Topspin Networks and Topology in Loop Quantum Gravity

Thirteenth Marcel Grossman Meeting, Stockholm

Christopher Duston (Florida State University)

July 2012

Outline of Talk

- Why do we care about topology in quantum gravity?
- Topspin networks
- (Exotic Smoothness)
- Incorporating topspin networks into LQG
- Area and (Hamiltonian) operators
- (Topspin Foam)

Topology in Quantum Gravity: Why?

- Unlike geometry, topology is not part of the background independence of GR; we must specify a topology *a priori*.
  - A similar thing is true of dimension and differential stucture...
  - "GR is not a purely relational theory"[1]
- "Where does this topology come from?" A nice answer would be: the quantum nature of gravity.
- At the very least, we should be able to answer some questions:
  1. Does topology play a role in quantum gravity?
  2. If topology must be specified, when do we do that?
  3. Does it even matter? Can local measurements even detect different topologies?

[1]Smolin 2005, The Case for Background Independence

Topology in Loop Quantum Gravity

- At first glace, the canonical LQG approach to QG destroys all topological knowledge.
- Restricting from a 3-sphere to an embedded graph trades smooth information for descrete information; one can embed a given graph (spin network) in a large number of topologically inequivalent 3-manifolds.
- Of course, there are proposals: Causal sets theory, semiclassical limits of spinfoam models,...

Proposal: Topspin Networks

Allows one to track the topology of the spatial sections by upgrading spin networks to "topspin networks".

Topspin Networks[2]

Based on the following observation:

Alexander's Theorem (1920)

Any compact oriented 3-manifold can be described as a branched covering of $\mathbb{S}^3$, branched along a graph.

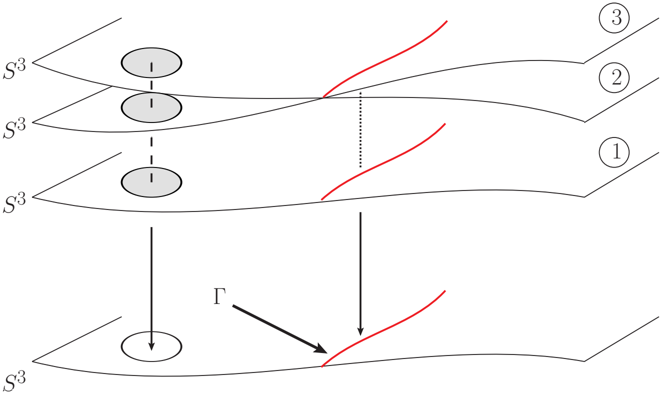
[2] Denicola D, Marcolli M and Zainy al-Yasry A 2010 Classical and Quantum Gravity 27(20)

Topspin Networks[2]

Based on the following observation:

Alexander's Theorem (1920)

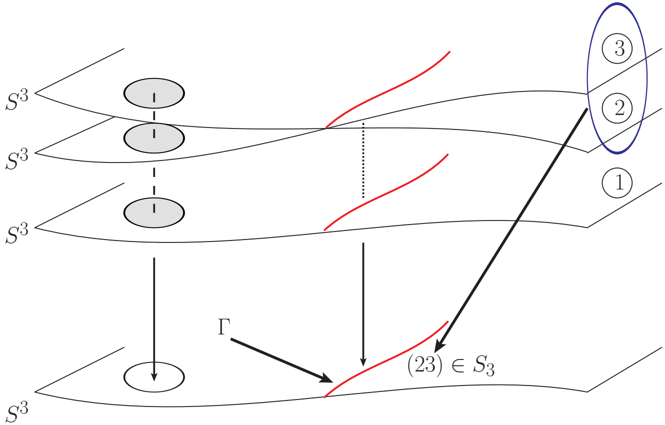
Any compact oriented 3-manifold can be described as a branched covering of $\mathbb{S}^3$, branched along a graph.



