Exotic Smoothness in Four Dimensions

The 6th Gulf Coast Gravity Meeting

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Outline

- Exotic Smoothness: Geometry and Physics
- Effects in Classical and Quantum Gravity
- Example: Semiclassical Calculation
- Example: One-loop effects
- Summary

Geometry and Physics

What does it mean for 2 manifolds to be "the same"?

1. Can answer this question by looking at the classes of mappings $f : M \to N$ between them:
   - A **homeomorphism** (continuous, invertible) f preserves topology ("sets")
   - A **diffeomorphism** (smooth) f preserves smooth structure.
2. We must be careful because *one does not imply the other*: Milnor[1] found homeomorphic but non-diffeomorphic sets of 7-spheres.

$\Longrightarrow$**Exotic Smooth Structures!**

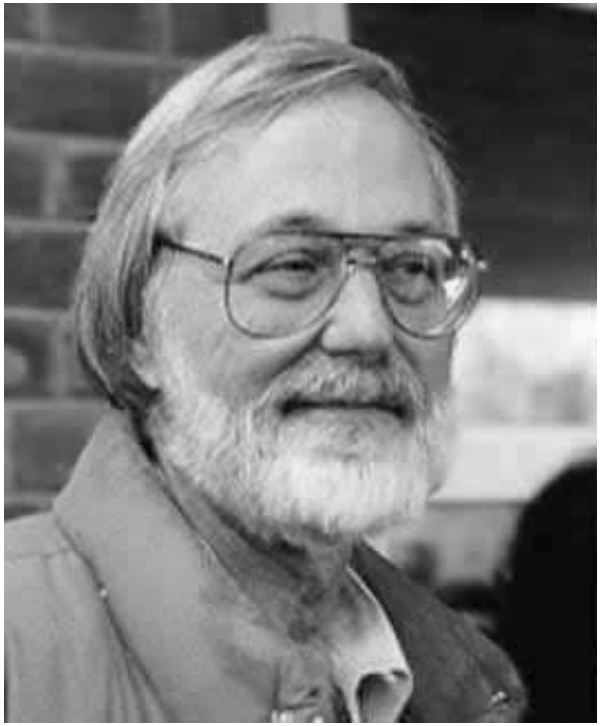

Figure: John Milnor, Fields Medal (1962), Abel Prize (2011) for Exotic 7-spheres.

[1] Milnor, Ann. of Math., Vol 64 (1956)

Geometry and Physics

Exotic smooth structures are gauge-inequivalent solutions to the Einstein field equations!

[2] Gompf, R. J. Diff. Geom. **21**, *1985*
Taubes, C., J. Diff. Geom. **25**, *1987.*

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The next natural question is "what kinds of manifolds can have exotic smooth structures?".

$\Longrightarrow$ For "Simplest Case": Euclidean space $\mathbb{R}^n$:

- For $n \neq 4$, unique smooth structure.
- For $n = 4$, there are an *uncountably infinite* number of exotic smooth structures [2]!

[2] Gompf, R. J. Diff. Geom. **21**, *1985*
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Effects in Classical and Quantum Gravity

What kinds of physical effects could there be?

1. Dark Matter: There can be regions inside $\mathbb{R}^4$ which are exotic - these regions might alter null geodesics and mimic lensing!
2. Point-like smooth failures: All matter is exotic structure[3].
3. Quantum Gravity: A "complete set of states" must count exotic structures

[3] Asselmeyer, T. Class. Quan. Grav. 14 (1997)

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The challenge:

(a) Find a way to incorporate these exotic smooth structures into a physical theory, to test if they have measurable consequences.

(b) Find a metric on these - this is the hardest part, since the proofs usually involve algebraic methods (intersection forms, Donaldson or Seiberg-Witten invariants...).

[3] Asselmeyer, T. Class. Quan. Grav. 14 (1997)

Exotic Smoothness in Semiclassical Gravity

How to measure the effect: Semiclassical Gravity [4].

- Evaluate the Euclidean path integral at minima of the action; since exotic smooth structures are *classical solutions*, they will be included in this sum:

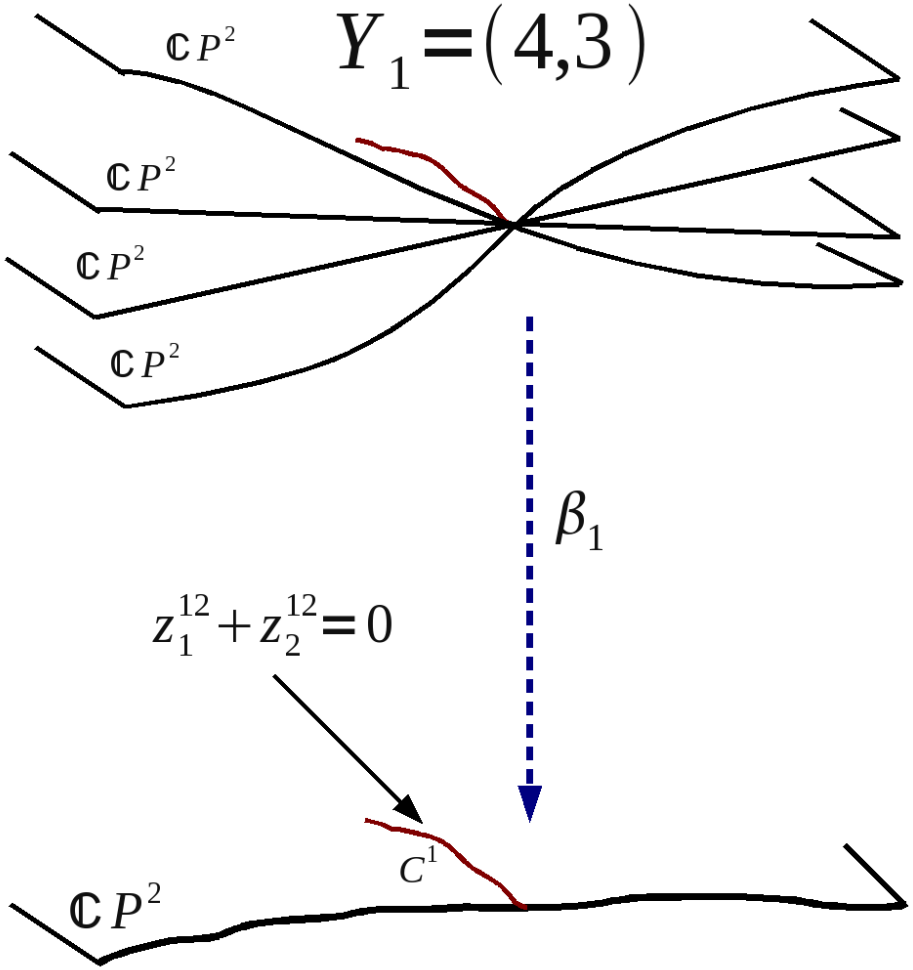
$$Z = \int_{[M,g]} exp\left(-\frac{1}{\hbar}S_E(M,g)\right) \longrightarrow \sum_i exp\left(-\frac{1}{\hbar}S_E(g^i)\right).$$

- This sum is now well-defined so one can use methods from QFT (generating functions, correlation functions) or statistical mechanics (expectation values) to look for a physical manifestation of the inclusion of exotic structures.

[4] Duston, International Journal of Geometric Methods in Modern Physics, Vol 8, No. 3 (2011)

Example: Iterated Branched Covers of $\mathbb{C}P^2$

Branched Covering Space:



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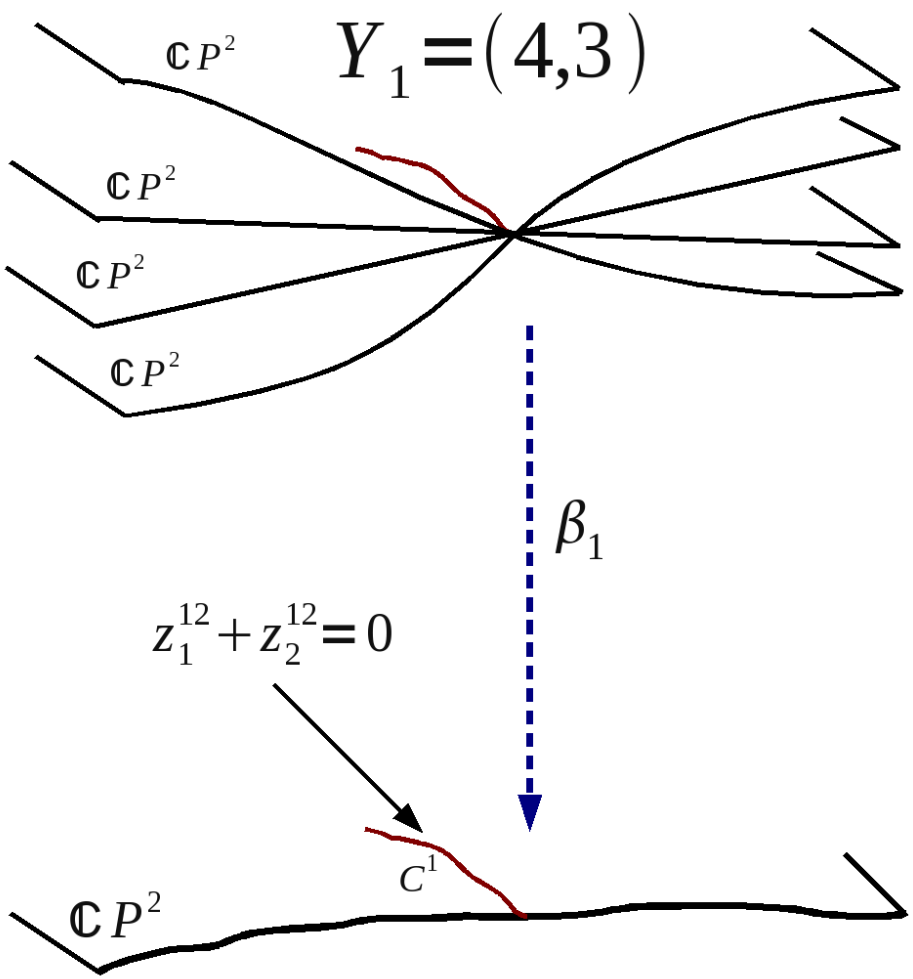

