EIT/FE REVIEW - MATHEMATICS

Algebra

1. Adding Fractions - find a least common denominator:

\[ \frac{1}{p^q} + \frac{1}{pq} = \frac{q + p}{p^q} \]

Never split a fraction whose denominator is not a monomial:

\[ \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c} \] but \( \frac{c}{a + b} \neq \frac{c}{a} + \frac{c}{b} \) in general.

2. Exponents - use the exponent laws \((x^{a+b} = x^a x^b, x^{-a} = 1/x^a, \text{ and } (x^a)^b = x^{ab})\) to simplify

\[ \frac{a^3 b^2 c}{c^2 (a^2 b - 1)^4} \] and \( \sqrt{xy^2} \cdot \sqrt{x^2 y} \)

3. Rationalizing (to remove the \( \sqrt{c} \) in \( \sqrt{a} - \sqrt{b} \)) from a denominator, multiply by \( \sqrt{a} + \sqrt{b} \) over \( \sqrt{a} + \sqrt{b} \): rewrite the following to eliminate the \( \sqrt{c} \) in the denominator

\[ \frac{\sqrt{a} + 2}{3 - \sqrt{a}} \]

Solve for \( x \) in \( \sqrt{5x + 1} = \sqrt{2x + 11} + 1 \).

4. Factoring: \( x - a \) is a factor of \( f(x) \) if and only if \( f(a) = 0 \).

\[ x^2 - a^2 = (x - a)(x + a) \]
\[ x^3 \pm a^3 = (x \pm a)(x^2 \mp ax + a^2) \]
\[ x^n - a^n = (x - a)[x^{n-1} + \cdots] \]

5. Roots of Polynomials: \( P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n x + a_n = 0 \). If \( P(x) \) has a rational root \( p/q \), then \( p \) is a factor of \( a_n \), and \( q \) is a factor of \( a_0 \). This suggests trial candidates which can be checked by substitution or synthetic division.

Example: \( P(x) = 2x^3 + x^2 + 5x - 3 = 0 \) has candidates roots of \( \pm 3, \pm 1, \pm 5/2, \) and \( \pm 1/2 \).

6. Roots of a quadratic: \( ax^2 + bx + c = a[(x + b)/(2a)]^2 - (b^2 - 4ac)/(4a^2) \) (completing the square), so the roots are:

\[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

7. Binomial Expansion - with

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

then

\[ (a + b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots \]

8. Simultaneous Linear Equations: \( a_1 x + b_1 y = c_1 \) and \( a_2 x + b_2 y = c_2 \) represent equations for two straight lines. Elimination (successive substitution) can be used to solve for their intersection point.

9. Determinants:

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \]
10. Eigenvalues: if \( Ax = \lambda x \) for a 2 by 2 matrix \( A \) and a 2-vector \( x \), then \( \lambda \) satisfies
\[
det(\lambda I - A) = 0.
\]
A special case is the 2 by 2
\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\]
whose eigenvalues are roots of the quadratic
\[
(\lambda - a)(\lambda - d) - bc = 0.
\]

11. Matrix operations: be able to add, multiply, and invert matrices.

12. Nonlinear Equations - example: solve simultaneously
\[
3x - y = 2, \quad \text{and} \quad x^2 + y^2 = 25.
\]
Example: two pipes can fill a reservoir in 4 hours 48 minutes. The larger pipe can do it alone in 4 hours less than the smaller one alone. Find the times it can be filled by each pipe alone.

13. Inequalities - examples: solve for \( x \) such that
\[
\frac{x + 1}{3x + 2} \leq 3 \quad \text{or} \quad x^2 - 5x + 10 > 4.
\]

14. Logarithms: \( \log_a p = q \) is equivalent to \( a^q = p \). Recall that \( \log(m/n) = \log m - \log n \) and \( \log x^r = r \log x \).

15. Complex Numbers: \( i = \sqrt{-1} \), write \( z = a + bi = r e^{i\theta} = r(\cos \theta + i \sin \theta) \) with
\[
r = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan \theta = y/x.
\]
For any integer \( n \)
\[
(a + ib)^n = r^n (\cos n\theta + i \sin n\theta).
\]
These can be "rationalized" too: to remove the \( i \) from \( a + bi \) in a denominator, multiply by \( a - bi \) over \( a - bi \).
Example: find \( r \) and \( \theta \) for \( z = 1 + i \).

Analytic Geometry and Trigonometry

Straight lines in two dimensions have the form \( Ax + By + C = 0 \), which can be written as \( y = mx + b \) when \( B \neq 0 \).

Conic sections satisfy the equation \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \). Special cases are (1) a circle \( (x - x_c)^2 + (y - y_c)^2 = r^2 \), (2) an ellipse \( (x - x_c)^2/a^2 + (y - y_c)^2/b^2 = 1 \), (3) a hyperbola \( (x - a)^2/A^2 - (y - b)^2/B^2 = 1 \), and (4) a parabola \( (x - x_c)^2 = 4p(y - y_c) \). Know their graphs.

Trig functions - know the graphs of sine, cosine, tangent. Geometric definitions from a right triangle are
\[
\sin = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \text{and} \quad \tan = \frac{\text{opposite}}{\text{adjacent}}.
\]
Recall that \( 2\pi \) radians = 360°.

