n=1}^{10,000} \frac{1}{n^2}$, and $\sum_{n=1}^{10^6} \frac{1}{n^2}$. Do these values seem to be moving towards a final value? If so, we say the series $\frac{1}{n^2}$ "converges" to a final value and call this

[1]For more on sums and "Sigma Notation", see wikipedia.org/wiki/Summation

[2] For more on infinite series, see wikipedia.org/wiki/Infinite_series

value “ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ”.

c. Now, let’s assume that the sums are converging to $\frac{\pi^2}{k}$ for some positive integer k. Try to find k as follows:

   i. First, add the constant π into python. Type the following command and run it (we’ll go over what this does later):

   ```
   from math import pi
   ```

   Try typing `pi` into the prompt and pressing enter. You should see `3.141592653589793`. **Call me over if something goes wrong.**

   ii. You can now use π in your expressions as if it were a number. To practice, find $\pi^2 + 3$, by entering `pi**2 + 3`.

   iii. If $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{k}$ then $k = \pi^2\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1}$. Make a guess for k by finding $\pi^2\left(\sum_{n=1}^{10^6} \frac{1}{n^2}\right)^{-1}$ (Hint: use your answer to part b above).

d. Find a good estimate for $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Interestingly, there is no known simple formula for this number[3].

3. Using python, you can find good guesses for all sorts of tricky infinite sums.

   a. Write a program to find $\sum_{a=0}^{30}\sum_{b=0}^{30}\left[\frac{1}{3^a 5^b}\right]$.

   b. Based on part a, what do you think $\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\left[\frac{1}{3^a 5^b}\right]$ is? Can you express this as a simple fraction?

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[3] If you look up different derivations of the formula for part iii (http://en.wikipedia.org/wiki/Basel_problem), you may be able to glimpse why the methods applied to squares don’t work on cubes. Without calculus, however, you’ll have to take many premises on faith.

Answers

1.
   a. **`sum([ n+1 for n in range( 1000 ) ])`** You should obtain `500500.` If you got `499500`, make sure you're adding all the numbers including 1000.
   b. **`sum([ (n+1)**2 for n in range( 1000 ) ])`** You should get `333833500`.
2.
   a. **`sum([ 1.0 / (n+1)**2 for n in range( 10 ) ])`**
   b. Replace `10` in the expression above with the upper bound of summation. For instance, $\sum_{n=1}^{10^6} \frac{1}{n^2}$ becomes **`sum([ 1.0 / (n+1)**2 for n in range( 10**6 ) ])`**
   c.
      i. If something isn't working, let me help you out. This part shouldn't give you trouble.
      ii. You should get `12.869604401089358`.
      iii. **`pi**2 / sum([ 1.0/(n+1)**2 for n in range(10**6) ])`** gives me `6.0000036475628455`, so it looks like $k = 6$. This is in fact correct. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
   d. **`sum([ 1.0 / (n+1)**3 for n in range( 10**6 ) ])`** I get
3.
   a. **`sum([ 1.0 / ( 3**a * 5**b ) for a in range( 30 ) for b in range( 30 ) ])`** gives me `1.8749999999999918`
   b. It looks like the sum might be $1.875 = \frac{15}{8}$. This is in fact correct.