[2] Denicola D, Marcolli M and Zainy al-Yasry A 2010 Classical and Quantum Gravity 27(20)

There is a very nice way to represent this branched n-fold covering space by labeling the graph $\Gamma \hookrightarrow \mathbb{S}^3$ with permutation elements $\sigma \in S_n$ (and compatibility conditions on the vertices).

There is a very nice way to represent this branched n-fold covering space by labeling the graph $\Gamma \hookrightarrow \mathbb{S}^3$ with permutation elements $\sigma \in S_n$ (and compatibility conditions on the vertices).



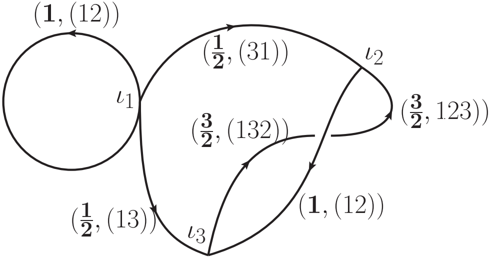
This tells us how to "stitch together" the sheets of the cover Σ, and knowledge of Γ and $\{\sigma\}$ now completely specifies the topology of Σ.

The Point:

By adding these permutation labels to spin networks, we can track both the topology and the geometry of the spatial sections.

A topological spin (**Topspin**) network $(\Gamma, \rho, \iota, \sigma)$ over a Lie group G consists of

- A spin network (Γ, ρ, ι) with embedded graph $\Gamma \hookrightarrow \Sigma$, G labels ρ and intertwiners ι;
- Permutation labels $\sigma \in S_n$ on the edges of Γ.



[a] CD 2011, IJGMMP (08),Asselmeyer-Maluga 2010,CQG (27)

The Point:

By adding these permutation labels to spin networks, we can track both the topology and the geometry of the spatial sections.

A topological spin (**Topspin**) network $(\Gamma, \rho, \iota, \sigma)$ over a Lie group G consists of

- A spin network (Γ, ρ, ι) with embedded graph $\Gamma \hookrightarrow \Sigma$, G labels ρ and intertwiners ι;
- Permutation labels $\sigma \in S_n$ on the edges of Γ.

Can think of this as a modification of the construction of LQG:

- Foliate spacetime into arbitrary spatial sections Σ_t.
- Realize these spatial sections as branched covers over $\mathbb{S}^3$.
- Add $SU(2)$ labels to the branch cover to encode the geometry.
- Determine appropriate fields and operators for LQG.

[a] CD 2011, IJGMMP (08), Asselmeyer-Maluga 2010, CQG (27)

The Point:

By adding these permutation labels to spin networks, we can track both the topology and the geometry of the spatial sections.

A topological spin (**Topspin**) network $(\Gamma, \rho, \iota, \sigma)$ over a Lie group G consists of

- A spin network (Γ, ρ, ι) with embedded graph $\Gamma \hookrightarrow \Sigma$, G labels ρ and intertwiners ι;
- Permutation labels $\sigma \in S_n$ on the edges of Γ.

Thinking ahead to a sum-over-states approach:

$$\sum_{\{(W,g):\partial W = M_1 \cup \bar{M}_2\}} \int \mathcal{D}[g] e^{iS[g]} \rightarrow \sum_{(\Gamma,\rho,\iota,\sigma)} \int \mathcal{D}[g] e^{iS[g]}$$

- A sum over cobordism W does not specify differential structure uniquely...**exotic smoothness**, which can modify observables[a].
- Alexander's theorem in 4D guarantees completely knowledge of the branch cover, including smooth structure...**no exotic smoothness**!