Vectors - in 3 dimensions \( a = (a_1, a_2, a_3) = a_1i + a_2j + a_3k. \)
Length of \( a = |a| = \sqrt{a_1^2 + a_2^2 + a_3^2} \),
\[
a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 = |a||b| \cos \theta.
\]
\[ \mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}, \]

and

\[ |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta, \]

where \( \theta \) is the angle between the vectors.

Parametric representations - a straight line through \((x_0, y_0, z_0)\) parallel to \(\mathbf{a}\) can be described by the points \((x, y, z)\) satisfying

\[ x = x_0 + a_1 t, \quad y = y_0 + a_2 t, \quad z = z_0 + a_3 t. \]

The points \((x, y, z)\) on a plane through \((x_0, y_0, z_0)\) perpendicular to \(\mathbf{a}\) satisfy

\[ a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0) = 0. \]

Other coordinate systems -
- Polar: \(x = r \cos \theta, \quad y = r \sin \theta. \)
- Spherical: \(x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi. \)

**Calculus**

**Derivative definition:**

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \]

Know geometric interpretation (slope of a tangent line), derivative formulas for sums, products, quotients, chain rule, sine, cosine, exp, ln, power function \((x^n)\), inverse tangent. Be able to find local max and min, global max and min, concavity. Know applications to motion: velocity and acceleration.

Approximation: \(f(x + \Delta x) \approx f(x) + \Delta x f'(x)\).

Integration: know formulas for sums, geometric interpretation (area), simple substitutions, integration by parts.

**Fundamental Theorem of Calculus:** if \(F'(x) = \int_a^x f(t) \, dt\) then \(F''(x) = f(x)\). Conversely, \(\int_a^b f(t) \, dt = F(b) - F(a)\) where \(F\) is an anti-derivative of \(f\), i.e., \(F' = f\).

Applications: areas, volumes, work, centers of mass (centroids), moments of inertia.

Partial derivatives: for rates of change of functions with more than one independent variable.

Taylor series: you should know the standard series for \(\sin x, \cos x, e^x\), and the ‘geometric series’ \(1/(1 + x)\).

**Differential Equations**

Recognize linear vs. nonlinear, constant coefficient vs. variable coefficient. Be able to find solutions to simple homogeneous linear equations, particular solutions for simple inhomogeneous equations, steady-state solutions.

**Laplace transforms** - bring a table!

**Probability and Statistics**

The number of permutations of \(n\) events taken \(r\) at a time (when the order is important) is \(p(n, r) = n!/((n - r)!). \) If the starting point is unknown (as in a ring) then \(p(n, r) = (n - 1)!/(n - r)!\). The number of combinations of \(n\) things taken \(r\) at a time (order not important) is \(C(n, r) = n!/r!(n - r)!\).

If data \(x_1, x_2, \ldots, x_N\) is gathered during an experiment, then the arithmetic sample mean (or sample average) is

\[ \bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{N}. \]
The sample median is the middle value when sorted and \(N\) is odd. When \(N\) is even the sample median is the average of the two middle values after sorting. The sample mode is the value that occurs most frequently. The standard deviation (an estimate of variability) is

\[
s = \sqrt{\frac{(x_1 - \bar{x})^2 + \cdots + (x_N - \bar{x})^2}{N - 1}}
\]

\[
= \sqrt{\frac{x_1^2 + \cdots + x_N^2 - N\bar{x}^2}{N - 1}}.
\]

The sample variance is \(s^2\).

Probability functions - a random variable \(x\) has a value which is a probability. It is said to be discrete if there are only a finite number of possible values: \(X_1 < X_2 < \cdots < X_N\). The value of the probability of \(X_i\) is denoted by \(P(X_i)\).

If \(x\) is continuous then a probability density function \(f(x)\) is defined so that

\[
\int_{a}^{b} f(t) \, dt = \text{the probability that } t \text{ is in } [a, b].
\]

Probability distribution functions - the probability distribution function \(F(X_n)\) of the discrete probability function \(P(X_i)\) is defined by

\[
F(X_n) = \sum_{k=1}^{N} P(X_k) = P(X_i \leq X_n).
\]

A probability distribution function \(F(x)\) for a continuous \(x\) is given by

\[
F(x) = \int_{-\infty}^{x} f(t) \, dt.
\]

Means and variances - for a density function \(f(x)\) the expected value \(g(x)\), denoted by \(E\{g(x)\}\), is defined as

\[
E\{g(x)\} = \int_{-\infty}^{\infty} g(x) f(x) \, dx.
\]

The mean \(\mu\) of the distribution and variance \(\sigma^2\) are given by \(E\{x\}\) and \(E\{(x - \mu)^2\}\).

Common distributions -

Binomial: let \(F(x)\) be the probability that \(x\) occurs in \(n\) independent trials and \(p\) be the probability of success; then

\[
F(x) = C(n, x) p^x (1 - p)^{n-x}.
\]

Normal: with mean \(\mu\) and variance \(\sigma^2\), this has density function

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
\]

Theory and testing - The Central Limit Theorem states that when sampling from a distribution with finite variance \(\sigma^2\) and common mean \(\mu\), as the sample size \(N\) increases, \(\bar{x} = \frac{1}{N} \sum X_i\) has as a limiting distribution the normal with mean \(\mu\) and variance \(\sigma^2/N\).

\(t\)-tests are used to test means, \(\chi^2\)-tests are for testing single variances, and \(F\)-tests are for testing two variances.