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THE EFFECTIVE CONTENT OF SURREAL ALGEBRA

JACOB LURIE

Abstract. This paper defines and explores the properties of several effectivizations of the structure of surreal numbers. The construction of one of previously investigated systems, the metadyadics, is shown to be effectively equivalent to the construction of the surreals in $L_{\omega_1^{CK}}$. This equivalence is used to answer several open questions concerning the metadyadics. Results obtained seem to indicate that the metadyadics best capture the notion of a recursive surreal number.

§1. Introduction. In this paper we investigate to what extent the study of the surreal numbers can be made effective. Conway's induction defining the class **No** can be considered a more general form of the inductive construction of the ordinals. The constructive ordinals have been studied in great detail and many of the useful ordinal operations can be performed effectively on their notations. The study of recursive versions of the surreal numbers was begun by Harkleroad in [5]. He shows that effectivizations of the two natural processes by which surreals have been built (cuts and sign sequences) are not equivalent. This may be contrasted with the result that the recursive ordinals are equal to the constructive ordinals; somehow the extra "shape" of the surreals makes them more difficult to handle algorithmically. Here we continue this study, defining "recursive" surreals in other ways and probing more deeply into their algebraic and order properties. We answer questions left open by Harkleroad concerning the effectivization by cuts. In order to understand better the effectiveness of operations on the surreals, stronger notions of recursiveness on games are investiaged.

§2. Overview. In the next section we summarize the material assumed and the notation we will be using. In the section following that, we define the basic objects which will occupy our attention throughout the paper (in particular, the metadyadics), summarize what is known about them, and deduce a few simple relations between them. The fifth section is devoted to the properties of E, which is perhaps the least well understood of the three classes of recursive surreals we will study (and was not considered by Harkleroad), as well as some general facts about recursive approximations of real numbers. The sixth section is a brief outline of the theory of the surreal numbers, developed in an arbitrary admissible set, which includes generalizations of some concepts discussed in [1]. Connections between

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this general theory and the effective theory is established in the seventh section, where we prove our basic characterization of the metadyadics and answer some of Harkleroad's questions about them. In the eighth section, we generalize some results of [3] concerning the fixed point sets of certain surreal-valued functions, and establish effective versions of the same results. The ninth section combines results of the sixth and seventh sections to demonstrate the computability of certain algebraic operations on the surreals, and includes one result of pure recursion-theoretic interest. In the tenth section, we define more restrictive notions of computability for the surreal numbers, with respect to which the arguments given in the previous sections are inapplicable (leading to a large number of as yet unanswered questions). The eleventh section explores this notion in the (perhaps simpler) case of impartial games, and in general develops the effective theory of such games: this theory leads to a natural system of notations for the recursive ordinals, with respect to which metarecursive functions are quite easily defined. In the twelfth section we explore some of the universal properties of the classes of recursive surreals, in particular, we (roughly) characterize the structure of the metadyadics up to metarecursive isomorphism. Finally, we provide a list of some of questions left open by this study.

§3. Notation and terminology. We assume an understanding of the basic concepts of recursion theory, the constructive ordinals, the hyperarithmetic hierarchy, and the class of surreal numbers. The concepts of admissible set recursion theory (in particular metarecursion) and the more general notion of a game as in [3] will also be useful but not essential. We use Church's thesis freely; algorithms and constructions will be presented in an informal manner.

Greek letters (α, β, γ) will denote ordinals.

O is the set of notations for the constructive ordinals.

$<_O$ is the ordering on constructive ordinal notations.

$+_O$ is effective ordinal addition on O.

$|x|$ is the ordinal coded by x for $x \in O$.

φ_n is the nth partial recursive function.

W_e is the eth recursively enumerable set.

K is the halting set. A' is the jump of A. H_a is the ath H-set, as in [6].

L is Gödel's constructible hierarchy.

L_α is the initial segment of L constructed before stage α.

Dyadics, or dyadic rationals, are rational numbers of the form $a/2^n$. **No** is the class of surreal numbers.

If x is a surreal, x^+ denotes the surreal obtained by adding $+$ to the end of the sign sequence for x; similarly for x^-.

For surreals x and y, $x \preceq y$ if and only if the sign sequence defining x is an initial segment of the sign sequence defining y.

If x is a surreal given by a cut, x^L is a typical left convergent to x, and x^R a typical right convergent, as in [3]. Following Conway, we write $x = \{x^L | x^R\}$. We also use this notation for sets: $x = \{L|R\}$ means that x is the simplest surreal between every element of L and every element of R.

If A is any set or function being constructed in ordinal stages A^α, then $A^{<\alpha} = \bigcup_{\beta<\alpha} A^\beta$.

Ordinal multiplication will be written on the left. That is, $\alpha\beta$ consists of α copies of β.

$\mathbf{On}_2$ is the field structure on the ordinals defined in [3].

If S is a set of ordinals, $\mathrm{mex}(S)$ is the least ordinal not in S.

§4. **The classes I, E, and C.** In this section we define the three most important classes of "recursive surreal numbers" and begin to explore their algebraic properties. Two of these, whose index sets we will call I and C, were considered by Harkleroad. The construction of I is an effectivization of the surreals by sign sequences. The construction of C is the simplest effectivization by cuts. E is obtained by modifying the construction of C to make it more effective.

DEFINITION 4.1. Let

$$I = \{ n : (\exists k \in O)(\forall x)[\varphi_n(x) \downarrow \iff x <_O k] \wedge [\mathrm{range}(\varphi_n) \subseteq \{0,1\}] \}.$$

Then each element of I can be considered to code a function defined on an initial segment of the ordinals (via O). Define a function s_n from the ordinals to $\{+,-\}$ by

$$s_n(\beta) = \begin{cases} + & \text{if } (\exists x)[|x| = \beta \wedge \varphi_n(x) \downarrow = 0] \\ - & \text{if } (\exists x)[|x| = \beta \wedge \varphi_n(x) \downarrow = 1]. \end{cases}$$

Let $s^I(n) = \lambda x\, s_n(x)$. Then s^I can be viewed as a function from I to $\mathbf{No}$. $n \in I$ is a code for the surreal $s^I(n) \in \mathbf{No}$.

I is clearly Π^1_1-complete (let $\varphi_{f(x)}$ converge only $<_O$-below x and always equal 0. Then f is a one-reduction of O to I). I, via s^I, codes those surreals which have computable sign sequences. Harkleroad ([5]) shows that multiplication and addition cannot be represented by effective operations on I-indices. However it is possible to pass effectively from O-notations for ordinals to I-notations of the corresponding surreals, and from computable real notations to I-notations for the same real.

I provides a recursive analog of the surreals in the same sense that O provides a recursive analog of the ordinal numbers. This can be made more precise as follows: We construct a set of notations $O^\dagger$, an ordering $<_{O^\dagger}$, and a function $s^{O^\dagger}$ from $O^\dagger$ to $\mathbf{No}$ inductively as follows:

DEFINITION 4.2. $O^\dagger$ is the smallest set such that:

- $1 \in O^\dagger$, $s^{O^\dagger}(1) = 0$.
- If $n \in O^\dagger$ then $3n \in O^\dagger$, $s^{O^\dagger}(3n) = s^{O^\dagger}(n)^+$, $n <_{O^\dagger} 3n$.
- If $n \in O^\dagger$ then $3n+1 \in O^\dagger$, $s^{O^\dagger}(3n+1) = s^{O^\dagger}(n)^-$, $n <_{O^\dagger} 3n+1$.
- If $(\forall x)[\varphi_n(x) \downarrow \in O^\dagger \wedge \varphi_n(x) <_{O^\dagger} \varphi_n(x+1)]$ then $3n+2 \in O^\dagger$, $s^{O^\dagger}(3n+2) = \lim_x s^{O^\dagger}(\varphi_n(x))$, and each $\varphi_n(x) <_{O^\dagger} 3n+2$.

This definition is almost identical to the inductive definition of O except that it allows for two different types of successor. As a consequence, the length of a surreal can be effectively obtained from its $O^\dagger$ index. This is not true for I, but this is essentially the only difference.

THEOREM 4.3. *There are recursive functions $f(x,y)$, $g(z)$, and $h(z)$ such that if x is an I-index for some surreal via a function defined on O below y, then $f(x,y)$ is an $O^\dagger$-index for the same surreal, and if z is an $O^\dagger$-index for some surreal, then $g(z)$ is an I-index for the same surreal, coding a function defined on O below $h(z)$.*

PROOF. We define f by transfinite recursion on y as follows:

$$f(x,1) = 1,$$
$$f(x,2^y) = 3f(x,y) + \varphi_x(y),$$
$$f(x,3\cdot 5^y) = 3e+2$$

where e codes the function $t(k) = f(x,\varphi_y(k))$. f is easily shown to have the desired properties. h is equally simple:

$$h(1) = 1,$$
$$h(3z) = h(3z+1) = 2^{h(z)},$$
$$h(3e+2) = 3\cdot 5^i$$

where i codes the function $t(k) = h(\varphi_e(k))$. With g we must take a bit more care. We first define a two-place function $g(z,a)$ and then let $g(z)$ be the index for the function $g(z,a)$ (where z is treated as a constant). $g(z,a)$ is defined by effective transfinite recursion on z. Intuitively $g(z,a)$ is the ath sign in the surreal coded by z. $g(z,a)$ will converge only if there is such a sign, that is, $a <_O h(z)$, assuming that z is a proper index.

$$g(3a,a) = 0,$$
$$g(3a+1,a) = 1,$$
$$g(3b,a) = g(b,a) \quad \text{if } b \neq a,$$
$$g(3b+1,a) = g(b,a) \quad \text{if } b \neq a,$$
$$g(3e+2,a) = g(\varphi_e(k),a)$$

for any k such that the right hand side converges. Any such k will do assuming that $3e+2$ is in $O^\dagger$, since in any case the algorithm can only converge when the unique successor of a is found. ⊣

The requirement of having a recursive sign sequence is extremely restrictive. The problems are hinted at in the proof of the equivalence of constructive and recursive ordinals. This proof is effective in only one direction; there is no effective way to pass from recursive well-orderings to their constructive ordinal heights. What is done instead is to achieve a clumsy bound on how large such a height might be. Such a bound is sufficient since an arbitary truncation of a constructive ordinal remains constructive. Here the situation is much more delicate. Information content is as important as length. While surreals with I-indices are closed (effectively) under arbitrary truncations, it is not enough to simply be "big", the sign sequence must also agree with the surreal being constructed. In light of this it is not surprising that the surreal analog of the equivalence between recursive and constructive ordinals fails. The next class of surreals we will describe, C, is the most important. It is the

surreal analog of the recursive ordinals, and we will have much to say about it in the future. The definition is due to Harkleroad.

DEFINITION 4.4. For recursive ordinals α, define

$$C^{\alpha} = \{ 2^x 3^y : W_x \subseteq C^{<\alpha} \wedge W_y \subseteq C^{<\alpha} \wedge (\forall m \in W_x)(\forall n \in W_y)[(m,n) \in LT^{<\alpha}] \},$$

$$LT^{\alpha} = \{ (2^q 3^r, 2^s 3^t) \in C^{<\alpha} : (\exists m \in W_r)[(m, 2^s 3^t) \in LE^{<\alpha}] \vee (\exists n \in W_s)[(2^q 3^r, n) \in LE^{<\alpha}] \},$$

$$LE^{\alpha} = \{ (2^q 3^r, 2^s 3^t) \in C^{<\alpha} : (\forall n \in W_t)[(2^q 3^r, n) \in LT^{<\alpha}] \wedge (\forall m \in W_q)[(m, 2^s 3^t) \in LT^{<\alpha}] \}.$$

Let C be the union of each C^{α}, and $<_C$ the union of each LT^{α}. Define the function s^C from C to **No** by $s^C(2^x 3^y) = \{ s^C(a) | s^C(b) \}$ where a ranges over W_x and b over W_y.

