Online Appendix of "Dynamic Signaling with Dropout Risk"

Francesc Dilme* Fei Li†

February 6, 2015

In this document we address two extensions. In Appendix A we extend the benchmark model by allowing type-dependent dropout risk. In Appendix B we extend the original binary-type model to a multiple-type model.

A Type-Dependent Dropout Risk

Here we extend our benchmark model by allowing the seller's dropout rate to be correlated with his type. There are three relevant cases: (1) $\lambda_H > \lambda_L \geq 0$, (2) $\lambda_L > \lambda_H > 0$, and (3) $\lambda_L \geq \lambda_H = 0$.

A.1 $\lambda_H > \lambda_L \geq 0$ Case

The first case we consider is $\lambda_H > \lambda_L \geq 0$; that is, the H-seller exogenously drops out at a higher rate than the L-seller. The following lemma implies that the equilibrium set in this case coincides with the benchmark model when $\lambda = \lambda_H$:

Lemma A1. *Assume that $\lambda_H > \lambda_L \geq 0$. Then, (s^L, s^H, w, p) is an equilibrium if and only if it is also an equilibrium in the benchmark model with $\lambda = \lambda_H$.*

Proof. We first prove that Lemma 1 in our paper (which holds when $\lambda_H = \lambda_L$) is still valid when $\lambda_H \geq \lambda_L$. Let T be the last period t before T^L where $s_t^L \leq s_t^H$. In this case

$$p_{T+1} \leq p_T \leq w_T \ .$$

*Department of Economics, University of Bonn; Email: fdilme@uni-bonn.de.
†University of North Carolina, Chapel Hill; Email: lifei@email.unc.edu.

Furthermore, since the L-seller is voluntarily dropping out at time $T+1$, this implies $w_T \leq w_{T+1} - c_L$. Nevertheless, since $s^L_{T+1} \geq s^H_{T+1}$, we have $w_{T+1} \geq p_{T+1}$, which is a contradiction, because

$$w_{T+1} \leq p_{T+1} \leq p_T \leq w_T \leq w_{T+1} - c_L \ .$$

So, when $\lambda_H \geq \lambda_L$, it is still true that $s^L_t > s^H_t$ in all periods before T^L. Therefore, relaxing the constraint $\lambda_L = \lambda_H = \lambda$ to $\lambda_L \leq \lambda_H = \lambda$ does not introduce new equilibria. Trivially, it does not destroy any equilibria, since in the model $\lambda_L = \lambda_H = \lambda$, in all equilibria, $s^L_t > \lambda$ for all equilibria and period $t \leq T^L$. □

The intuition behind this lemma is the following. In our original model the endogenous dropout rate of the L-seller is positive in all periods before the last. So, the constraint $s^L_t \geq \lambda$ is never binding in equilibrium. Therefore, all equilibria from the benchmark model for $\lambda = \lambda_H$ are also equilibria for the case $\lambda_H > \lambda_L \geq 0$. On the other hand, for any equilibrium in the case where $\lambda_H > \lambda_L$, the incentive compatibility for the L-seller imposes that $s^L_t \geq \lambda_H - \lambda_L$. Is then easy to show that an equilibrium exists in the benchmark model where the L-seller's dropout rate is equal to $s^L_t - (\lambda_H - \lambda_L)$ exists.

A.2 $\lambda_L > \lambda_H > 0$ Case

We now assume that s^L_t is no lower than $\lambda_L > \lambda_H$. As we see in Figure 1, this constraint may be potentially binding in two connected regions, one for large t (and w) and the other for intermediate values. In any equilibrium, when this constraint is binding, both types strictly prefer to wait. Unlike our benchmark model, the equilibrium belief p_t still goes up in this regions since $\lambda_L > \lambda_H$. So, the equilibrium characterization in the benchmark model cannot survive for some parameters. Fortunately, the following theorem shows that the equilibrium characterization in the benchmark model still works when λ_L is not significantly larger than λ_H.

In order to compare models, we denote the set of parameters as $(p_0, c_L, c_H, \lambda_L, \lambda_H)$. Note that in our original problem the set of parameters is $(p_0, c_L, c_H, \lambda, \lambda)$.

