The Partition Function for Semiclassical Gravity and Cosmic Strings

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Path Integrals in Quantum Gravity

- In relativistic quantum theory, path integrals give us a direct connection between the classical actions $S[\phi]$ and the quantum amplitudes Z.
- In quantum field theory, path integrals

$$Z = \int \mathcal{D}\phi \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

are taken over the entire *history* of the particles.

- In (Euclidean) quantum gravity, one would expect something like:

$$Z = \int \mathcal{D}g \exp\left(-\frac{1}{\hbar} S[g_{\mu\nu}]\right), \qquad S[g_{\mu\nu}] = \frac{1}{\kappa} \int R[g_{\mu\nu}] dV$$

- In the standard model, the symmetry group of $S[\phi]$ is a Lie group G; for gravity, the symmetries of $S[g_{\mu\nu}]$ are related to the geometry of the underlying manifold.
- Goal: Formulate a path integral in terms of geometry rather then the metric.

- What are the degrees of freedom that correspond to the geometry?
- The gauge symmetries of general relativity are **diffeomorphisms** - invertible smooth functions

$$f : M_i \rightarrow M_j$$

for smooth 4-manifolds $M_i,\ M_j$.

- The dominant terms in the path integral would be the classical solutions $g^i_{\mu\nu}$ corresponding to diffeomorphism-inequivalent geometries M_i:

$$\boxed{Z = \sum_i \exp\left(-\frac{1}{\hbar} S[g^i_{\mu\nu}]\right)}.$$

- This is called the **Semiclassical Partition Function**.

Specifying Diffeomorphism Classes

- A complete specification of diffeomorphism classes (**smooth structures**) is an unsolved problem in dimension 4 (Fields and Abel medals for Milnor, Freedman, Donaldson, Witten, ...)
- We can sidestep this problem by reparametrizing any 4-manifold as a branched cover[1]:

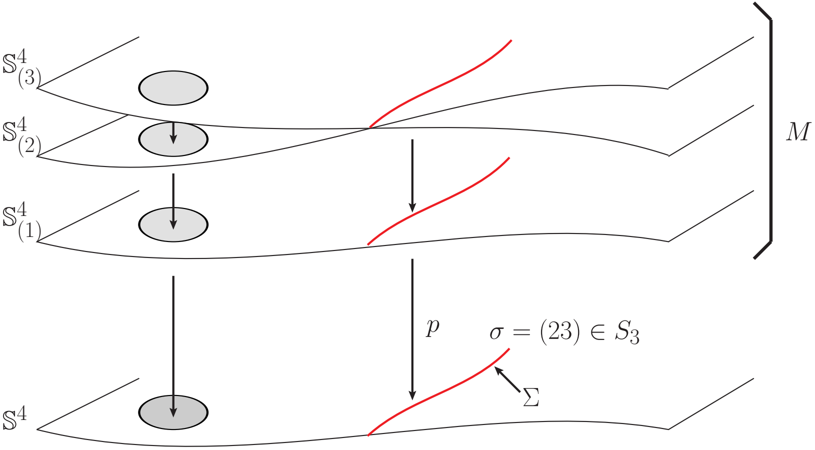
The Alexander-Piergallini Theorem

Any compact oriented 4-manifold can be described as a branched covering of $\mathbb{S}^4$, branched along an embedded surface.