It is a consequence of the induction that s^C is well defined. C and its ordering are Π^1_1-complete. Harkleroad shows that there is an effective embedding of I into C that is the identity on the surreals coded, that is, there is a recursive function f such that $(\forall x \in I)[f(x) \in C \wedge s^I(x) = s^C(f(x))]$. Harkleroad also shows that negation, addition, and multiplication can be performed effectively on C-indices. A similar argument shows that the same holds for ω-exponentiation. Recall that $\omega^x = \{ 0, r\omega^{x^L} | s\omega^{x^R} \}$, where r and s range over the positive real numbers. It is sufficient to allow r to range over the natural numbers and s to range over the reciprocals of positive natural numbers. Since I effectively includes the computable reals and C effectively includes I, let f and g be recursive functions so that $f(n)$ is a C-index for n, and $g(n)$ is a C-index for $1/n$. Now define $h(2^s 3^t) = 2^q 3^r$ where W_q includes $f(0)$ and all products (which, as has been shown, can be performed effectively) of $f(n)$ and x for $x \in W_s$. Similarly W_r will be the recursively enumerable sets of all indices of products of $g(n)$ and y, for $n > 0$ and $y \in W_t$. It is easy to see that $(\forall x \in C)[h(x) \in C \wedge s^C(h(x)) = \omega^{s^C(x)}]$. We will postpone a more thorough discussion about the closure of C under various operations until later. Note that the definitions of multiplication and addition are simple effectivizations of the usual definitions. As a consequence, we will find that the elements entering C before some ε-number form a ring in which the ring operations may be carried out effectively. We will use this fact later. Now we define one more class of surreals, E. E is defined by an induction similar to the one defining C. There are various ways to weaken this induction. While C allows one to take arbitary recursively enumerable sets of convergents, we may want to make the construction more effective by requiring that certain additional requirements be met. For example, we might require that the convergents be enumerated in order. Or if there is some greatest left convergent or least right convergent, we might require that that convergent to be specified as such. Or we might require that there be some sort of evidence that each left convergent is below each right convergent. Any combination of these and possibly other requirements would produce a particular

effectivization of the surreals. We want to single out one, E, which has some particularly nice properties. We will require that E satisfy the requirement involving evidence, in a manner which we shall now make precise. E will be a subset of C, and code surreals in the same manner. Hence it is automatic that one can pass effectively from E-indices to C-indices. We define relations $<_L$ and $<_R$ on C as follows: $x <_L y$ if x is a left convergent to y. We then take the transitive closure of this relation. Similarly, $x <_R y$ if y is a right convergent to x. Again, we take the transitive closure of this relation. Now we define $\leq_{\text{eff}}$. $x \leq_{\text{eff}} y$ just if $(x = y) \vee (x <_R y) \vee (x <_L y) \vee (\exists z)[z <_L y \wedge x <_R z]$. Note that $\leq_{\text{eff}}$ is not transitive! This is very important and we *do not* take the transitive closure. The idea here is that in order to have $x \leq_{\text{eff}} y$ we must be able to find evidence of this. Since by assumption, x and y are valid surreals, such evidence must be valid. However if we were to take the transitive closure, it would allow us to attempt to locate z with x as a left convergent and y as a right convergent, indepenently of z's status as a surreal. If we took the transitive closure only of the restriction of $\leq_{\text{eff}}$ to C we would not have this problem, but the search for "proof" of an inequality would become highly ineffective. Essentially E is the largest subset of C such that if $x \in E$ every convergent y to x is also in E, and if x is in E, $x^L \leq_{\text{eff}} x^R$. This is effectively equivalent to the definition we will adopt, however we place an additional restriction that greatly simplifies the second condition. We require that if $x \in E$ and $y <_L x$ then y is a left convergent to x, and the analogous statement for right convergents. This simplifies the definition of $x \leq_{\text{eff}} y$: $x = y$, or x is a left convergent to y, or y is a right convergent to x, or they share a convergent z to the right of x and the left of y. Clearly this is a subset of the E defined without this restriction, and given any x in the unrestricted E we can perform a hereditary "filling out" on x to bring it into E. If $x, y \in E$ and $x \leq_{\text{eff}} y$ then we say that z is the evidence of the inequality just when $(x = y = z) \vee (z = x <_L y) \vee (x <_R y = z) \vee (x >_R z >_L y)$. Clearly z is uniquely defined.

§5. The properties of E. In this section we explore the properties of E. What justifies our interest in E? An analysis of Harkleroad's effective embedding of I into C reveals that it factors through E, via the canonical (identity) inclusion of E into C. Later results will show that C is quite noneffective in its properties, and is perhaps more (the indices for) a set of hyperarithmetic surreals than recursive surreals. E might then be a better recursive analog of the surreals. Unlike I, E does have at least one nice algebraic property.

THEOREM 5.1. *There is a partial recursive function $f(x, y)$ such that, if x and y are in E, $f(x, y) \in E$ and $s^C(f(x, y)) = s^C(x) + s^C(y)$. Furthermore, if $x' \leq_{\text{eff}} x$ and $y' \leq_{\text{eff}} y$ then $f(x', y') \leq_{\text{eff}} f(x, y)$.*

PROOF. We define $f(2^q3^r, 2^s3^t) = 2^x3^y$ where

$$W_x = \{\, f(a, 2^s3^t), f(2^q3^r, b), f(a, b) : a \in W_q, b \in W_s \,\},$$
$$W_y = \{\, f(c, 2^s3^t), f(2^q3^r, d), f(c, d) : c \in W_r, d \in W_t \,\},$$

the restriction of Harkleroad's definition of addition on C, plus some extra terms to ensure that it satisfies the third condition to be a member of E. The second property is the easier to verify. Let x'' be the evidence that $x' \leq_{\text{eff}} x$ and y'' be the

evidence that $y' \leq_{\text{eff}} y$. Then it is not difficult to see that $f(x'', y'')$ is evidence that $f(x', y') \leq f(x, y)$. To see that E is closed under f, note that the second claim is applicable to the convergents on each side of $f(x, y)$. ⊣

This is essentially an effectivization of the proof that addition of surreals is well defined. In order for the sum to stay in E, not only must the required inequalities hold, but there must be evidence that they hold. An analogous proof for multiplication appears to be more difficult because the E-index for the sum of an E-indexed surreal and its additive inverse (E-indexed) is in general a very complicated notation for 0. We do not know if the surreals with C-indices properly include those with E-indices. However the surreals with E-indices do properly include those with I-indices. For example it is not difficult to see that the halting real has an E-index. In fact, every arithmetic real has an E-index. First we note that an arbitrary truncate of a surreal with an E-index has an E-index. The same proof works for any class of surreals built by induction. The proof is nonconstructive; we do not get an index for the truncate. Effective truncation of I is immediate, for C it will follow from later results. We do not know if elements of E can be truncated effectively.

LEMMA 5.2. *Suppose x and y are surreals with $x \preceq y$. If y has an E-index, so does x.*

PROOF. Consider the first stage at which some z extending x has an E index. Then no convergent to z extends x, so x is between all left convergents and right convergents to z. Thus $z \preceq x$, so $x = z$. ⊣

DEFINITION 5.3. Let f be a partial function from ω to the real numbers. Let A be an arbitrary set. We define a partial function $u_A^{1,f}$ from ω to the real numbers as follows: $u_A^f(e) = \sup_j f(\varphi_e^A(j))$ if φ_e^A is total and the supremum exists. Similarly, let $l_A^{1,f}(e) = \inf_j f(\varphi_e^A(j))$ if φ_e^A is total and the infimum exists. For $n > 0$, define $u_A^{n+1,f} = u_A^{1,l_A^{n,f}}$, and $l_A^{n+1,f} = l_A^{1,u_A^{n,f}}$. Real numbers in the range of $u_A^{n,f}$ will be called *nth upper approximable* in (f, A). Similarly reals in the range of $l_A^{n,f}$ will be called *nth lower approximable* in (f, A).

The simple cases are when A is recursive and f is some effective numbering of the dyadic rationals (or all of the rationals, or the real algebraic numbers, ...). In that case, a real number is 1st upper approximable and 1st lower approximable if and only if it is computable. There is obviously a duality between upper and lower approximability. From here on we will make statements in only one form, the dual statements being assumed. Also note that we could just as well require that the suprema be taken over increasing sequences in case f is a numbering of the type described above because it follows by induction that the function $\max(a, b)$ is computable. We could also deal only with total functions by taking infima and suprema over r. e. sets instead of the ranges of total functions, but this does not concern us here.

LEMMA 5.4. *Let f be an effective numbering of the dyadics. Let $A \subseteq \omega$. Then for each real number r, r is 1st upper approximable in (f, A') if and only if it is 2nd upper approximable in (f, A). The equivalence is uniform.*

Proof. Let r be 1st upper approximable in (f, A'). Let x_i be a sequence whose supremum is r which is recursive in A'. Let x_i^k be an A-recursive approximation to the sequence x_i, so that $\lim_k x_i^k = x_i$. Let $y_i^j = \inf_{k>j} x_i^k$. It is not hard to see that the y_i^j are uniformly 1st lower approximable in (f, A'). The supremum over all y_i^j is r. So we can find 2nd upper approximable index (in (f, A)) for r. Conversely, let r be 2nd lower approximable in (f, A). Let $r = \sup_i x_i$, where $x_i = \inf_j y_i^j$, with y_i^j uniformly recursive in A. For each dyadic d, A' recursively compute whether some sequence y_i^j never drops below d as j increases. If so, then enumerate d. Clearly r is the supremum of such d. ⊣

Lemma 5.5. *Let f be an effective numbering of the dyadics. Let $A \subseteq \omega$. Then a real number r is $(n+1)$st upper approximable in (f, A) if and only if it is nth upper approximable in (f, A'). This can be made uniform.*

Proof. By induction on n. $n = 1$ is handled by the previous lemma. For the inductive step merely invoke the inductive hypothesis (in dual form) and use the uniformity to convert approximating sequences from one form to another. ⊣

Theorem 5.6. *Let f be an effective numbering of the dyadics. For any set A and real number r, the following are effectively equivalent:*

1. *r is $(n+1)$st upper-approximable in (f, A).*
2. *r is 1st upper-approximable in $(f, A^{(n)})$.*
3. *r is the measure of some class $S \in \Sigma_1^{A^{(n)}}$ (if $r \in [0, 1]$).*
4. *r is the measure of some class $S \in \Sigma_{n+1}^A$ (if $r \in [0, 1]$).*

Proof. First note that it suffices to prove this for $r \in [0, 1]$, since approximability is preserved by the addition of dyadic rationals.

$(1) \Longrightarrow (2)$: By repeated application of the above lemma.

$(2) \Longrightarrow (3)$: Suppose that r is a limit of an $A^{(n)}$ recursive sequence of dyadics d_i, which we assume to be in $[0, 1]$. For each d_i we define a predicate $P_i(C)$ so that the class of all C satisfying this predicate has measure d_i. Furthermore for $d_i \leq d_j$ we ensure that the solutions to P_i are solutions to P_j. Then the class $S = \{ C : (\exists k)[\bigvee_{j<k} P_j(C_k)] \}$ is $\sum_1^{A^{(n)}}$ and has the desired measure. The P_is are defined as follows: Let $d_i = \sum_j 1/2^{a_j+1}$, with $\max_j a_j = a = a_l$. Then for $k > a$, let $P_i(C_k)$ be $(\exists j < l)(\forall b < a_j)[b \notin C_k \wedge a_j \in C_k]$. Clearly P_i has the desired properties.

$(3) \Longrightarrow (4)$: Follows from $\Sigma_1^{A^{(n)}} \subseteq \Sigma_{n+1}^A$.

$(4) \Longrightarrow (1)$: By induction on n. For $n = 1$ let $C \in S$ just if $(\exists k)Q(C_k, A_k, k)$ where Q is primitive recursive. For each j compute the measure r_j of all C which are allowed into S (which are finitary conditions) via $k < j$. Then the measure of S is $\sup_j r_j$. The inductive case is almost identical. Let $C \in S$ just if $(\exists k)Q(C, k)$ where Q is Π_n^A. Then we can effectively compute nth lower approximable indices for each r_j, the measure of those C entering S via $k < j$. Then $r = \sup_j r_j$ is $(n+1)$st upper approximable. ⊣

Note that the equivalence of (3) and (4) is not obvious because $\Sigma_1^{A^{(n)}}$ is properly contained in Σ_{n+1}^A (for classes).

LEMMA 5.7. *Suppose we can uniformly compute E-indices for reals infinitesimally close to r_i, for $i \in \omega$. Then we can effectively find an E-index for a surreal infinitesimally close to* $\sup_i r_i$.

PROOF. Let e be an index for a recursively enumerable set of E-indices for surreals infinitesimally close to each r_i, along with every left convergent to those surreals. Define by recursion $f(z) = 2^e 3^k$, where k is an index for the image under f of the numbers below z. It is not difficult to see if j is an index for the image of f, then $2^e 3^j$ is an index for a surreal infinitesimally close supremum of the e_i. Verification that $2^e 3^k \in E$ is straightforward. ⊣

THEOREM 5.8. *Any real with an arithmetic sign sequence has an index in E.*

PROOF. Let r have a sign sequence recursive in $\emptyset^{(n)}$. Let f be some recursive numbering of the dyadics. Then r is 1st upper approximable in $(f, \emptyset^{(n)})$, thus $(n+1)$st upper approximable in $(f, \emptyset)$. Repeated application of the preceding lemma and its dual imply that we can find an E-index for a surreal s infinitesimally close to r. If r is a dyadic then r certainly has an E-index. Otherwise, the truncation of s after the first ω terms is r and we have shown (nonconstructively) that it has an E-index. ⊣

COROLLARY 5.9. *There are surreals with E-indices which do not have I-indices.*

PROOF. The sign sequence corresponding to the characteristic function of any arithmetic, nonrecursive set has an E-index but no I-index. ⊣

It does not appear to be easy to modify the preceeding proof to get arbitrary arithmetic sign sequences into E, to get arbitrary hyperarithmetic sequences of length $\leq \omega$, or to make the proof effective. However it is clear that E contains reals with non-arithmetic sign sequences; we can construct a real by diagonalizing against all arithmetic reals using the uniformity of passing from notations to arithmetic reals to E-indices for infinitesimally close surreals. We have the following inclusions, both effective: $I \subset E \subseteq C$ (We abuse notation by writing X for the surreals which have codes in X.)

§6. Surreal algebra in A. The theory of the surreal numbers can be developed in set theories weaker than ZFC. In particular, we consider the axioms describing admissible sets (in particular, admissible initial segments of the constructible universe). Such sets model the Kripke-Platek axioms (unless we are working with ur-elements, but since surreals are functions from ordinals to $\{+,-\}$, all the action we care about will be happening in the pure part of an admissible set anyway). Except in the trivial case of the hereditarily finite sets (where the theory of the surreals is also quite trivial), the ordinal ω will exist. In this section we work in a fixed admissible set A, which models the axiom system KP+Infinity. It is not our purpose here to redevelop the theory of the surreal numbers under these weakened conditions. Most of the work is already done for us. For example, once we see that the A-finite surreals are closed under addition, we can immediately conclude that addition is commutative and associative with 0 as identity, etc. The effectiveness of operations on the surreals (in an A-recursion theoretic sense) will be considered. The relevance of this to the recursive surreals is that, as we shall show, the surreals

with indices in C correspond exactly to those which are $L_{\omega_1^{CK}}$-finite. This is the key to answering most of Harkleroad's questions. First we note that the equivalence of the cut and sign sequence approaches to the theory of the surreal numbers can be made effective. As this is the most basic result in the theory of the surreals, we present the argument in full. We assume some simple formalism of the construction of the surreals by cuts: a cut is an ordered pair of sets of previous cuts. The particular formalization is not so important, but this form will be handy since we can employ recursion over $\in$ in the following result.