[a] CD 2011, IJGMMP (08), Asselmeyer-Maluga 2010, CQG (27)

Modifying LQG for the Topspin Case[3]

- Upgrading to topspin networks adds a finite symmetry to the theory generated by the permutation σ_I on each edge e_I by requiring that the tetrad fields be the same on sheets that collide.
- Symmetry group is now $SU(2) \times P_I$
- Need an algebra associated to this group for our fields; must contend with $T(P_I) = 0$ and $\int\sqcap(2)$ is not assocative. Thus we use

$$\mathcal{A} = \mathcal{U}(\mathfrak{su}(2)) \otimes \mathbb{C}P_I.$$

1. $\mathcal{U}(\mathfrak{su}(2))$ is the **universal enveloping algebra**, has a basis of polynomials in the generators of $\mathfrak{su}(2)$.
2. $\mathbb{C}P_I$ is the **group algebra** for $P_I \subset S_n$, with elements $\sigma = \sum_g a(g)g$, $a \in \mathbb{C},\ g \in P_I$.

[3]CD 2011,1111.1252

Modifying LQG for the Topspin Case[3]

- Upgrading to topspin networks adds a finite symmetry to the theory generated by the permutation σ_I on each edge e_I by requiring that the tetrad fields be the same on sheets that collide.
- Symmetry group is now $SU(2) \times P_I$
- Need an algebra associated to this group for our fields; must contend with $T(P_I) = 0$ and $\int\sqcap(2)$ is not assocative. Thus we use

$$\mathcal{A} = \mathcal{U}(\mathfrak{su}(2)) \otimes \mathbb{C}P_I.$$

1. $\mathcal{U}(\mathfrak{su}(2))$ is the **universal enveloping algebra**, has a basis of polynomials in the generators of $\mathfrak{su}(2)$.
2. $\mathbb{C}P_I$ is the **group algebra** for $P_I \subset S_n$, with elements $\sigma = \sum_g a(g)g$, $a \in \mathbb{C},\ g \in P_I$.

The universal enveloping algebra is isomorphic to the algebra of differential operators, so in some sense we are "upgrading the fields to operators earlier"

[3] CD 2011,1111.1252

Classical Algebra

For the classical algebra, we want the analogy of the "fields and holonomy"

- **Fields:** Use $\mathcal{A}$-valued one forms $E_a^{i\mu}$.
- **Holonomy:** "Ashtekar connection"

$$A_a^{i\mu} = \Gamma_a^i \sigma^\mu + \beta K_a^{i\mu},$$

But since $\mathcal{A}$ is no longer a Lie algebra, we can't use the usual exponential map. Instead

$$P_I \xrightarrow{\quad \mathcal{A} \quad} \mathcal{U}(\mathfrak{su}(2)) \xrightarrow{\quad exp \quad} GL_\lambda$$

- The composition $exp \circ \mathcal{A}$ is the usual exponential map on a $(\lambda + 1)$-dimensional representation of $\mathfrak{su}(2)$ polynomials.

Area Operator

Constructing operators for this approach is somewhat more complicated, since there are more possibilities for gauge invariance. Normally, one looks for things like

$$E^i E^j \kappa_{ij} \sim j(j+1) \text{ (Casimirs)}$$

The quadradic Casimirs naturally lie in the center $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ ("commuting elements") of the universal enveloping algebra. Since our fields can be arbitrary polynomials, we can have single second-order operators

$$\hat{\mathcal{O}}\tau,\ \tau = \kappa_{ij} E^i E^j,\ \tau \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})).$$

Thus, anything in $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ could be a gauge-invariant operator. For comparison with the usual theory, we can consider a first-order area operator,

$$^{(1)}A(S) = \sqrt{^{(1)}E_a^{i\mu}(S)^{(1)}E^{ak\nu}(S)\kappa_{ij}\kappa_{\mu\nu}}dx^1 \wedge dx^2.$$

The group action $p^{-1}(\Gamma)/P_I$ has m_I orbits and so above edge e_I, the fields actually take values in

$$\mathcal{U}(\mathfrak{g})^{m_I};$$

in other words m_I "counts the collisions". Thus;

$$^{(1)}A(S) = 4\pi l_p^2 \beta \sum_{I=1}^{N} \sqrt{j_I(j_I+1) \cdot m_I^2}.$$

with m_I as the number of independent sheets over edge e_I.