Theorem A1. *Fix some parameters $p_0 \in (0,1)$, $0 < c_H < c_L$ and $\lambda \in (0,1)$. Then there exists $\varepsilon > 0$ such that the set of equilibria of the model with parameters $(p_0, c_L, c_H, \lambda_L, \lambda)$ is independent of λ_L in the region $[\lambda, \lambda + \varepsilon]$.*

Proof. We first prove that if $\varepsilon > 0$ is small enough, the model with parameters $(p_0, c_L, c_H, \lambda+\varepsilon, \lambda)$ does not have more equilibria than the model with parameters $(p_0, c_L, c_H, \lambda, \lambda)$. Note that Lemma 3 in the paper still holds (the H-seller can imitate the strategy of the L-seller). Now we prove a

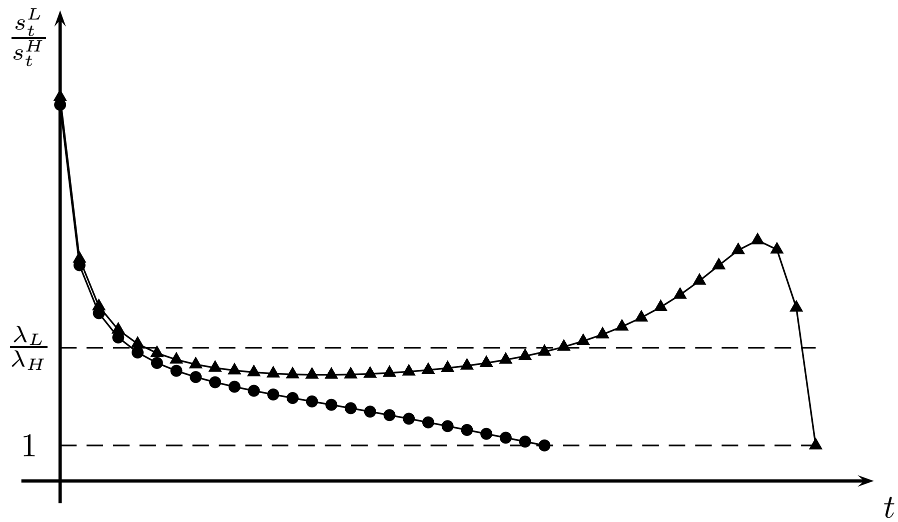

Figure 1: Endogenous dropout rate of the L-seller.

result analogous to Lemma 1. Assume that the L-seller does not voluntarily drop out in period t, so his dropout rate is $\lambda + \varepsilon$. We then have two cases:

1. First assume that the dropout rate of the H-seller is larger than $\lambda + \varepsilon$, so $w_t > p_t$. In this case, since Lemma 3 is still valid (i.e., $V_t^L \leq p_t$) the L-seller strictly wants to drop out, so we obtain a contradiction.

2. Assume now that $s^H \in [\lambda, \lambda + \varepsilon]$. In this case $p_{t+1} = p_t + O(\varepsilon)$ and $w_t = p_t + O(\varepsilon)$, so $w_t - p_{t+1} = O(\varepsilon)$. Then, using the same logic we used in the proof of Lemma 1, we have

$$w_t \leq W_t^L \leq V_{t+1}^L - c_L \leq p_{t+1} - c_L \ .$$

   Therefore, $w_t - p_{t+1} \leq -c_L$. But this is inconsistent with $w_t - p_{t+1} = O(\varepsilon)$.

This proves that, if $\varepsilon > 0$ is small enough, the model with $\lambda_H = \lambda$ and $\lambda_L = \lambda + \varepsilon$ does not have more equilibria than it does when $\varepsilon = 0$.