THEOREM 6.1. *There are A-recursive functions f and g such that if x is an A-finite cut, $f(x)$ is an A-finite sign sequence for the same surreal, and $f(x) = 1$ if x is not a cut. Similarly if x is an A-finite sign sequence, $g(x)$ is an A-finite cut for the same surreal, and $g(x) = 1$ if x is not a sign sequence.*

PROOF. Essentially this consists of noticing that the constructions given in [4] are of a fairly simple nature. g is defined as follows. First, we check that x is actually a sign sequence. This is easily seen to be Δ_1 definable ("x is a function, the domain of x is an ordinal, and the range of x is $\{+,-\}$") and thus we are justified in making a definition by cases and defining $g(x) = 1$ for x not satisfying this. We define a function $g'(x,\alpha)$ for sign sequences $x \in A$ and ordinals $\alpha \in A$. Temporarily, if x is a sign sequence, x_β is x restricted to β (that is, the first β signs of x). g' passes from the sign sequence x to a cut form for x_α. g' is defined by transfinite recursion. $g'(x,\alpha) = \{g_1(x,\alpha) | \, g_2(x,\alpha)\}$ where

$$g_1(x,\alpha) = \{\, g(y) : (\exists \beta < \alpha)[y = x_\beta \wedge x(\beta) = +]\,\}$$

and

$$g_2(x,\alpha) = \{\, g(y) : (\exists \beta < \alpha)[y = x_\beta \wedge x(\beta) = -]\,\}.$$

g_1 and g_2 involve the taking of a set (those x_β with $x(\beta)$ having a specified value) under an A-recurisve function (which is simultaneously being recursively defined) and are thus sets in A, by Σ_1-collection. The functions g_1, g_2, and g' are easily seen to be Σ_1 definable in g. Also note that the length of a sign sequence is A-recursively computable, as follows: length$(x) = \bigcup_{y \in x} \pi_1(y)$, where π_1 denotes projection onto the first coordinate. Thus $g(x) = g'(x,\alpha)$ is A-recursive by Σ-recursion. g computes the "canonical cut representation" of a sign sequence, as described in [4]. f is a bit more complicated. As mentioned above, we assume some appropriate formalism of the cut forms of surreals. We will define f by recursion over $\in$, which requires that the notions of left convergent and right convergent be well-founded relations which collapse to sets. As before, we define the function $f'(x,\alpha)$, this time intended to represent $f(x)_\alpha$. f' will be defined by recursion on α and will assume that previous values of f are given. Note that for sign sequences x, y and ordinals β, the relation $x_\beta = y_\beta$ is Δ_1-definable. For $\alpha = 0$, let $f(x,\alpha) = \emptyset$. Given x, $\alpha + 1$, let a be the image of the set of all right convergents y to x such that $f(x,\alpha) = y_\alpha$ under f. The formation of this set is justified by Δ_1-seperation. Similarly let b be the image of the set of all such left convergents. Now let $f'(x,\alpha+1) = f'(x,\alpha)^+$ if $x_\alpha \in a$ or $(\exists a' \in a)[a'(\alpha) = +]$. Similarly let $f'(x,\alpha+1) = f'(x,\alpha)^-$ if $f'(x,\alpha) \in b$ or $(\exists b' \in b)[b'(\alpha) = -]$. If both these conditions hold (again a Δ_1 condition) let $f'(x,\alpha+1) = 1$. If neither hold, let $f'(x,\alpha+1) = f'(x,\alpha)$. For limit ordinals α,

let $f'(x, \alpha) = 1$ if $(\exists \beta < \alpha)[f'(x, \beta) = 1]$ and $\bigcup_{\beta<\alpha} f'(x, \beta)$ otherwise. Now note that for a given (proposed) cut x, the rank of x is a bound on the length of the corresponding sign sequence. So define $f(x) = f'(x, \text{rank}(x))$. f is seen to have a (very complicated) Σ_1-recursive definition, and to have the desired properties. ⊣

Let us remark that some other definitions, usually equivalent, fail in this context. For example, thinking of A-finite surreals as corresponding to "games in A" is not accurate. Even if these "games in A" are actual games, it is not clear that the positions admit an A-recursive ordinal rank without additional conditions on A.

Corollary 6.2. *The set of A-finite cuts is A-recursive. The ordering and equality relations on A-finite cuts are A-recursive.*

Proof. x is a surreal if and only if $f(x) \neq 1$. $x < y$ if and only if $f(\{x|y\}) \neq 1$. $x = y$ (as surreals) if and only if $f(x) = f(y)$. ⊣

If f is any function (possibly many-place) on surreals which is given by a simple recursive definition, then it is not hard to show that the surreals in A are closed under f, and f can be carried out A-recursively. For example, addition of surreals is given by $f(x, y) = x+y = \{x^L+y, x+y^L | x^R+y, x+y^R\}$. The cut defining $x+y$ is Δ_0 definable in the cuts defining x, y, and the previous values of f. Hence f is A-recursive. This breaks down in the case of division, square roots, and exponentiation for the hereditarily finite sets, since the operations are no longer finitary and involve taking the image of ω under some recursively defined function (to get a sequence). However for any other admissible set, ω is A-finite and thus its image remains A-finite. In particular, ω-exponentiation is A-recursive and we could easily carry out the classical argument to show that every A-finite surreal has an A-finite power series in ω, and passing to this representation is A-recursive. All essential features of the surreals remain unchanged by truncation at an arbitary admissible set. From this representation theorem we could conclude that the A-finite surreals form a real closed field. The proof of this would be slightly tedious and no different from the argument given in [3] or [4], so we omit it here. There is a bit of effectiveness missing in the previous statement. The A-finite surreals do form a real closed field, but this says nothing about the difficulty of finding the roots of a given polynomial of odd degree (although this would come out A-recursive if we were to go into the details of the proof). However the A-finite surreals are A-recursive, hence A-recursively enumerable. Evaluation of the polynomial is A-recursive because addition and multiplication are A-recursive. Thus we need only to search through all possible roots until one is found, and identified (since equality to zero is A-recursive). Real closure guarantees that the search will halt. This argument is general, given an (index for an) A-recursive function that is guaranteed to have a root, we can A-recursively find that root (uniformly in an index for the function, or, in general, A-recursively enumerate all roots). Thus we have:

Corollary 6.3. *For every odd $n \in \omega$ there exists an A-recursive function $f_n(c_0, c_1, c_2, \ldots, c_n)$ such that if each c_i is the surreal $s_i \in A$, then $f_n(c_0, c_1, c_2, \ldots, c_n)$ codes a surreal $x \in A$ such that $c_0 x^n + c_1 x^{n-1} + \cdots + c_0 = 0$. Furthermore, the functions f_n are uniformly A-recursive in n.*

The surreal numbers form a proper class in some set theories, but strictly speaking we cannot talk about the surreals in ZFC without some sort of metalinguistic contortions. Usually it is easier to ignore these difficulties, or to restrict ourselves to the study of a sufficiently large set of surreals. In [4] Gonshor demonstrates that for most purposes, at least all algebraic purposes, the surreals of length below some cardinal are sufficient. The surreal numbers form a constructible class; in fact $L_\alpha \cap \mathbf{No}$ consists of precisely the surreals that can be constructed in L_α. This effectively truncates the construction of the surreals at an arbitary admissible stage, and thus constitutes in some sense a finer analysis than truncating at cardinal stages. It includes as a special case ($A = H(\kappa)$) truncation at arbitrary cardinal stages. It also includes as a special case the construction of the "recursive" surreals, as we shall soon see. A good number of results in [1] do not concern the class of all surreals, but those of length bounded by some cardinal. As mentioned above, this is a special case of considering the A-finite surreals for some admissible set A. Certain concepts that Alling introduces can be considered the relativization of classical concepts to the admissible set in question. It is not surprising, then, that these results can be obtained in a more general (and more motivated) way.

DEFINITION 6.4. A set of A-finite surreals is *A-open* if it can be written as an A-finite union of principal open intervals (intervals of the form (a, b)). (Note that each such set is bounded, since the collection of A-finite surreals cannot be covered by an A-finite set of intervals. One might wish to generalize this by allowing some of these intervals to be unbounded, but this point is not essential.) If K is some set of A-finite surreals, then a subset of K will be called *A-open relative to K* if it is the intersection of K with an A-open set. We call a set K of A-finite surreals *A-compact* if any A-finite cover of K by A-open sets has a finite subcover. We call a set C of A-finite surreals *A-connected* if for any pair U_0, U_1 disjoint subsets of C, A-open with respect to C, which cover C, one U_i is empty.

It is clear that an A-finite union of A-open sets is A-open. If we imagine that A is in some sense a universe of "all sets", then these are just ordinary topological notions. In general, as the note above indicates, the A-open sets do not even generate a topology. Other differences abound. For example, an A-compact set need not be A-closed (see [1] for an example). However, these concepts become very interesting when applied to the surreals. Note that the collection of all A-finite surreals is A-compact and A-connected by our definition, for trivial reasons (this would be true even if A-open sets were allowed to be unbounded, in the manner indicated above, but those facts require the argument below). Things are less obvious for bounded sets, but the obvious questions are settled below. The proof is taken, in essence, from [1], but some modifications are necessary because an arbitrary subset of an A-finite set need not be A-finite.

THEOREM 6.5. *The closed interval* $[0, 1]$ *of A-finite surreals is A-connected and A-compact.*

PROOF. First we prove connectedness. Let U and V be a disjoint open cover of $[0, 1]$. Suppose both are nonempty. Then, say $x \in U$ and $y \in V$, and we assume without loss of generality that $x < y$. Then U and V are a disjoint open cover of $[x, y]$, neither member of which is empty. In this case we replace $[x, y]$ with

$[0,1]$. So without loss of generality, assume $0 \in U$, $1 \in V$. Write U as $\bigcup_i U_i$, where the union is A-finite and the U_i are principal open intervals. Let U' be the set of all x such that $[0,x]$ is contained in a finite union of U_i. Clearly one can A-recursively determine whether or not $x \in U'$, by an A-finite search through all finite unions of indices i. Note that if $U_i \cap U' \neq \emptyset$, then $U_i \subseteq U'$, by adding U_i to the finite union that guarantees any member of itself is a member of U'. Then one can A-recursively determine if $U_i \subseteq U'$ by testing the membership of any member of U_i. Since U' is a union of the U_i, it is an A-finite union of the U_i, since there are A-finitely many U_i and $\{\, i : U_i \subseteq U' \,\}$ is A-recursive. Thus U' is A-open, and it is clearly closed downward. Now let $V' = V \cup \bigcup_{U_i \cap U' = \emptyset} U_i$. V' is a union of an A-open set and an A-finite union of intervals, and is thus A-open. Clearly $U' \cup V' = U \cup V = [0,1]$ and $U' \cap V' = \emptyset$. Write $U' = \bigcup_i (a_i, b_i) \cap [0,1]$, $V' = \bigcup_j (c_j, d_j) \cap [0,1]$. If some $b_i > c_j$, then $U' \cap V' \neq \emptyset$. If $b_i < c_j$ for all i, j, then, since the collection of b_is and c_js are both A-finite, there exists (by the cut construction) a surreal s satisfying $b_i < s < c_j$ for all i, j. Then $s \in [0,1]$ but s is not covered by $U' \cup V'$. Otherwise $b_i = c_j = x$ for some i, j. Then if any $c_k < x$, or any $b_k > x$, we have $U' \cap V' \neq \emptyset$. On the other hand, if every $c_k \geq x$, and every $b_k \leq x$, then x is in neither U' nor V', again, contrary to assumption. Connectedness is proven. Now, we prove compactness. Let U_i be an A-finite cover of $[0,1]$ by A-open sets. Since each such set is an A-finite union of principal intervals, we assume without loss of generality that each U_i is an interval. We introduce an equivalence relation $\sim$ on $[0,1]$: $x \sim y$ if there exist finitely many intervals U_i such that their union contains $[x,y]$. This is clearly an equivalence relation, each equivalence class of which is A-open (relative to $[0,1]$). To see this, note that the checking relation involves only searching through all finite subsets of an A-finite set, and thus is A-recursive. In addition, if $x \sim y$ and y, $z \in U_i$, then $x \sim z$ (this is seen by adding U_i to the collection covering $[x,y]$), so every equivalence class is an A-recursive union of U_i, of which there are A-finitely many, which is therefore an A-finite union. We claim that $[0,1]$ is an entire equivalence class (and this implies the desired result). Let U be a nonempty equivalence class. The union V of all other equivalence classes is an A-finite union of A-open sets and is thus A-open. Then U and V are disjoint A-open subsets of $[0,1]$ which cover $[0,1]$, so by connectedness one of them is empty. Since U is nonempty, it follows that $U = [0,1]$. ⊣

We will provide an interpretation of this result later. Let us just remark that the above theorem is a generalization of Alling's results for ξ-topologies on restricted classes of surreals. We conclude this section with an alternative way of looking at the cut construction of the surreals.

DEFINITION 6.6. Let S be an (A-finite) order relation. Let s be an (A-recursive) function which assigns to each (A-finite) convex subset K of S an element $k \in K$. We call s an *(A-)selector* if it satisfies the *consistency condition*: $K \subseteq K'$ and $s(K') \in K$ then $s(K) = s(K')$. We call an A-selector *perfect* if it admits an extension to a selector.

An example is an A-finite subset of the A-finite surreals which is closed downward under the formation of initial segments, in this case we can take $s(K)$ to be the

simplest surreal in K. For many admissible sets A, this situation is general, as we make precise in the following theorem. The first lemma makes no mention of admissible sets.