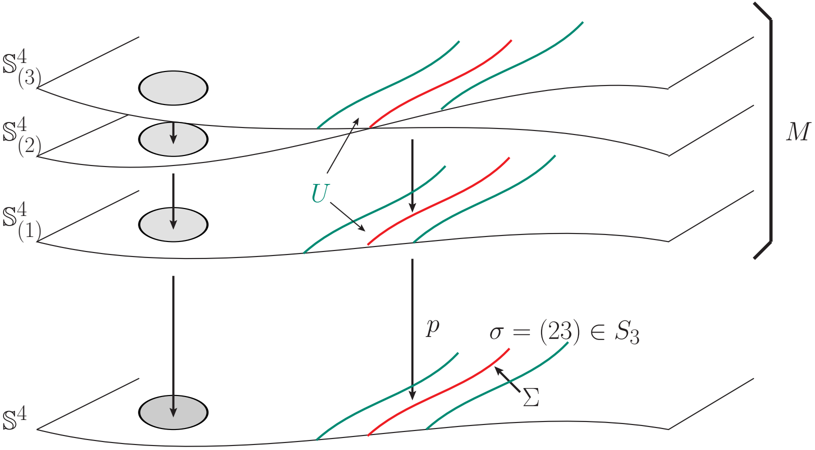
- This works for dimension $n > 2$ smooth manifolds branched over a $n - 2$ subcomplex of $\mathbb{S}^n$;
- We actually get complete *topological* information from this theorem too, not just geometric.

[1] Alexander, Bull. Amer. Math. Soc. 26 (1920), Piergallini, Topology 34 (1995) no. 3, 497-508

Illustrating the Alexander-Piergallini Theorem



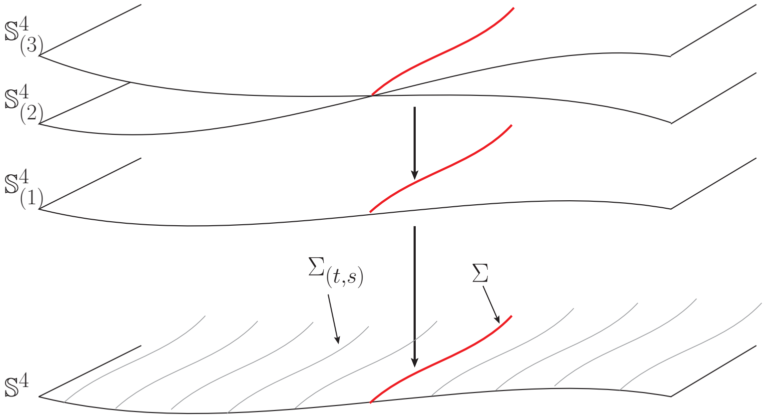
Illustrating the Alexander-Piergallini Theorem



$$Z = \sum_{(\Sigma_i, \sigma_i)} \exp\left(-\frac{1}{\hbar}\left[n S_{\mathbb{S}^4}(\Sigma_i) + S_U(\Sigma_i)\right]\right).$$

Representing the Surfaces Σ

- To perform the action integral, we can construct a codimension 2 foliation of $\mathbb{S}^4$ via the surfaces $\Sigma_{(t,s)}$ (*a la* ADM), now parametrized by $(t,s) \in \mathbb{R} \times \mathbb{R}$:



[a]Konopelchenko and Landolfi, J. Geom. Phys. 29 (1999) no. 4, 319-333.

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- We can explicitly write the embedding $\Sigma_{(t,s)} \hookrightarrow \mathbb{S}^4$ for each (t,s) in local coordinates (X^1, X^2, X^3, X^4):

$$X^1 = \frac{1}{2}\int(\bar{\psi}_1\bar{\psi}_2 - \phi_1\phi_2)dz + c.c \qquad X^3 = \frac{1}{2}\int(\phi_1\bar{\psi}_2 + \bar{\psi}_1\phi_2)dz + c.c.$$

$$X^2 = \frac{i}{2}\int(\bar{\psi}_1\bar{\psi}_2 + \phi_1\phi_2)dz + c.c \qquad X^4 = \frac{i}{2}\int(\bar{\psi}_1\phi_2 - \phi_1\bar{\psi}_2)dz + c.c$$

- These coordinates are functions of spinors which satisfy a set of Dirac equations:

$$\partial_z\psi_1 = p\phi_1 \qquad \partial_z\psi_2 = \bar{p}\phi_2$$

$$\partial_{\bar{z}}\phi_1 = -\bar{p}\psi_1 \qquad \partial_{\bar{z}}\phi_2 = -p\psi_2.$$