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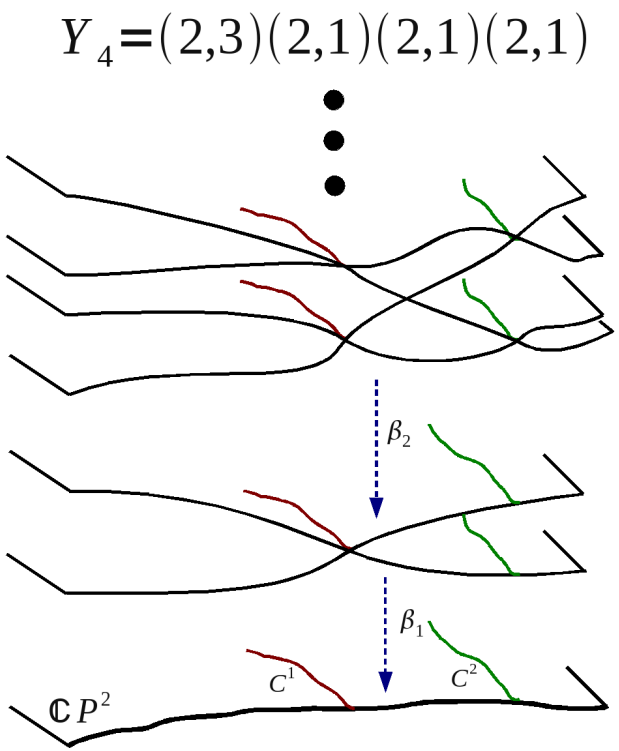


Using these integers can we calculate some **invariants** and determine when two such spaces are exotic (as these two are).

Finding the Metric on the Covers

1. As covering spaces there is map $\beta_r : Y_r \to \mathbb{C}P^2$; this map allows us to <u>determine the metric</u> on each Y_r by pulling back the metric from $\mathbb{C}P^2$ (the Fubini-Study metric h_{ab}).
2. It turns out the geometry on the covers is unchanged under this pullback:
   - They remain Einstein with constant scalar curvature ($R = 12 = 4\Lambda$)
   - The volume of a cover Y_r is simply $V = \mathfrak{D} V_0$ where $V_0 = 2\pi^2$ is the volume of $\mathbb{C}P^2$ and $\mathfrak{D}$ is the total degree.
   - Volume of $Y_1 = (4,3)$ is $4 \cdot 2\pi^2$
   - Volume of $Y_4 = (2,3)(2,1)(2,1)(2,1)$ is $2^4 \cdot 2\pi^2$.
3. Thus the (Euclidean, Einstein-Hilbert) action is

$$S[h_{ab}] = -\frac{c^3}{16\pi G}\int (R - 2\Lambda) d\mu[h_{ab}] = -\frac{c^3\Lambda}{8\pi G}\mathfrak{D} V_0, \quad \mathfrak{D} := d_1 \ldots d_r$$

Expectation Value of Volume

The expectation value of volume in the semiclassical regime would be given by

$$\langle V \rangle = \frac{\sum_i V_i e^{-S[h_i]/\hbar}}{\sum_i e^{-S[h_i]/\hbar}} = \langle \mathfrak{D} \rangle V_0$$

For the example Y_1 and Y_4, using the above analysis we find $\langle \mathfrak{D} \rangle = 16$, the total covering degree of the largest space.

Thus, we find that in the first approximation the expectation value of volume is dominated by the largest space. This was true for all of the 1000+ exotic families we found.