LEMMA 6.7. *Let S be an order-relation, let s be a selector. There exists a set H of surreals which is closed downward under the formation of initial segments and an bijection f from H to S which preserves order, such that $s(K) = f(x)$ where x is the simplest member of $f^{-1}(K)$. H and f are, in fact, uniquely determined.*

PROOF. We define f by induction. Let a be a surreal. Assume for all initial segments a' of a, $f(a')$ has been defined. Let K be the set of all x such that for each initial segment $a' < a$, $f(a') < x$, and for each $a' > a$, $f(a') > x$. Then let $f(a) = s(K)$, if K is nonempty. Otherwise, let $f(a)$ be undefined. f is clearly order preserving, and thus injective. We claim that if $a = \{a^L|a^R\}$ is any cut representation of a, then $f(a) = s(\{x : f(a^L) < x < f(a^R)\})$. The proof of this is straightforward: this is clearly true, by construction, for the canonical cut representation of a by initial segments in itself. Because any cut representation of a is cofinal in the canonical one ([4]), we see that the set on the right hand side is contained in the K used to define $f(a)$, and thus the claim follows the consistency condition. We now claim that f is surjective. Suppose it were not surjective. Then there exists some $x \in S$ not in the range of f. Let $L = \{y < x : y \in \text{range}(f)\}$ and $R = \{y > x : y \in \text{range}(f)\}$. Let $a = \{f^{-1}L|f^{-1}R\}$. Then $f(a)$ is defined, since x lies in the appropriate interval. Either $f(a) = x$, contradicting the nature of x, or $f(a)$ is an initial segment of x, and thus lies in either L or R, contradicting the nature of these sets. Thus f is surjective. Since the domain of f is the preimage of a set, it is itself a set, H. Now, if some other f' and H' satisfied the conditions, then $g = f^{-1}f'$ is a bijection from H' to H preserving order and initial segments. We claim this map is the identity on $H = H'$. Suppose $x \in H'$ with $g(x) \neq x$. We assume without loss of generality that no initial segment of x has this property. Then, since $g(x)$ must lie between all left and right convergents to x, we see $x \preceq g(x)$. The same argument shows $g(x) \preceq x$. Thus $x = g(x)$, so g is the identity, and uniqueness is established. ⊣

LEMMA 6.8. *Let S be an A-finite linear order, let s be an A-selector for S. For the moment, denote by $[x, y]$ the nonempty closed interval between x and y, regardless of the relative order of x and y. Define $x \preceq y$ if $x = s([x, y])$. Then $\preceq$ is an A-recursive partial order on S, with no A-finite descending chains. For any A-finite convex K, $s(K)$ is a $\preceq$-least member of K. s is perfect if and only if $\preceq$ is wellfounded. If s is perfect, then its extension to a selector is unique.*

PROOF. This is straightforward, by repeated application of the consistency condition. Clearly $x = s([x, x])$, so $x \preceq x$. Assume $x \preceq y$ and $y \preceq z$. We check transitivity. Assume $[x, z] = [x, y] \cup [y, z]$. By the consistency condition,

$$s([x, z]) = \begin{cases} s([x, y]) = x & \text{if } s([x, z]) \in [x, y] \\ s([y, z]) = y & \text{if } s([x, z]) \in [y, z]. \end{cases}$$

In the latter case, $y \in [x, z]$, so $[x, y] \subseteq [x, z]$, and since $s([x, z]) = y \in [x, y]$, $y = s([x, y]) = x$. It follows that $s([x, z]) = x$. Otherwise $[x, z]$ is contained in

$[x, y]$ or $[y, z]$. In the first case, since $s([x, y]) \in [x, z]$, we again have $s([x, z]) = x$. In the second case, since $[x, y] \subseteq [y, z]$ as well, and $s([y, z]) = y \in [x, y]$, we have $y = s([x, y]) = x$, so $s([y, z]) = y = x \in [x, z]$, so $s([x, z]) = s([y, z]) = y = x$. In any event, we have $x \preceq z$. Now, if $x \preceq y$ and $y \preceq x$, we have $x = s([x, y]) = y$. It follows that $\preceq$ is a partial order. A-recursiveness is evident from the definition. Now suppose x_i is an infinite $\preceq$-decreasing sequence. Let K_i be the smallest convex set containing $\{x_j\}_{j \in i}$. Let $K = \bigcup_i K_i$. Clearly K is A-finite if x_i is an A-recursive sequence. So let $x = s(K)$. Then x lies in some $[x_i, x_j]$. Without loss of generality $i < j$. Since s is a selector, it follows that $x = x_i$. Then it follows that $x = s([x, x_k])$ for all x_k, so $x_i \preceq x_k$, a contradiction if $k > i$. Now, if K is any A-finite convex set, for any $y \in K$ we have that $[s(K), y]$ is a subset of K containing $s(K)$, whence $s([s(K), y]) = s(K)$ by the consistency condition, from which we obtain $s(K) \preceq y$. If s is perfect, then by the preceeding lemma, we can assume S is a set of surreals closed downwards under initial segments. In that case, $\preceq$ is just the ordinary initial segment ordering, which is clearly wellfounded. Conversely, assume $\preceq$ is wellfounded. We claim any convex set K contains at most one $\preceq$-minimal member. For if it contained two, say, x and y, then $[x, y] \in K$, whence $[x, y]$ is an A-finite subset of K, and $s[x, y] \in K$ is a lower bound for both. So if we define $s'(K)$ to be the unique $\preceq$-minimal member of K for an arbitrary convex set K, we have the desired extension s' of s. Since any closed interval $[x, y]$ is A-finite, it follows that the $\preceq'$ defined by s' coincides with $\preceq$ for any s' extending s, and thus s' is uniquely determined. $\dashv$

The next lemma is of more general interest.

LEMMA 6.9. *Let a be a surreal whose length is bounded in A. Suppose a can be written as $\{L|R\}$ where L and R are A-recursively enumerable collections of A-finite surreals. Then a is A-finite.*

PROOF. We think of a as a $\{+, -\}$ valued function on an ordinal α. The set of all pairs $\{s, \beta\}$, where $s \in \{+, -\}$ and $\beta \in \alpha$, is clearly A-finite. Thus we need only show that the sign sequence of a is A-recursive, from which it follows that a is an A-recursive subset of an A-finite set, hence A-finite. Let $\beta \in \alpha$. We A-recursively determine $a(\beta)$ as follows. Note that for any surreal $x = \{L|R\}$, x is the least upper bound (relative to $\preceq$) of surreals of the form $\{l|r\}$, over all choices of finite (even 0 or 1 element) subsets $l \subseteq L$, $r \subseteq R$. Then, since the set of I such surreals is clearly A-recursively enumerable, we see that in order to compute $a(\beta)$, we need only search through I until we find $i \in I$ with $i(\beta)$ defined, and then $a(\beta) = i(\beta)$. $\dashv$

Note that the proof given above is ineffective, in that one cannot determine α from indices for L and R. This is actually quite important. The above constuction resembles the "first step" in the construction of C, if we take A to be the collection of hereditarily finite sets. In that case it is clear that little can go wrong, since the A-finite surreals are exactly the surreals with finite sign sequences. For general A this is less clear, but the above lemma indicates that for special kinds of cuts, shortness implies simplicity.

THEOREM 6.10. *Let A be an admissible set such that every A-finite partial order which contains no A-finite (arbitrary) descending chains admits an A-recursive rank.*

Let S be an A-finite linear order and s a (perfect) A-selector for S. Then there exists a unique A-finite collection of surreals and an A-recursive bijection as in the first lemma above.

PROOF. It follows from the preceding lemma that the relation $\preceq$ defined from a (perfect) A-selector has no A-finite (arbitrary) descending chains. By the condition above then, we see that S under $\preceq$ admits an A-recursive ordinal rank. Thus $\preceq$ is wellfounded, and s is perfect. Let f be the function defined in the first lemma. It is not difficult to see that f is A-recursive, when restricted to A-finite surreals. The only unclear point is the surjectivity of this restriction. Suppose the restriction is not surjective. Then there exists some x in the H of the first lemma which is not A-finite. Choose such an x minimal with respect to $\preceq$, and let y be its image. Then let $L = \{ z \in \text{range}(f) : z < y \}$ and $R = \{ z \in \text{range}(f) : z > y \}$. Then since f is partial A-recursive, L and R are A-recursively enumerable. Hence their preimages L' and R' under f are A-recursively enumerable. But by definition $x = \{L' | R'\}$. Since x has length equal to the rank of y under $\preceq$, which is an ordinal in A, the third lemma applies to show that x is A-finite, contrary to assumption. ⊣

Note that the condition above is satisfied by the admissible collections of sets of hereditarily bounded cardinality. In the case we are most interested in, $L_{\omega_1^{CK}}$, the perfection condition on s is necessary. For any nonstandard recursive ordinal (that is, a recursive linear order with no hyperarithmetic descending chains which is not a wellorder), if we define s to be the smallest member of any hyperarithmetic convex set, then s is an imperfect selector, and clearly the ordering does not correspond to any recursive surreals. Imperfect selectors in linear orders correspond, in fact, to "nonstandard recursive surreals". These surreals correspond to recursive games which do not necessarily end, but have no infinite play by hyperarithmetic strategies. Perhaps a theory of such surreals could be developed, relying only on apparent wellfoundedness and not on an actual existence of ranks. This seems possible: addition and multiplication admit essentially non-inductive definitions directly in terms of games. And numbers are simply games whose set of positions admits a linear order and a selector with appropriate properties, which could be relativized to other admissible sets. It is rather amazing that the one simple axiom for selectors has such startling implications. Let us remark that the above proof, applied to selectors for arbitrary sets instead of convex sets in some linear order, and without the consistency condition, parallels to a large degree the proof of the well-ordering theorem. Of course, the consistency condition is necessary to demonstrate the uniqueness of f.

§7. The metarecursive case. In this section we consider the case of the admissible set $L_{\omega_1^{CK}}$. Surprisingly enough, this set of surreals corresponds to those with notations in C; this correspondence can be made effective. C thus inherits a rich algebraic structure. As this section and later results will show, C seems to best capture the notion of a "recursive surreal number". The inclusion of the surreals with C-indices in $L_{\omega_1^{CK}}$ is clear, by the results of the last section, $L_{\omega_1^{CK}}$ contains precisely those surreals with hereditarily hyperarithmetic sign sequences, and hence hereditarily recursively enumerable sign sequences are included as a special case.

However we have two other objectives here: To make this correspondence uniform and to get a bound on the complexity involved in passing from C to $L_{\omega_1^{CK}}$. First we need some sort of indices for these surreals. Since these are exactly the surreals with hyperarithmetic sign sequences (because hyperarithmetic $\iff$ Δ^1_1, [6]), we use indices similar to I-indices. Since these new indices will turn out to be effectively equivalent to C-indices, we will call the index set $C^\dagger$.

DEFINITION 7.1. $C^\dagger = \{(n,a) : (\exists b <_O a)[\mathrm{dom}(\varphi_n^{H_a}) = \{c : c <_O b\}]\}$. $s^{C^\dagger}_{(n,a)}(\beta) = \varphi_n^{H_a}(b)$ where $|b| = \beta$, if such b exists. This is defined on an initial segment of the ordinals (below ω_1^{CK}) and we define a surreal $s^{C^\dagger}(x)$ to be the surreal with βth sign $+$ if $s_x^{C^\dagger}(\beta)$ converges to 0, and $-$ if it converges to anything else.

This is analogous to the definition of I except here we are allowing use of the power of the hyperarithmetic hierarchy.

LEMMA 7.2. *There are partial recursive functions $f(x)$ and $g(x)$, such that if $x \in C$ then $(f(x), g(x)) \in C^\dagger$, and they code the same surreal which is of length $< |g(x)|$. Furthermore, if x enters C before stage β, then $|g(x)| < 2\omega^\beta$.*

PROOF. We define these functions by effective transfinite recursion. Let x enter C at stage $\gamma < \beta$. Then by induction we can effectively find indices for sign sequences for each convergent recursive in H_ks for some ks coding ordinals $\leq 2\omega^\gamma$. Form the uniform sum of these notations, call it b. Then $|b| \leq \omega^\beta$, and the convergents have sign sequences uniformly recursive in H_b. Each of these sign sequences is defined only on an initial segment of O of length at most $2\omega^\gamma$, hence with an oracle to H_b we can uniformly convert these into sign sequences below b in O. Now we will compute the sign sequence of x below b. The terms below k in the sign sequence, where k codes the ordinal $a\omega + c$ for $c \in \omega$, will require an oracle to $H_{b+_O k+_O d}$, where d is the code for c. This is done recursively as follows. Suppose we have already computed the sign sequence below k. With an oracle to $H_{2^{2^{b+_O k+_O d}}} = H_{b+_O(k+_O 1)+_O(2^d)}$ we can determine which convergents have sign sequences which agree with x up to k. These sign sequences are all recursive in H_b, thus with an H_{2^b} oracle we can determine which are defined at k, whether any $+$'s appear among the left convergents or any $-$'s appear among the right convergents, and thus define the appropriate sign. Hence the kth term is found uniformly recursively in $H_{b+_O 2^k+_O 2^d}$, as desired. Since k can be restricted to b which is necessarily a limit ordinal, the whole process is recursive in $H_{b+_O b}$. Since $|b| \leq \omega^\beta$, we have the desired result. Let $g(x) = b$ and $f(x)$ be an index for the algorithm described above. ⊣

The other direction is a bit more surprising. Most of the surprise is contained in the following preliminary result:

LEMMA 7.3. *There is a partial recursive function $f(n,a)$ such that if $a \in O$ and $|a| < \beta$ then $f(n,a) \in C$, $s^C(f(n,a)) = 0$ if $n \in H_a$, $s^C(f(n,a)) = 1$ if $n \notin H_a$, and $f(n,a)$ enters C before ε_β.*

PROOF. The definition of f is by effective transfinite recursion on a. By definition, $H_1 = \emptyset$, so let b be a notation for 1 in C. Then define $f(n,1) = b$.