This is called the **generalized Weierstrass representation**[a]

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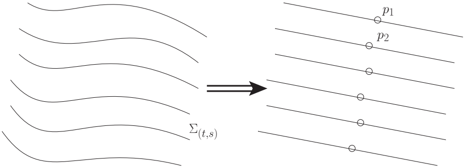
Flattening Surfaces and Cosmic Strings



- It turns out that such surfaces can generally be flattened if one includes a singular point p (called a *conical point*) on each[2]:

$$R_\Sigma \longrightarrow R|_p.$$

"Moving all the curvature to a point p".

[2] Troyanov, Trans. Amer. Math. Soc. 324 (1991), no. 2, 793-821

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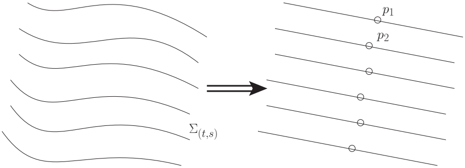

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- The metric near such a point is that of a cone with angular coordinate $0 \leq \phi \leq 2\pi(\beta + 1)$,

$$ds^2 = dr^2 + (\beta + 1)^2 r^2 d\phi^2.$$

This is *exactly* the form of the metric transverse to a **cosmic string**!

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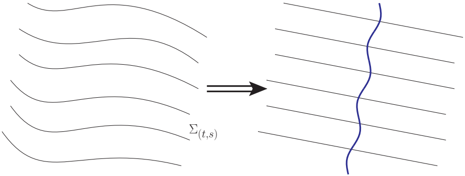

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- Adding up the contribution to the action from each string (with worldsheet metric γ_i), assuming the strings do not interact;

$$Z = \sum_{\sigma} \int \mathcal{D}\phi \mathcal{D}\psi \exp\left[\frac{4\pi n}{\kappa\hbar} \sum_i \beta_i \int \sqrt{-|\gamma_i|} dA\right] \times$$

The Complete Partition Function

- "Summing over all Σ" is replaced by integrating over all spinors ϕ, ψ;
- Adding up the contribution to the action from each string (with worldsheet metric γ_i), assuming the strings do not interact;
- Adding in the contribution from the extrinsic curvature via the Codazzi equation.

$$Z = \sum_{\sigma} \int \mathcal{D}\phi \mathcal{D}\psi \exp\left[\frac{4\pi n}{\kappa\hbar} \sum_i \beta_i \int \sqrt{-|\gamma_i|} dA\right] \times$$

$$\times \exp\left[-\frac{2}{\kappa\hbar} \int [Q_1 \bar{Q}_1 + Q_2 \bar{Q}_2 - (p^2 + \bar{p}^2)] dV\right].$$

Here the Hopf fields Q_i are currents of the spinor fields, *i.e.*

$$Q_1 = \frac{1}{2}\left[\frac{\psi_2 \partial_z \bar{\phi}_2 - \bar{\phi}_2 \partial_z \psi_2}{|\psi_2|^2 + |\phi_2|^2} + \frac{\phi_1 \partial_z \bar{\psi}_1 - \bar{\psi}_1 \partial_z \phi_1}{|\psi_1|^2 + |\phi_1|^2}\right].$$

Summary

- We have a semiclassical partition function for gravity which includes *all* classical solutions ("no exotic smoothness")
- This form of the partition function should be explicit enough for calculations such as expectation values or propagators.
- In this approach there is a natural connection between semiclassical gravity and non-interacting cosmic strings.

Summary

- We have a semiclassical partition function for gravity which includes *all* classical solutions ("no exotic smoothness")
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- In this approach there is a natural connection between semiclassical gravity and non-interacting cosmic strings.

What Can We Do?

- Treat as a generating functional and study semiclassical a) gravity through (ϕ, ψ) or b) strings through γ_i?
- Propagators of the geometry of the string worldsheet?
- Interaction terms between ϕ and ψ?