One-loop Approximation

- It is easy to extend this result to include one-loop contributions: expand the metric about each minima and use Zeta-function renormalization to deal with divergences [5].
- This results in a partition function which is now a sum over semiclassical states, weighed by a factor which depends on the conformal scale Ω:

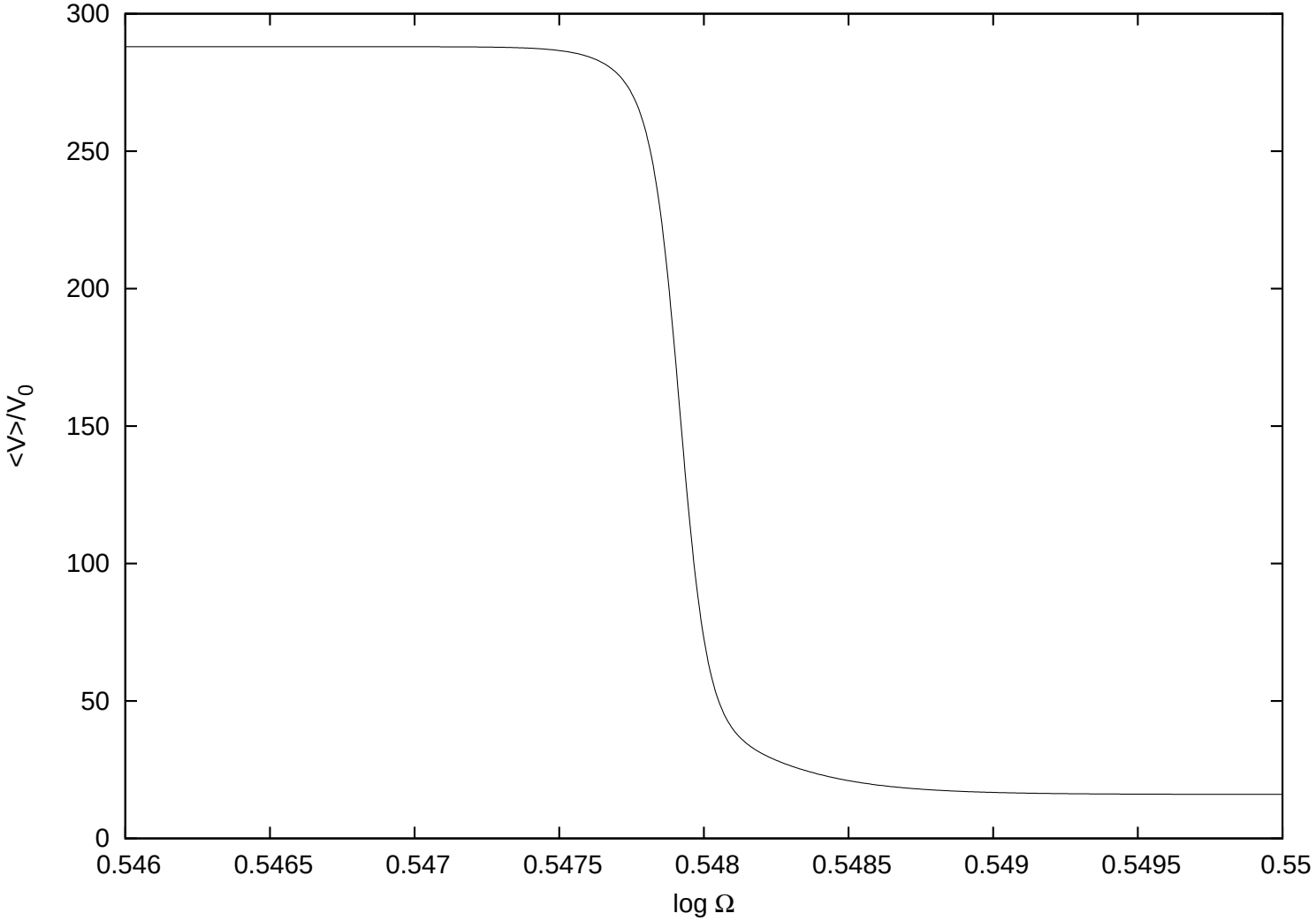
$$Z = \sum_i W_1[h^i, \Omega] \exp\left(-\frac{1}{\hbar} S[h^i]\right), W_1 = \exp\left(-\frac{1}{2}\frac{73}{60}\log \Omega\Lambda^2\right).$$

- Repeating the analysis of the expectation value of volume, we find a strong dependence on the conformal scale factor...

[5] Gibbons, Hawking, Perry, Nuc. Phys. B138 (1978)

One-loop Approximation

Now using a set of exotic spaces with total degrees $\mathfrak{D}_i \in \{16, 50, 72, 200, 288\}$:



Conclusion and Speculation

- This study represents the first time where a physical observable (volume) is shown to be dependent on the smoothness structure of the manifold.
- The effect is seen at the one-loop level (phase transition?)
- Are there more physically interesting spacetimes where a similar semiclassical analysis can be performed?
- Using techniques such as this can one find a metric on exotic $\mathbb{R}^4$?

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