The limit case is equally easy. Let $n = (c, d)$. Then $n \in H_{3\cdot 5^e}$ if and only if $c \in H_{\varphi_e(d)}$. So define $f((c,d), 3\cdot 5^e) = f(c, \varphi_e(d))$. The successor case is the most complicated. Let $a = 2^k$, $|a| = \gamma$. Then $H_a = H_k'$, which can be put into $\Sigma_1^{H_k}$ form. Write this as $n \in H_a$ if and only if $(\exists m)[Q(S_m, m, n)]$ where S_m is a strong finite index for the restriction of H_k to m, and Q is some primitive recursive predicate. Q can be put into Diophantine form: $Q(x, y, z)$ if and only if $(\exists w_1, w_2, \dots, w_n)[P(x, y, z, w_1, \dots, w_n) = 0]$ where P is a polynomial with integer coefficients. n is known, and given m we can calculate a surreal equal to S_m entering C before $\varepsilon_{\gamma+1}$ using $f(x,k)$ already defined by induction (this uses the fact that at ε-number stages, C is closed under the ring operations). Then for all m, w_1, w_2, etc., we can calculate an index for the value of the polynomial P as a surreal, call this p. Find an index for $1 - s^C(p)^2$, and collect all the indices produced into a recursively enumerable set W_i. Let j be such that $W_j = \emptyset$. Then $2^i 3^j \in C$, and it codes a surreal which is 0 if the polynomial P has no roots (in which case each left convergent is below zero, and there are no right convergents), and 1 if P has a root (in which case some left convergents are zero, the rest negative, and there are still no right convergents). Let $f(n,a)$ be an index for $1 - s^C(2^i 3^j)$. Then $f(n,a)$ enters C at stage $\varepsilon_{\gamma+1} < \varepsilon_\beta$ at the latest. ⊣

Lemma 7.4. *There is a recursive function g such that for all $(n,a) \in C^\dagger$ coding x, $b <_O a$, $g(n,a,b) \in C$ and codes the initial segment of that surreal below $|b|$. In particular $g(n,a,a)$ codes x. Furthermore if $|a| < \beta$ then $g(n,a,b)$ enters C no later than stage ε_β.*

Proof. We use the function f defined in the previous lemma. We will let $g(n,a,b) = 2^x 3^y$ where W_x and W_y are recursively enumerable sets enumerated as follows. Enumerate all $c <_O b$. For each c we can compute the convergent $d = g(n,a,c)$. The only question is whether it is a left convergent or a right convergent, and to determine this, we need to compute $\varphi_n^{H_a}(c)$. We obviously cannot do this, but using the function f defined above we can effectively produce a C index for surreals s and s' so that $s = 1$ if the sign corresponding to c exists and is $+$, and zero otherwise, $s' = 1$ if the sign is a $-$, and zero otherwise.. Now if $s = 1$ we want to put d into W_x and if $s' = 1$ we want to put d into W_y. This we cannot do, but we can do something almost as good. Find a C index a' for the ordinal coded by a, which is clearly larger than the surreal we are trying to produce. Then it will do no harm to throw the additive inverse of a' into W_x, or a' itself into W_y. So find an index for $s^C(d)s - s^C(a)(1-s)$ and put it into W_x. Similarly put an index for $s^C(d)s' + a'(1-s)$ into W_y. Then $2^x 3^y$ codes the desired surreal. Furthermore, since all of the surreals involved entered C before stage ε_β and C below this stage is closed under the ring operations, $2^x 3^y$ will enter C no later than ε_β. ⊣

Combining these lemmata, we obtain

Theorem 7.5. *It is possible to pass effectively between C and $C^\dagger$ indices for surreals. Furthermore, if β is an ordinal such that $\beta = \varepsilon_\beta$, then the surreals in $C^\dagger$ defined recursive in H_b below b for $|b| < \beta$ are mapped to surreals entering C before β and vice versa.*

Two very natural concepts of simplicity for surreal numbers seem to have very little to do with each other. In Conway's hierarchy, a surreal x is simpler than a surreal y if it has a shorter sign sequence. Recursion-theoretically, we might say that x is simpler than y if the sign sequence for x could be reduced to the sign sequence of y in some effective manner. However, there are short surreals (of length ω) of arbitarily large recursion-theoretic complexity in the hyperarithmetic hierarchy (the characteristic functions of H-sets). There are also very long sequences which are fairly simple; every recursive ordinal has a recursive sign sequence, yet these grow arbitrarily long. A recursive surreal's "birthday" in C provides a useful medium; those born earlier are required to be both short and simple. Thus a C-birthday fuses together these two very different notions of simplicity. It it closely related to the order of constructibility of a surreal (its rank in the constructible hierarchy). We are now in a position to answer some questions left open in [5]. Harkleroad asked if there were a nice characterization of the collection of sign sequences of surreals with codes in C. As we have shown, these are precisely the hyperarithmetic sign sequences. Harkleroad also asked if the stage at which a surreal entered C must be its length, or "birthday" in the ordinary construction. This is clearly false. Choose some b large in O, let s be the surreal whose sign sequence has length ω and codes the characteristic function of H_b. Then s has a C-index, and if b is larger than some ε-number, x must enter C after that ε-number. Thus the real numbers, which enter **No** quite soon in the ordinary construction, enter C arbitrarily late in its construction. In the trivial case of the last section, $\alpha = \omega$, L_α does not satisfy the axiom of infinity and thus the results gathered are no longer valid. In fact all surreal numbers of finite length are constructed (since every finite set is recursive), these correspond to the dyadic rationals. In recursion theory, next step beyond ω is ω_1^{CK} (metarecursion theory). Henceforth we shall refer to surreals with indices in C as metadyadics.

§8. Trees and fixed points. The sign sequence approach to the theory of surreal numbers identifies a surreal with a function from an ordinal to the set $\{+,-\}$ of signs. The class of all surreal numbers can thus be thought of as a large transfinite tree. Thinking in this way leads to a number of interesting fixed point results in the theory of the surreals. The objective of this section is to formulate the classical and effective versions of these results. We first provide a general framework within which fixed point results may be obtained. As before, A is an arbitrary admissible set satisfying the axiom of infinity. Let α be the least ordinal not in A.

DEFINITION 8.1. Let S be an A-finite set. Let T be a nonempty, A-recursive set of A-finite functions from ordinals β to S.

(1) T is a *tree* if for any A-finite family $\{f_i\}$ of functions in T which are compatible on α, their amalgamation $\bigcup_i f_i$ is in T, and T has a least element.

(2) The elements of a tree T are called *words*.

(3) The *length* of a word is the ordinal β that it is defined on.

(4) Any tree T can be given the initial segment ordering $\preceq$. This ordering induces a topology on T, with f a limit point of $S \subseteq T$ if f can be written as a union $\bigcup_i f_i$ of compatible f_is in S.

(5) A function from T to T which is monotonic with respect to the initial segment order and continuous with respect to the induced topology is called a *word function*.

(6) If $w \in T$, $x \in S$, then w^x is the function obtained by defining $w^x(\gamma) = w(\gamma)$ for $\gamma < \beta$ and $w^x(\beta) = x$, where β is the length of w.

(7) If T has the property that for any $x \in S$, $w \in T$, there is a least $y \in T$ such that $w^x \preceq y$, then T is said to be *good*.

(8) If a word function f has the property that $f(x)^x \preceq f(x^x)$ then f is said to be *good*.

A tree can thus be identified with a subtree of the A-finite words in alphabet S. A good tree will be called *full* if it can be identified with the image of this tree under a good word function.

THEOREM 8.2. *If f is a continuous word function on T, then f has a least fixed point. The fixed points of f are a tree. If f is good then this tree is good.*

PROOF. The A-recursiveness of f easily demonstrates the A-recursiveness of the fixed-point set. Let 0 be the least element of T. Clearly $0 \preceq f(0)$. By monotonicity $0 \preceq f(0) \preceq f(f(0)) \preceq \cdots$. Let x be the limit of this (clearly A-recursive) sequence. Then x is a least fixed point. Hence the fixed points are nonempty and have a least element. Furthermore if $\{w_i\}$ is any ascending sequence of fixed points, then $f(\lim_i w_i) = \lim_i f(w_i) = \lim_i w_i$, so the limit is a fixed point. Thus the fixed points form a tree. Let f be good. Let w be a fixed point of f, and x be any element of S. Then let $w_x = \lim_i f^i(w^x)$. w_x is the least fixed point extending w^x, and thus T is good. ⊣

If T is a full tree, then it contains a tree isomorphic to the tree of all A-finite sequences in S; we identify them via the isomorphism. Let T' be a good subtree of T such that for any words w and w' in T', their common initial segment is in T'. Then define g as follows: $g(0)$ is the least element of T', where 0 is the empty function; $g(w^x) = w'$, where w' is the least word in T' extending $g(w)^x$; and $g(\lim_i w_i) = \lim_i g(w_i)$. Then, by construction, g is a good word function that carries the T into T'. We claim that g is onto T'. First note that the inverse of g is A-recursive. For to compute $g^{-1}(w)(\beta)$ for $w \in T'$, we need only determine the next symbol in w after $g(w')$ where w' is the initial segment of $g^{-1}(w)$ before β. It follows that the image of g is closed in T. Suppose $w \in T'$ is a word minimal among those that are not in the image of g. Let w' be the longest initial segment of $w = g(v)$ that is in the image of g. w' must exist since the image of g is closed in T'. By assumption $w' \prec w$. Let $w'^x \preceq w$. Then $g(v^x)$ is the least element of T' starting with w'^x, hence it is an initial segment of w. $g(v^x)$ is contained in the image of g, is below w and above w', contradicting the choice of w'. Conversely, any full tree must be closed under the formation of common initial segments, since the common initial segment of $f(x)$ and $f(y)$ is $f(z)$ where z is the common initial segment of x and y for any good word function f. Hence we have proven:

THEOREM 8.3. *A good tree is full if and only if it is closed under the formation of common initial segments.*

In the case of the surreal numbers there is a very nice process which generates word functions. Suppose we are given an A-recursive equivalence relation $\sim$ on the A-finite surreals, such that $(a \sim c) \wedge (a < b < c)$ implies $a \sim b \sim c$. Such an equivalence relation will be called *convex*, as the equivalence classes it defines are convex. We will call an equivalence relation *doubly convex* if it is convex and

$(a \sim c) \wedge (a \preceq b \preceq c)$ implies $a \sim b \sim c$. Now every convex equivalence class has a simplest member, since if $x \sim y$, then their common initial segment z is between them, hence $z \sim x \sim y$. We will call the elements which are the simplest members of their equivalence classes *simple*. Furthermore, suppose that all the equivalence classes are an A-finite union of principal open intervals (that is, they are A-open). Then we have the following:

THEOREM 8.4. *The set S of all simple surreals (relative to some doubly convex A-recursive equivalence relation $\sim$ for which each equivalence class is A-open) forms a good, full tree.*

PROOF. First, we verify the A-recursiveness of simplicity. Given a surreal a, it is simple precisely if it is not equivalent to any truncate of itself. Since testing equivalence is A-recursive and the truncates of a are A-finite, this is immediate. Now note that if x is in some equivalence class S, the simplest member of S is a truncation of x. Let x be simple. Let the equivalence class containing y, for some $y \preceq x$, be an A-finite union of intervals (l_i^y, r_i^y). The set of surreals extending x is not A-open. So, since the equivalence class of x extends x, but cannot contain all y extending x. Hence there is some z extending x that is not equivalent to any truncate of x. The same argument, applied to elements extending x^+ and x^-, shows that we can find such z on either side of x. Consider all z extending x^+ that are not equivalent to any truncate of x. These z are properly between the equivalence class of x and the equivalence classes of all truncates of x that are larger than x, consequently so are the common initial segments of any two such z's. Thus we can find a simplest z extending x^+ that is not equivalent to any truncate of x, this must be the simplest element in its equivalence class. Thus, for any x which is the simplest in its equivalence class, there are least z and z' extending x^+ and x^-, respectively, that are simple. Let z_i's be the simplest members of their equivalence classes, all compatible. Then the simplest member s of the equivalence class of $z = \lim_i z_i$ is a truncate of z, if it is a proper truncate then some z_i lies between s and z, and thus the simplest member is z_i. However, z_{i+1} is the simplest member of its equivalence class, and since $z_i \preceq z_{i+1} \preceq z$, we cannot have $z \sim z_i$. Thus z is simple. So the simple elements form a good tree. Furthermore, let x and y be simple and z be their common initial segment. Suppose z is not simple. Let s be the simplest member in the equivalence class of z. We assume without loss of generality that $z > x$. Then let s' be the simplest simple surreal extending s^+. This must be a truncate of both x and y and hence a truncate of z, thus, since $s \preceq s' \preceq z$, we have $s \sim s'$, which is a contradiction. Thus z is simple, and the tree of simple elements is full. ⊣