- This the *same result* we would have gotten by calculating the area of the surface $p^{-1}(S)$ intersecting with the usual spin network in the spatial section!
- Since when we represent the surface Σ_t as a branched cover we identify some edges e_I of the spin network m_I times, we would get

$$A(p^{-1}(S)) = 4\pi l_p^2 \beta \sum_{I=1}^{N} m_I \sqrt{j_I(j_I+1)}.$$

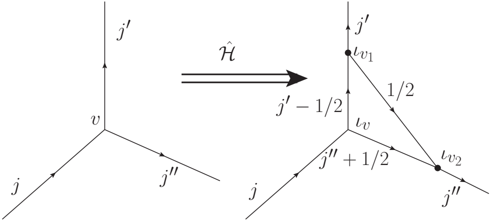
- Of course, in the usual theory we don't track the identifications so we wouldn't see this...

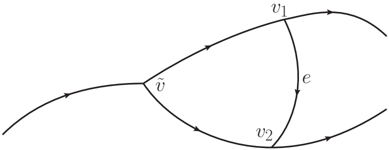
Summary

- Alexander's theorem can be used to express any closed 3-manifold as a branched cover of $\mathbb{S}^3$ over a graph.
- A topspin network is a spin network with added topological labels.
- These topspin networks can be incorporated into LQG by extended the symmetry group to $SU(2) \times S_n$.
- The exact form of the spectrum of the area operator depends on the topologty of the spatial section.
- The Hamiltonian is distinct from the usual one by virtue of the covering moves.
- The full **diffeomorphism invariance** must be studied, and must be done from the perspective of the geometric covering moves.
- **Topspin Foams:** It is easy to generalize Alexander's theorem to 4 dimensional branched coverings of $\mathbb{S}^3 \times [0,1]$ over a 2-complex; "time evolution of topspin networks". Summing over all possible branch loci would be a topspin foam model.

Hamiltonian Operator

The Hamiltonian we choose (Lorentzian, real connection, transforms covariantly under extended diffeomorphisms) by "making extraordinary vertices":

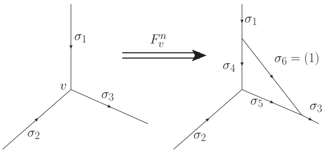




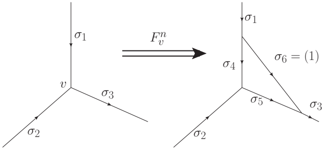
It is known[4] that the action of the Hamiltonian produces disjoint spin networks - each spin network ϕ_n with n extraordinary edges has a unique source network ϕ_0.

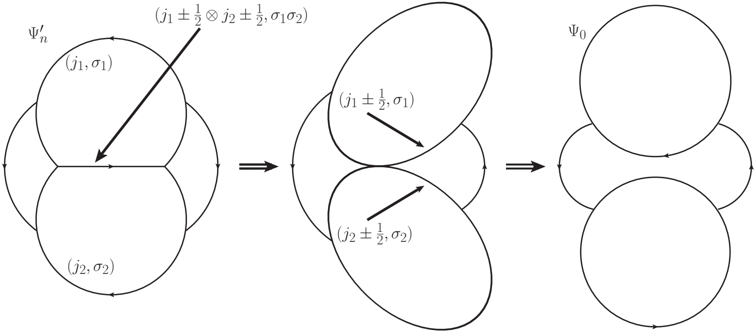
[4] Thiemann T 1998, Classical and Quantum Gravity 15(4)

For the action of the Hamiltonian on topspin networks, we choose the simplest possibility (in a similar manner to the geometric labels). It turns out that under the covering moves, one can remove extraordinary edges:



For the action of the Hamiltonian on topspin networks, we choose the simplest possibility (in a similar manner to the geometric labels). It turns out that under the covering moves, one can remove extraordinary edges:





Thus, the action of the Hamiltonian for topspin networks is fundamentally different from the usual one for spin networks (in the category-theoretic sense).