Let's prove the converse, that is, that if $\varepsilon > 0$ is small enough, the model with parameters $(p_0, c_L, c_H, \lambda, \lambda)$ does not have more equilibria than the model with parameters $(p_0, c_L, c_H, \lambda+\varepsilon, \lambda)$. Assume by contradiction that there exists a strictly decreasing sequence $\{\varepsilon_n > 0\}_{n\in\mathbb{N}}$ converging to 0 such that, for each n, there exists an equilibrium in the model with parameters $(p_0, c_L, c_H, \lambda, \lambda)$ and some t_n reached with positive probability such that $s_{t_n}^L \in [\lambda, \lambda + \varepsilon_n)$. This implies $p_{t_n+1} =$

$p_{t_n} + O(\varepsilon_n)$ and $w_{t_n} = p_{t_n} + O(\varepsilon_n)$, so $w_{t_n} - p_{t_n+1} = O(\varepsilon_n)$.[1] So,

$$w_{t_n} = W^L_{t_n} = V^L_{t_n+1} - c_L \leq p_{t_n+1} - c_L \ .$$

This, again, is a contradiction. □

A.3 $\lambda_L \geq \lambda_H = 0$ Case

When the H-seller is not forced to leave the market, the main mechanism used in our benchmark model is not present here. Indeed, in our benchmark model, Lemma 1 establishes that the L-seller mimics the H-seller who is exogenously forced to drop out in order to save the high cost of signaling. In this case, the fact that the H-seller exogenously drops out implies that early dropout cannot be punished too much, constraining the off-the-path-of-play beliefs of the buyers. This is no longer true when $\lambda_H = 0$, so the set of equilibria is qualitatively different from the $\lambda_H > 0$ case, where we recover most of the equilibria of the static signaling model.

B Multiple Types

Now we consider the $N > 2$ types case in which $\theta \in \{1, 2, 3, ..., N\}$ with a prior p_0^θ, where $\sum_{\theta=1}^N p_0^\theta = 1$. The θ-seller has a cost of signaling c^θ, $c^\theta > c^{\theta+1}$. The value of the θ-good is Y^θ, $Y^\theta < Y^{\theta+1}$. All types exogenously drop out with probability λ.

The equilibrium concept is has the same spirit as the one for the two-types model (Definition 1), but we adapt it to address the fact that now we have many types. Note that buyers' offers depend only on the expected value of the good of a seller that drops out, and not on other moments of the value distribution. This fact helps us to keep our definition simple:

Definition B1. *An equilibrium is a strategy profile $(s^\theta)_{\theta=1,...,N}$, a wage process w and N belief sequences p^θ, for $\theta = 1, ..., N$, such that:*

1. *the θ-seller chooses $s^\theta \in [\lambda, 1]$ to maximize his expected payoff given w;*
2. *if a seller drops out in period t, buyers offer*

$$w_t = \frac{\sum_{\theta=1}^N p_t^\theta s_t^\theta Y^\theta}{\sum_{\theta=1}^N p_t^\theta s_t^\theta} \ ; \quad \textit{and} \tag{1}$$

[1] With some abuse of notation, p_{t_n} and w_{t_n} denote the corresponding posteriors in the n-th equilibrium of the sequence.

3. *when it is well defined, p_t^θ is updated according to Bayes' rule*

$$p_{t+1}^{\theta} = \frac{p_t^\theta(1-s_t^\theta)}{\sum_{\theta'=1}^{N} p_t^{\theta'}(1-s_t^{\theta'})} \ . \tag{2}$$

Let T^θ be the last time the θ-seller is in school. The following theorem characterizes the properties of the equilibria in a multiple-type model.

Theorem B2. *Under the previous assumptions, in any equilibrium:*

1. *in each period t, there is at most one type that is indifferent to dropping out or continuing to signal;*
2. *more productive types signal longer, $T^\theta \leq T^{\theta+1}$;*
3. *there is positive voluntary dropout in all periods; and*
4. *the wage w_t is concave in t.*

Proof. 1. Assume that, in period t, there are two types $\theta_1, \theta_2 \in \Theta$, with $c^{\theta_1} < c^{\theta_2}$, and both are indifferent between dropping out or continuing to signal. Let τ_1 and τ_2 denote, respectively, the stopping times of the continuation strategies that make sellers indifferent to dropping out or remaining.[2] Then, we have

$$w_t = \mathbb{E}[w_{\tau_{\theta_2}} - c^{\theta_2}\tau_{\theta_2}] \geq \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_2}\tau_{\theta_1}] > \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_1}\tau_{\theta_1}] = w_t \ .$$

The first (weak) inequality is from the optimality of the θ_2-seller. The strict inequality is because $\mathbb{E}[\tau_{\theta_1}] > 0$ and $c^{\theta_1} < c^{\theta_2}$. The equalities come from the fact, for each $i \in \{1,2\}$, the i-seller id indifferent between dropping out (and getting w_t) or staying and following τ_i. Therefore, we have a contradiction.