As a consequence, the simplest members of the equivalence classes are the image of some good word function. As examples we have the cases of the equivalence relation "commensurable with" on the positive surreals, the word function in this case is ω^x. If the equivalence relation is "finite difference", the resulting map carries x to the xth purely infinite number, given by replacing each sign in x by a block of ω copies of the same sign. In both cases we can look at the fixed points of the word function generating the simple members. In the first case these are the generalized ε- numbers described in [3], and can again be generated by a good word function. Suppose that T is a tree and that we have a partial function n from ω to T. n is

thought of as a coding of members of T by elements of ω. Of course, the example we have in mind is when $S = \{+, -\}$, then T is the tree of all A-finite surreals, and n is some function from a set of indices for "recursive surreals" (I, E, or C). We can then talk about recursive word functions defined on the image of n. Our objective now is to obtain an effectivization of the results of this section. Since the n we will be considering will be defined on a Π^1_1 set, it will do little good to try to effectivize the equivalence of full trees and the images of word functions. Instead, we will consider only the restrictions of word functions to the range of n. So, the two obvious questions are: Given a good recursive word function, is its tree of fixed points the image of a recursive word function? If an equivalence relation is presented nicely enough, must the simple elements be the range of a recursive word function? In the case of I we must place restrictions on the order properties of the word function. I-indices are not well suited to this sort of thing; we must use $O^\dagger$-indices instead. Let f be a good word function on the A-finite surreals, let g be the good word function whose image is the fixed points of f. Let f' be a recursive function from $O^\dagger$ to $O^\dagger$ such that f' represents f on $O^\dagger$, in the sense that $s^{O^\dagger} \circ f' = f \circ s^{O^\dagger}$. We also assume that f' clearly exhibits the goodness of f: we require f' to be monotonic with respect to the order $<_{O^\dagger}$. We will represent g by a recursive g' as follows. If $f'(0) = 0$, then set $g'(0) = 0$. Otherwise $0 <_{O^\dagger} f'(0) <_{O^\dagger} f'(f'(0)) <_{O^\dagger} \dots$, so let $g'(0) = 3e + 2$ where e is an index for this sequence. In general, we calculate g' as follows. If $f'(3g'(e)) = 3g'(e)$ then $g'(3e) = 3g'(e)$ (that is, $3e$ is a fixed point), otherwise $g'(3e) = 3i + 2$ where i is an index for the sequence $f'^n(3g'(e))$ (the next fixed point above $3e$). The case of $3e + 1$ is handled in exactly the same manner. $g'(3e + 2) = 3i + 2$ where i is an index for the function $g' \circ \varphi_e$. This is essentially a definition by effective transfinite recursion with cases $+$ and $-$. It is not difficult to see that g' is well defined on $O^\dagger$ and that it represents g. However, it is not clear that the image of $s^{O^\dagger} \circ g'$ is all the fixed points of f contained in the image of $s^{O^\dagger}$. To see that this is nontrivial, one could consider the case of $f(x) = \omega^x$. The fact that there is some recursive f' representing f on $O^\dagger$ follows from the simple description of ω-exponentiation on sign sequences given in [4], which lends itself easily to definition by effective transfinite recursion. In the case of C, these arguments are unnecessary, for the metadyadics coincide with the set of $L_{\omega_1^{CK}}$-finite surreals. This has a number of "effective" consequences: given an equivalence relation such that the equivalence classes are recursively generated convex sets, the corresponding map of surreals is recursive. The results of the next section will show that the action of f on the metadyadics is equivalent to the action of some recursive function on their C-notations. This provides another proof that ω-exponentiation on the metadyadics is computable.

§9. **Operations on the metadyadics.** In this section we give an alternate characterization of the metarecursive functions. We apply this characterization to show that all familiar operations on the surreal numbers can be performed effectively on metadyadic notations (in an ordinary recursion-theoretic sense).

DEFINITION 9.1. Let S be a Π^1_1 subset of ω. A function f from S to ω is *locally hyperarithmetic* if there exist partial recursive functions g and h such that $(\forall x \in S)[h(x) \in O \wedge f(x) = \varphi_{g(x)}^{H_{h(x)}}(x)]$.

If $S = \omega$, then this is equivalent to S being hyperarithmetic. However, if $h(x) \in O$ is not required for all x, then the range of h need not be bounded in O. In particular, if $S \subseteq O$ then we can think of f as a partial function from ω_1^{CK} to ω.

LEMMA 9.2. *A function f is locally hyperarithmetic if and only if it is metarecursive (that is, the graph of f is Π_1^1).*

PROOF. $\Longrightarrow$: $(x, y) \in f$ if and only if $\varphi_{g(x)}^{H_{h(x)}}(x) \downarrow = y$ and $x \in S$. Since the first condition can be one-reduced to $H_{2^{h(x)}}$, which 1-reduces to O uniformly in x, the graph of f is Π_1^1, hence f is metarecursive.

$\Longleftarrow$: Let $f : S \to \omega$ be a metarecursive function. Then the graph of f is metarecursively enumerable. Let A be its graph (considered as a subset of $S \times \omega$) and B be the complement of its graph relative to the domain S. Then A and B are both Π_1^1 and their union is $S \times \omega$. We construct partial recursive functions g and h that satisfy the hypotheses. Choose 1-reductions i and j of A and B to O, respectively. Take any $x \in S$. Then for any $n \in \omega$, (x, n) is in $A \cup B$. Sacks [6] shows that we can effectively find a b_n so that b is larger than some stage at which (x, n) is observed to enter A or B via one of the reductions, i or j. Let b be the uniform sum of the b_n. Define $h(x) = 2^{2^b}$. Then $\{y : |y| <_O |b|\}$ is recursive in $H_{h(x)}$. Thus, with an oracle to $H_{h(x)}$ we can determine if $f(x) = y$ for any $y \in \omega$. Such y exists, search until one is found. Let $g(x)$ be an algorithm describing this search. Then $f(x) = \varphi_{g(x)}^{H_{h(x)}}(x)$, as desired. ⊣

COROLLARY 9.3. *If f is any partial metarecursive function from ω_1^{CK} to C, then there exists a partial recursive function j such that $(\forall x \in O)[j(x) \downarrow \in C]$, and $j(x)$ codes the same surreal as $f(|x|)$.*

PROOF. We will use $C^\dagger$ indices (as we have shown, it makes no difference). f is partial metarecursive and thus can be made locally hyperarithmetic via some g and h. For any x, we can compute $f(x) = (n, a)$ with the oracle $H_{h(x)}$. Thus we can find a hyperarithmetic index for the set $\{a\}$, and with that effectively compute a bound b for a in O. Now, let $j(x) = (m, b +_O h(x))$, where m codes the following algorithm: First, using the oracle $H_{b+_O h(x)}$, compute $\varphi_{g(x)}^{H_{h(x)}}(x) = (n, a)$. Now use the algorithm provided by n, which is recursive in H_a, hence in $H_{b+_O h(x)}$, (by Spector uniqueness, [6]) to compute a sign sequence defined below a in O. Finally, with the oracle, effectively convert this sign sequence to one that converges below b instead. Then $j(x)$ codes the desired surreal. ⊣

We are now in a position to answer another question posed by Harkleroad (in [5]): Can division be performed effectively on the C-notations for surreals? The answer, and much more, is contained in the following.

COROLLARY 9.4. *All the operations of surreal algebra (addition, multiplication, reciprocation, exponentiation, ω-exponentiation, square roots, finding of roots to polynomials of odd degree) can be carried out effectively on C-indices.*

PROOF. We can pass metarecursively from C-indices for metadyadics to sets in $L_{\omega_1^{CK}}$, where the operations can be carried out metarecursively, then pass to a hyperarithmetic sign sequence, then to a $C^\dagger$ index, then to a C index. The series

of compositions results in a metarecursive function from C to C. By the previous corollary, there is a recursive function which is equivalent. ⊣

This argument also applies to the truncation of sign sequences at arbitrary stages which was needed in the last section. The compactness properties of surreals in an admissible set can now be interpreted as a sort of "recursive compactness" property of the metadyadics. We state this as follows.

COROLLARY 9.5. *Let W be a recursively enumerable set of ordered pairs (x, y) which code intervals $(s^C(x), s^C(y))$ of metadyadics. Suppose these intervals cover* $[0, 1]$. *Then finitely many such intervals cover* $[0, 1]$.

Indeed, W could be taken to be any hyperarithmetic set. So the recursive surreals, while certainly not compact in the order topology, are compact in an effective sense. It is natural to ask if the same is true of other recursive generalizations of the compact real interval $[0, 1]$. For the most natural of these, the recursive reals, this is actually false.

THEOREM 9.6. *There exists a recursively enumerable set of intervals (a, b) (where a and b are dyadic rationals) which covers every recursive real in* $[0, 1]$ *but such that no finite subset of these intervals covers every such real.*

PROOF. We will construct such a set satisfying the first hypothesis which does not cover the entire interval $[0, 1]$. Then, if some finite subset of the intervals covers the recursive reals, it follows from the density of the recursive reals in $[0, 1]$ that all of $[0, 1]$ is covered, implying a contradiction. The set W of intervals is constructed as follows. At stage s, for $i < s$, run the ith program (which might define a recursive real) for s steps. If this program gives an approximation to a real, correct to within an interval I_i between dyadics of length less than $1/2^{i+2}$, then enumerate this interval into W. Then W is a union of intervals I_i with measure $\leq 1/2^{i+2}$, and thus has measure $\leq 1/2$, and so does not cover $[0, 1]$. On the other hand, for any recursive real r, there is a program i which approximates it, and at some stage s, an interval covering r will be enumerated into W. ⊣

In view of this, the metadyadics seem to be a natural "recursive compactification" of the recursive reals. (The compactness does not require all the metadyadics; in particular, the hyperarithmetic reals would also suffice.) The construction by cuts can be naturally viewed in this way: to ensure that $[0, a_i)$ and $(b_i, 1]$ do not cover $[0, 1]$ unless some finite subcover does (implying some $b_j \leq a_i$), we must find some "number" between each a_i and each b_i, which are recursively enumerable sets of previously constructed "numbers". From this point of view, the "recursive compactness" of the metadyadics (and, more generally, the A-compactness of the A-finite surreals) is not surprising.

§10. Recursive games and R-good functions. The proof that reciprocation can be performed effectively on C-indices may seem like "cheating". In some sense, the algorithm provided does not actually do the reciprocation, but codes the instructions for the reciprocation into the index produced. On the other hand, the definitions given of addition and multiplication on C do not "cheat" in this sense. The difference can be made precise, and that is the objective of this section. We begin by defining the class G of recursive games. Take two numbers (indices) e and i. These can be

taken as codes for recursively enumerable relations $\subseteq \omega \times \omega$, which we will denote $\vdash_e$ and $\vdash_i$. If the union of these two relations is wellfounded, then $(e, i) \in G$. We think of a member of G as a game. A position in the game is some number $n \in \omega$. The left player may move from a to b just if $a \vdash_e b$ and the right player just if $a \vdash_i b$. The wellfoundedness condition guarantees that this game must end no matter how it is played. G is clearly Π^1_1. Two games (e, i) and (e', i') are equivalent just if $W_e = W_{e'}$ and $W_i = W_{i'}$. If a game happens to be (hereditarily) a number, then it is a recursive number. Since this construction can be carried out in $L_{\omega_1^{CK}}$, the resulting number must be a metadyadic. It is easy to embed C into G by a function that is the identity on surreals. However, G by no means exhausts the class of games in $L_{\omega_1^{CK}}$. Before continuing the study of recursive games, we prove one entirely recursion-theoretic result. We assume the reader is familiar with the concept of an enumeration reducibility.

LEMMA 10.1. *Let f be a recursive function such that whenever $W_x = W_y$, $W_{f(x)} = W_{f(y)}$. Then f induces an enumeration reducibility (on the recursively enumerable sets). An index e for this reducibility can be found effectively from an index for f.*

PROOF. The enumeration reducibility (think of it as a function e on sets) is given by $a \in e(B)$ if and only if for some finite subset $W_x \subseteq B$, $a \in W_{f(x)}$. This constitutes an enumeration reducibility: in order for a to enter $e(B)$, we must have finite evidence (W_x) in B, and since f is recursive, the reduction is effective. Now we show that $W_{f(x)} = e(W_x)$ for all x. First, let $W_x \subseteq W_y$. Then define g so that $W_{g(k)} = W_y$ if $k \in K$, $W_{g(k)} = W_x$ otherwise. Suppose $a \in W_{f(x)}$. If $a \notin W_{f(y)}$, then we would have $k \notin K$ if and only if $a \in W_{g(k)}$, hence K would be co-r. e. Thus $a \in W_{f(y)}$, so $W_{f(x)} \subseteq W_{f(y)}$. So f is monotonic in the action it induces on recursively enumerable sets. Now let $x \in W_{f(y)}$. Define g by letting $W_{g(k)}$ be the set of the first n elements to enter W_k, where n is the largest initial segment of ω on which φ_x converges, and $W_{g(k)} = W_y$ if φ_k is total. Now the condition $x \in W_{f(g(k))}$ is recursive in K, thus cannot hold if and only if φ_k is total. Thus for some k, $W_{g(k)}$ is finite (since φ_k is not total) and $x \in W_{f(g(k))}$. Thus f is compact in the action it induces on recursively enumerable sets. Now $x \in W_{f(x)}$ if (by monotonicity) and only if (by compactness) $x \in W_{f(y)}$ for some y for which W_y is a finite subset of W_x. But this is the description given above. ⊣

The above result indicates that there is some strength to the notion of a computable function on the recursively enumerable sets. Two obvious definitions, via a function on indices and via a reducibility (of which enumeration is the most fundamental sort), turn out to be equivalent. Also domain-theoretically, the computable functions on 2^ω are precisely the enumeration reducibilities.

DEFINITION 10.2. An n-ary recursive function f is *R-good* if it is well-defined on games; that is, $\vec{x} \in G^n$ implies $f(\vec{x}) \in G$ and if $\vec{x}$ and $\vec{x}'$ code the same games then so do $f(\vec{x})$ and $f(\vec{x}')$.

The above theorem gives us (since every candidate for a game with finitely many options satisfies the wellfoundedness condition, and we can consider the game (e, i) to be a recursively enumerable set of options)

COROLLARY 10.3. *Every R-good function acts as an enumeration reducibility on games. In particular, the action it induces is monotonic and compact.*

This can almost be taken as a definition. Indeed, we might want to eliminate the recursion-theoretical aspects of R-goodness and consider the more general notion of a G-good operation on games. This could be defined, in general, for arbitrary games, but for simplicity we limit ourselves to the countable case here.

DEFINITION 10.4. A game consists of two binary relations, R and L on $\omega \times \omega$ whose union is well-founded. $G' = (R', L')$ is a subgame of $G = (R, L)$, written $G' \preceq G$, if $R' \subseteq R$ and $L' \subseteq L$. The limit $(R, L) = \lim_i (R_i, L_i)$ of a set of games is defined by $R = \bigcup_i R_i$ and $L = \bigcup_i L_i$, if this is a game. The ordering $\preceq$ on games induces a topology. A function (possibly many place) F from games to games will be called G-good if F is $\preceq$-monotonic and continuous on the induced topology (in all its arguments).

Equivalently, F is G-good if and only if $F(\lim_i G_i) = \lim_i F(G_i)$ for any finite or countable directed set (or chain) of games G_i (in each argument). It follows that every R-good function is G-good, but not conversely. For example, the addition of any non-recursive game is G-good but not R-good. Many of the functions we work with in surreal algebra can be carried out by R-good functions. These include addition, multiplication, ω-exponentiation, constants, the identity, and concatenation of constant sign sequences on the left (whether or not this includes reciprocation or exponentiation is open). Note that our equivalence relation on games is rather small. This serves to eliminate the "cheating": by tightly controlling the form of a number one cannot code noneffective instructions into its construction. However, this equivalence relation is weaker in a sense; it does not force the R-good function to be well-defined on surreals (or, more generally, on games under their more general concept of equivalence). Indeed, any function given effectively by a uniform inductive definition can be performed by a R-good function on indices, even those which are not well-defined on surreals. As an example, we will demonstrate how this is done for addition. Let $f((e, i), (e', i')) = (e'', i'')$, where e'' is an index so that $(x, y) \vdash_{e''} (x', y')$ if and only if $(x = x' \wedge y \vdash_{e'} y') \vee (x \vdash_e x' \wedge y = y')$ and i'' is an index so that $(x, y) \vdash_{i''} (x', y')$ if and only if $(x = x' \wedge y \vdash_{i'} y') \vee (x \vdash_i x' \wedge y = y')$. Since the definition of the function involves only the relations coded by the corresponding indices, it is easy to see that this is well-defined on games, and agrees with the definition of surreal addition (or with the more general game addition). The above construction applies whenever we have an operation defined by a uniform positive inductive definition. Whether or not some sort of converse holds is unclear; R-good functions can certainly depend on detailed structural properties of a game, but only in very simple ways. We do not know of any R-good function well-defined on surreals except those given by simple inductive definitions like this one. More generally we can make sense of the notion of a recursive function on arbitrary countable games. These will be precisely the restrictions of enumeration reducibilities to the structures which are games (that is, satisfy the wellfoundedness condition); restricted to recursive games they are precisely the R-good functions. If we ignore the general games and look only at those games which are numbers, then we have the concept of a computable function on the countable surreal numbers. The above argument will

show that the ring operations, ω-exponentiation, and many other simple operations are computable. The computability of division and exponentiation is open; however it seems unlikely that reciprocation is possible because any enumeration reducibility must be monotonic and thus defined on the empty game since this is a subgame of every other game, giving meaning to $1/0$. There is no contradiction here, but it seems unnatural.

§11. Good functions on impartial games. In this section we consider the notions of the last section in the case of impartial games. Recursive impartial games are all of the form $(e, e) \in G$; in general a countable game (R, L) is impartial if and only if $R = L$. We consider all games to begin at the position $0 \in \omega$. We can consider each game as a subtree of $\omega^{<\omega}$ in the following way: A finite sequence $x_1, x_2, \ldots, x_n$ is in the tree just if x_{i+1} is an option from x_i, where x_0 is taken to be 0. Conversely with each wellfounded subtree of $\omega^{<\omega}$ we can associate a game in which the positions are finite tuples $(x_1, x_2, \ldots, x_n)$ where the only possible moves are from $(x_1, \ldots, x_{n-1})$ to $(x_1, x_2, \ldots, x_n)$ where the latter tuple is in the tree. Although these correspondences are not inverses they are monotonic with respect to the orderings "subtree of" and "subgame of" and continuous with respect to the induced topologies. Then it is natural to call functions from trees to trees R-good and G-good just if the induced function from games to games is R-good or G-good. In general, any function given by a simple inductive definition will be G- good. We will not worry so much about the structure of the trees, instead functions will be defined by bracket definitions $f(x) = \{g_i(x)\}$, meaning that if x is a game, $f(x)$ is the game whose options (for each player) are given by exactly the $g_i(x)$ (which will typically depend more on the options of x). It should be clear, in the examples we will give, how to define a consistent choice of branches in order to make function G-good. An ordinal $\hat{G}$ (the Grundy number of G) can be assigned to each impartial game G. This ordinal is a complete invariant with respect to the natural equivalence relations on games. Addition and multiplication are well-defined on games; these operations give a natural field structure to the ordinals. We will write this field addition and multiplication as $\oplus$ and $\otimes$, respectively. The ordinals, under this field structure, will be referred to as $\mathbf{On}_2$. Let us summarize the relevant facts (proofs may be found in [3]):

- If G is an impartial game, $\hat{G}$ is the smallest ordinal which is not of the form $\hat{G}'$ where G' is an option of G.
- $\oplus$ and $\otimes$ can be considered as operations on games which are well defined on the associated ordinals. They are defined by a uniform induction so that their restriction to countable games is R-good.
- $\mathbf{On}_2$ is an algebraically closed field of characteristic 2. Ordinals with sufficiently strong closure properties are algebraically closed subfields of $\mathbf{On}_2$. In particular, ω_1 and ω_1^{CK} are algebraically closed.
- $\alpha \oplus \beta$ is can be obtained by writing α and β as finite sums of (ordinal) powers of 2 and adding without carrying. As a result, α is an additive subgroup of $\mathbf{On}_2$ just if α is a power of 2, and $\alpha 2^\beta \oplus \gamma = \alpha 2^\beta + \gamma$ if $\gamma < 2^\beta$.

Conway's "simplest extension theorems" provide evidence that in some sense the operations $\oplus$ and $\otimes$ are the simplest and most natural algebraic structure to give the

ordinals. However there are a number of less natural R-good and G-good functions on the ordinals (as invariants of impartial games).

EXAMPLE 11.1. Define f by induction on games. $f(G) = \{0, f(G')\}$ where G' ranges over all options of G. It is easily verified by induction that $\widehat{f(G)} = 1 + \hat{G}$. More generally, a simple modification shows that $f(\alpha) = \beta + \alpha$ is R-good for any recursive ordinal β, and G- good for any countable β.

EXAMPLE 11.2. Let c be arbitrary. 2^c is a subgroup of the additive group of $\mathbf{On}_2$. Define $f(G) = \{f(G') \oplus d\}$ where G' is an option of G and d ranges over ordinals $< 2^c$. It is easily verified by induction that $\widehat{f(G)} = \hat{G}2^c$. This function is G-good in general, if c is a recursive ordinal then it is R-good.

EXAMPLE 11.3. Let $f(\beta) = (1 + \beta\omega) \oplus (\beta\omega)$. Then $f(\beta) = 1$ if $\beta = 0$ and 0 otherwise. As a composition of R-good functions, f is R-good.

The preceding example allows us to "paste together" G-good functions defined on different sections of the ordinals. As an easy consequence, the zero-sets of G-good functions are a σ-algebra on ω_1 which contains all one-point sets, and thus all countable and cocountable sets. It also contains the set of limit ordinals, the set of successor ordinals, the set of limits of limits, and many others. A precise characterization of this σ-algebra remains open. However, a simplified version of the problem has an easy solution. We discuss this now. Suppose we consider only finite impartial games. Such games can be coded by natural numbers in such a way that the Grundy number is a computable function (the Grundy number of a finite game is finite). We wish to consider the possible restrictions of R-good functions to such games, under the assumption that finite games are closed under this operation (which is true for many familiar examples: addition and multiplication in particular). Note that any such function must be K-recursive: given a game G, its image under the R-good function in question yields an index for a finite, recursively enumerable tree. The structure of this tree can be determined K-recursively. The Grundy number is then effectively computable. Now, it is not clear that any K-recursive function can be so represented, but we can construct what *appears* to be an R-good function, so long as we restrict our attention to finite games. We outline how this is done: First, note that given a finite tree T, we can effectively construct a finite tree T' extending T with any desired Grundy number. We show this by induction on the height of the tree. If the tree is trivial, then we simply add branches with Grundy numbers i for $i < n$, and the resulting tree has Grundy number n. If the tree is nontrivial, we do the same, and to each existing branch we apply the inductive hypothesis, and grow it to a tree with Grundy number $n + 1$. Again, the resulting tree has the desired Grundy number n. Now continuity is a trivial matter among finite trees, so all we need to require of the function we construct is monotonicity. Enumerate the trees T_i effectively in some order extending the natural order on trees, so that T_i is a subtree of T_j implies $i \leq j$. Let f be the K-recursive function we wish to represent, so f is recursively approximable. We create a corresponding recursive approximation to our R-good function: we define $g(s, T_i)$ to be a tree extending $g(t, T_j)$ for $t \leq s$ and $i \leq j$ (excluding, of course, $t = s$ and $i = j$) which we assume, by recursion, to have been already defined, which has Grundy number equal to the recursive approximation of

f at stage s (such a tree exists by the above remarks). Now since the approximation to f changes only finitely often, it follows by induction that $g(s, T_i)$ is eventually constant for any i. The limit of $g(s, T_i)$ is clearly monotonic. If we define $h(T_i)$ to be an r. e. index for this limit tree, then $h(T_i)$ is effectively computable, and the desired R-good function. This also solves the similar problem for G-good functions, since the above construction can be relativized to an arbitrary set A, shows that an arbitrary function from ω to ω can be represented if we allow a powerful enough oracle. The analogues of the zero-sets above are thus the Σ_2^0 sets and all sets, for R-good and G-good functions, respectively. Note that in the first case, the zero-sets are not closed under complementation, since the argument that establishes such closure in the general case makes use of infinite ordinals. In any case, this hints that the sigma-algebra mentioned above might consist of all subset of ω_1. However, cardinality considerations show that this implies $2^\omega = 2^{\omega_1}$, since there are 2^{ω_1} sets requiring representation and only 2^ω functions to work with. Though this is not contradictory, we cannot hope to prove it, since if the continuum hypothesis holds the above equality is false. We now consider an analogue of the metadyadics for impartial games.

Definition 11.4. O_2 is the smallest set such that $W_n \subseteq O_2$ implies $n \in O_2$. For $n \in O_2$, let $\hat{n} = \operatorname{mex}\{ \hat{x} : x \in W_n \}$.

O_2 provides what is in some sense the most nonconstructive set of notations for the constructive ordinals. First, note that $\oplus$ and $\otimes$ can be induced by recursive functions on O_2 indices (this is routine). By an abuse of notation, we will write $x \oplus y$ for some notation z satisfying $\hat{z} = \hat{x} \oplus \hat{y}$ (and similarly for $\otimes$).

Theorem 11.5. *It is possible to pass from indices for recursively enumerable well-founded relations with distinguished starting points (and thus also O-notations) to O_2-notations for their heights, but not vice-versa.*

Proof. For the positive direction we proceed by induction. Suppose we have a recursively enumerable well-founded relation $<$ on ω with distinguished point s. The height of this relation from s is the smallest ordinal greater than the height from any distinguished point $s' < s$. Extending $<$ to an order relation (by taking the transitive closure), it is not difficult to see that these heights will range over all ordinals (and only those ordinals) less than the height from distinguished point s. Inductively (recursively) compute O_2 notations for all of these smaller ordinals, the result is the image of a recursively enumerable set under a partial recursive function and thus recursively enumerable, furthermore the effectiveness of this enables us to obtain an index e for it. Then $W_e \subseteq O_2$ hence $e \in O_2$, and $\hat{e}$ is the desired height. Suppose it were possible to pass effectively from O_2 indices for ordinals to recursively enumerable wellfounded relations with the same height. The set of all indices for recursively enumerable wellorders with height 0 from some distinguished point is co-r. e. Thus there is a co-r. e. set A, the inverse image of this set, whose intersection with O_2 is precisely those x with $\hat{x} = 0$. Let $a \in O_2$ with $\hat{a} = 0$. Let $W_{f(x)} = \{a\}$ just if x enters K, and $W_{f(x)} = \emptyset$ otherwise. Then each $f(x) \in O_2$, with $\widehat{f(x)} = 0$ if $x \notin K$, $\widehat{f(x)} = 1$ if $x \in K$. Since $\oplus$ can be performed effectively, we can find g with $\widehat{g(x)} = \widehat{f(x)} \oplus 1$. Thus $x \in K$ if and only if $g(x) \in A$, hence K is co-r. e., which is impossible. Thus it is impossible to pass effectively from O_2

indices for ordinals to indices for recursively enumerable wellorderings of the same height. ⊣

All of the R-good predicates on impartial games discussed above can be carried out effectively on O_2 indices. However, much less effective operations can be carried out as well. A sense in which O_2 notations are as non-constructive as possible is made precise in the next theorem.

LEMMA 11.6. *There is a partial recursive function* $h(a,x)$ *such that for* $a \in O$, $h(a,x) \in O_2$ *and* $\widehat{h(a,x)} = 0$ *if* $x \in H_a$, $\widehat{h(a,x)} = 1$ *if* $x \notin H_a$.

PROOF. h is defined by transfinite recursion on a as follows. If $a = 1$, let $h(1,x) = 1$ for all x. If $a = 3 \cdot 5^e$, let $h(a,(x,y)) = h(\varphi_e(y),x)$. The interesting case is the limit case, $a = 2^b$. Then $H_a = H_b'$ which can be put into $\sum_1^{H_b}$ form. By induction we can compute codes for natural numbers coding arbitrary initial segments of H_b. Thus to compute the characteristic function of H_a we need only show that the relations whose characteristic functions are computable are closed (effectively) under conjunction, complementation, existential quantification, and that these formulas contain the identity and successor relations. Let $f(\vec{x})$ and $g(\vec{x})$ be the characteristic functions of two many-place predicates. Then $f(\vec{x}) \otimes g(\vec{x})$ is a characteristic function of the conjunction of the two predicates, and $f(\vec{x}) \oplus c$ is the characteristic function for the complement of the first predicate when c is some notation for 1. Successor and identity relations are taken care of by the somewhat strange examples of R-good predicates mentioned earlier. For existential quantification, suppose that $f(x,\vec{y})$ is the characteristic function of some predicate. Let $g(\vec{y}) = n$ where $W_n = \{c, f(x,\vec{y}) : x \in \omega\}$. Then $g(\vec{y})$ codes the successor of the characteristic function projection of that predicate onto the first coordinate (where c is a code for 0). The image of g under a polynomial (necessarily recursive) relative to the $\oplus$, $\otimes$ field structure yields the desired characteristic function of the projection. ⊣

THEOREM 11.7. *Let* S *be a* Π_1^1 *set, let* f *be a metarecursive function from* S *to* ω_1^{CK}. *Then there is a partial recursive function* g *from* S *to* O_2 *so that* $\widehat{g(x)} = f(x)$. *An index for* g *can be obtained effectively from an index for* f.