2. Assume there exists $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 < \theta_2$ and $T^{\theta_1} > T^{\theta_2}$. Let τ_{θ_1} be the stopping time of the continuation strategy after T^{θ_2}, given by the strategy of θ_1. Then, note that

$$w_{T^{\theta_2}} \geq \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_2}\tau_{\theta_1}] > \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_1}\tau_{\theta_1}] \geq w_{T^{\theta_2}} \ .$$

This is clearly a contradiction. The first inequality comes from the optimality of the θ_2-seller choosing to drop out at T^{θ_2} (since he could deviate to mimic the θ_1-seller). The second inequality is given by the fact that since $\theta_1 < \theta_2$, $c^{\theta_2} < c^{\theta_1}$ and since $T^{\theta_1} > T^{\theta_2}$, $\mathbb{E}[\tau_{\theta_1}] > 0$. The last inequality comes from the optimality of the θ_1-seller choosing to drop out at $T^{\theta_1} > T^{\theta_2}$ (since he could deviate to mimic the θ_2-seller).

[2] For this proof, for a given strategy, it is convenient to use the random variable τ, which gives the duration of the game.

3. Define $\Theta_t \equiv \{\theta | T^\theta \geq t\}$ and $\theta_t \equiv \min \Theta_t$. We proceed as in the proof of Lemma 3 in the paper. Now we have

$$\begin{aligned}
\mathbb{E}_t[w_\tau | \tau \geq t] &= \sum_{\tau=t}^{\infty} \Pr(\tau, t) w_\tau = \sum_{\tau=t}^{\infty} \Pr(\tau, t) \frac{\sum_\theta Y^\theta s_\tau^\theta p_t^\theta \Pr^\theta(\tau, t)}{\Pr(\tau, t)} \\
&= \sum_\theta p_t^\theta Y^\theta \sum_{\tau=t}^{\infty} s_\tau^\theta \Pr^\theta(\tau, t) = \sum_\theta p_t^\theta Y^\theta \ ,
\end{aligned}$$

where $\Pr(\tau, t)$ and $\Pr^\theta(\tau, t) = s_\tau^\theta \prod_{t'=t}^{\tau-1} (1 - s_{t'}^\theta)$ are defined as in the proof of Lemma 3 in the paper.

Note that, by the previous result,

$$\sum_{\theta=\theta_t}^{N} p_t^\theta V_t^\theta = \mathbb{E}_t[w_\tau | \tau \geq t] - \sum_{\theta=\theta_t}^{N} p_t^\theta c^\theta \tau^\theta(t) < \mathbb{E}_t[w_\tau | \tau \geq t],$$

where $\tau^\theta(t)$ is the stopping time for the θ-seller conditional on reaching t. Since $V_t^\theta \leq V_t^{\theta+1}$ (since the $(\theta+1)$-seller can mimic the θ-seller at a lower cost), and $\sum_{\theta=\theta_t}^{N} p_t^\theta = 1$ we have that $V_t^{\theta_t} < w_t$.

Assume that in period t there is no voluntary dropout. In this case, $w_t = \sum_\theta p_t^\theta Y^\theta$. Since we just showed $V_{\theta_t} < \sum_\theta p_t^\theta Y^\theta$, the θ_t-seller strictly prefers to drop out, which is a contradiction.

4. Note that, by part 3 of this theorem, we have that $w_{t+1} - c^{\theta_t} \leq w_t$. Furthermore, $w_{t+1} - c^{\theta_{t+1}} \geq w_t$. This implies that $w_{t+1} - w_{t+1} \in [c^{\theta_{t+1}}, c^{\theta_t}]$. Since c^θ is decreasing in θ and, by part 2 of this theorem, θ_t is (weakly) increasing in t, w_t is concave in t.

$\square$

As we have shown, most features of the two-type model are preserved. Note, however, that under many types we have decreasing returns to signaling instead of linear ones, because lower types are skimmed out before higher types in equilibria. The equilibrium construction in multiple-type models is almost identical to that in the two-type model and thus is omitted.