PROOF. f is metarecursive hence (considered as a function from S to O_1) locally hyperarithmetic. By the effectiveness of what follows, it is sufficient to consider the case of hyperarithmetic f. The proof is a modification of the method used to show the effective equivalence of C and $C^\dagger$ indices. For a given x, we can effectively (by [6]) compute a bound $b \in O$ for the ordinal coded by $f(x)$. Let f be recursive in H_a, let $c = a +_O b$. Then we can recursively in H_c determine whether $|f(x)| = |d|$ for all $d <_O b$. The set of such d is recursively enumerable. For each d, compute an O_2-notation d' for an ordinal which is 1 if $|f(x)| = \hat{d}$ and 0 otherwise, this can be done effectively by the preceding lemma. Let

$$W_{g(x)} = \{b \otimes (1 \oplus d') \oplus e \otimes d' : d <_O b \wedge e \in W_d\}.$$

It is not difficult to verify that $\widehat{g(x)} = |f(x)|$. ⊣

§12. **Universal properties of the recursive surreals.** In [3] Conway proves that **No** is a universally embedding ordered field. This means that given any ordered fields A and B and ordered field morphisms f from A to **No** and g from A to B, there is a morphism h from B to **No** such that $h \circ g = f$. Note that this would be obviously false if we could take $A = \mathbf{No}$ and f to be the identity, unless g was surjective (and thus an isomorphism). The key point is that **No** is not a set, and this statement applies only when A and B are ordered fields that are actually sets. If we truncate the construction of **No** at some regular cardinal κ, then this property can be stated more precisely: It is true for any ordered fields A and B of size less than κ. So, in general, we have some "big" field of surreal numbers, and it has this property with A and B restricted to "small" fields. Conway also shows that this property characterizes **No** up to isomorphism. The proof of this depends on the fact that all proper classes are of the same size (in a set theory which allows such objects). In a more restricted setting we should say that the surreals are the smallest field with this property, or the only field (up to isomorphism) of a particular cardinality with this property. In this section we provide recursive analogues of this universal property. Consider the metadyadics. As we have shown, they form a countable real closed field. We will consider the ordering first. As an ordered set they form countable dense linear order and are thus order isomorphic to the rational numbers. Yet most of the complexity of C is in its order, since the field operations are effective. The order isomorphism cannot be made recursive or even metarecursive. We want to make precise a sense in which the metadyadics are not order isomorphic to the rationals. In order to do so we must take into account the recursion-theoretic properties of C, and thus consider not only ordered sets but ordered sets indexed by elements of ω. Now when C is considered as a set indexing the metadyadics, not every natural number codes a metadyadic, and every metadyadic has infinitely many codes. The complexity of C makes it impossible to get around this difficulty. We now consider indexed ordered sets.

Definition 12.1. An indexing is a set $A \subseteq \omega$ and a total preorder $\leq$ on A. The indexed objects of the indexing are the equivalence classes induced by the preorder.

For example, C and $LT^{<\omega_1^{CK}} = \leq_C$ are an indexing. The equivalence classes induced by the preorder are the indexed objects of the indexing set. In this case the indexed objects are the metadyadics.

Definition 12.2. An index morphism from an indexing $(A, \leq_A)$ to an indexing $(B, \leq_B)$ is a function f from ω to ω such that if $x \in A$, $f(x) \in B$, and $x \leq_A y$ if and only if $f(x) \leq_B f(y)$.

This implies that the morphism f is well-defined and injective on the indexed objects. Typically, we will be concerned with the recursion-theoretic properties of f, as well as its effect on the indexed objects. We want to consider various categories of indexed sets and morphisms, particularly those with slightly more structure; for example, indexed ordered fields or groups. Then we would consider only those morphisms which also represent field or group morphisms on the indexed objects. All of the proofs about C are basically the same, they rely on C's universal properties as an indexed ordered set, as well as its particular structural properties as a divisible group, (effectively) real-closed field, etc. Since these proofs are all essentially the

same we will include only the most difficult here, others may be obtained by omitting details (it is difficult to get them as corollaries).

THEOREM 12.3. *The following uniquely characterizes C as an indexed ordered field (up to metarecursive isomorphism):*

- *C and $\leq_C$ are Π^1_1. The field operations can be carried out effectively.*
- *If H and H' are hyperarithmetic indexed ordered fields, f is a hyperarithmetic indexed ordered field morphism from H to C and g is an hyperarithmetic indexed ordered field morphism from H to H', then there exists a hyperarithmetic indexed ordered field morphism (for which the indices can be found effectively) h from H' to C such that $h \circ g = f$.*

Furthermore, h can be chosen recursive in $\deg(f) \oplus \deg(g) \oplus \deg(\mathrm{range}(g))$ *(thus, if f and g are recursive and the range of g is recursive, h can be chosen to be recursive).*

PROOF. First we verify that C has these properties. C and its ordering are Π^1_1 by construction, and we have demonstrated that the field operations can be carried out effectively. Suppose we are given H, H', f, and g as above. Define $h(x) = f(y)$ if $g(x) = y \in \mathrm{range}(g)$. $h(x)$ on other values will be computed recursively, coding the same surreal as the metarecursive (and thus locally hyperarithmetic) $h'(x)$ which we now define. $h'(x)$ is computed nonconstructively (but metarecursively) as follows. We assume that h' has already been defined on a subfield of H', and after we know where x is sent we will decide the value of h' on all the new elements added to that subfield by x. First we define h' on all x algebraic over the subfield on which h' has already been defined (including the image of g). To add some algebraic r we find its minimal polynomial. H' is an ordered field, hence has characteristic zero, and is thus separable. Then we find all roots of this minimal polynomial and notations for all the metadyadic roots of its image in C. To each root r in H' there corresponds a root r' in C such that if h' has already been defined on x, then $x < r$ if and only if $h'(x) <_C r'$, because C is real-closed. Define $h'(r) = r'$, and extend h' to the subfield generated by the old subfield and r. The case when r is trancendental over the field defined so far is much simpler. Let S be the subfield of H' on which h' has been defined so far. Choose any number in C which is between the image of h' restricted to S below r and h' restricted to S above r, and is transcendental over the image of this subfield, call it r', and set $h'(r) = r'$. This completes the construction. To show that this uniquely characterizes $\mathbb{C}$ up to metarecursive isomorphism we use a back-and-forth argument. Let D and D' be any indexed ordered field with these two properties. Pick 1-reductions of D and D' to O. We define by recursion subsets $D^\beta \subseteq D$ and $D'^\beta \subseteq D'$ and indexed ordered field morphisms f^β from D^β to D'^β and g^β from $D'^{<\beta}$ to D^β, which we assume to be hyperarithmetic (uniformly in β). To form D^β, take $D^{<\beta}$ and the morphism $f^{<\beta}$ from $D^{<\beta}$ to $D'^{<\beta}$. $f^{<\beta}$ is hyperarithmetic, there is a hyperarithmetic morphism from $D^{<\beta}$ to D, (the inclusion), thus we can find a commuting hyperarithmetic morphism from $D'^{<\beta}$ to D, which we call g^β. Let D^β be the subset of D generated by the field operations from the image of g^β and those elements of D which reduce to O below β. Then consider the map $f^{<\beta}$ from $D^{<\beta}$ to D', and the inclusion of $D^{<\beta}$ into D^β; again we can find a hyperarithmetic morphism f^β from D^β to D'. We then let D'^β be the subset of D' generated by the field operations from the image of f^β and those elements of D' that reduce to O below β. Then set $f = \bigcup_\beta f^\beta$,

$g = \bigcup_\beta g^\beta$. It is not difficult to verify that f and g are inverses to each other, as uniform unions of hyperarithmetic functions they are metarecursive, and that they are defined everywhere in D and D'. Thus, D and D' are metarecursively isomorphic. ⊣

The preceding argument is capable of generalization, but not without conditions on the admissible set one is working in. Essential use was made above of a recursive well-ordering of the universe; such a well-ordering is in general available for admissible L_α, but seldom elsewhere. C has one further important recursion-theoretic property previously discussed. If f is any locally hyperarithmetic function to C, then there is some recursive g which is equivalent to f on the indexed objects (metadyadics). If we add this requirement to any D satisfying the above requirements, then we can construct a recursive functions f from C to D and g from D to C, such that $f \circ g$ and $g \circ f$ are equivalent to the identities on the indexed objects. However, these are not isomorphisms in the category of indexed ordered fields. Another bit of evidence that the class E has desirable properties is that it shares a similar universal property. We do not know that E is a field, so we can say nothing about its properties as an indexed ordered field. The group-theoretic analog of a real closed field would be an abelian divisible ordered group, and we do not know if surreals coded by E are closed even under division by 2. So any properties of E known at this point must be purely order-theoretic. We can hardly claim that this characterizes E up to isomorphism; we still do not know if E is effectively equivalent to C. However, in contrast to the previous construction which was highly ineffective, the following is more of a recursion-theoretic property. Here we consider E not with the ordering it inherits from the surreals, but with its usual effective ordering, $\leq_{\text{eff}}$. Thus the indexed objects are actually elements of E.

THEOREM 12.4. *Let R and R' be two recursive linear orderings, let f be a recursive order-preserving map from R into R' such that the image of f is recursive, and let g be a recursive order-preserving map from R into the structure $(E, \leq_{\text{eff}})$. Then there exists a recursive order-preserving map h from R' into $(E, \leq_{\text{eff}})$ such that $h \circ f = g$. Furthermore, an index for h can be found effectively from indices for R', f, g, and the image of f in R'. (An index for the ordering on R is unnecessary because its order is determined by the order of its image in R'.)*

PROOF. Let $<_R$ represent the recursive order on R'. h is defined as follows. If x is in the image of f, say $x = f(y)$, then $h(x) = g(y)$. This ensures that $h \circ f = g$. For other x, $h(x)$ is defined recursively. Assume $h(y)$ has been defined for all $y < x$ and all $y \in \text{range}(f)$. Then set $h(x) = 2^a 3^b$ where

$$W_a = \{ z : (\exists y \in H')[z \leq_{\text{eff}} h(y) \wedge y <_R x] \},$$
$$W_b = \{ z : (\exists y \in H')[z \leq_{\text{eff}} h(y) \wedge y >_R x] \},$$

where y ranges $\{ a : a < x \} \cup \text{range}(f)$. It is easy to verify that $h(x) \in E$ and f is order preserving. ⊣

§13. Conclusions and open problems. We have seen that the theory of the surreals remains rich even when truncated in the constructible hierarchy at arbitrary admissible levels. In particular, when truncated at ω_1^{CK}, there exist several alternative

characterizations that indicate that the surreals obtained, the metadyadics, correctly capture the notion of a recursive surreal number. In the case of the metadyadics, all of the familiar operations from surreal algebra can be performed effectively. An analysis of the meaning of this effectiveness leads to a more restricted notion that applies to any function defined on the countable surreals. The following interesting problems are left unsolved:

• Can anything be said about other classes of surreals obtained by weaking the inductive definition of C? In particular, are any of them equivalent to I, E, or C?

• Harkleroad has shown that the metadyadics properly contain the surreals with I-indices. As we have shown, it is possible to pass effectively from I-indices to E-indices to C-indices, and the surreals with E-indices properly contain those with I-indices. Do the metadyadics property contain the surreals with E-indices? If not, is it possible to pass effectively from C-indices to E-indices? We suspect that the inclusion is proper.

• Can truncation of sign sequences be performed effectively on E-indices? Are the surreals with E-indices closed under multiplication? If so, can multiplication be performed effectively on E-indices?

• Harkleroad showed that addition and multiplication could not be performed effectively on I indices. Are the surreals with I-indices closed under addition or multiplication? We suspect not.

• The general notion of a R-good function on the countable surreals has been introduced. It has been argued that any function with a "nice" inductive definition (note that this does not include division, which has a nonuniformity) is R-good. Are there examples of R-good functions which cannot be put into a "nice" inductive form? We suspect that if any exist, they are unnatural.

• Division can be performed effectively on the metadyadics; however because of a nonuniformity in the recursion defining reciprocation there does not seem to be any way to make it R-good, or even G-good. Is there a R-good or G-good function which, restricted to nonzero numbers, is the reciprocation function? We suspect not.

• If f is a good word function which is represented on $O^\dagger$ by some recursive, $<_{O^\dagger}$-monotonic f', and g is a good word function whose image is the fixed points of f, must it be true that the recursive function g' representing g on $O^\dagger$ has as its image indices for all fixed points of f which have $O^\dagger$ indices? We suspect so.

• Can a theory of "nonstandard surreals" be developed in an arbitrary admissible set, by considering some kind of direct limit of orderings with imperfect selectors?

• The class of "hereditarily recursively enumerable" surreals (that is, metadyadics) coincides with the set of hyperarithmetic surreals. Can this statement be generalized to more general admissible sets? That is, is the collection of hereditarily A-recursively enumerable surreals (when appropriately defined) the same as the collection of A'-finite surreals, where A' is the smallest admissible set containing A? It is clear that all such surreals are A'-finite, but the converse appears more elusive given the method by which it is proven in the special case here.

• All arithmetic reals have E-indices. Can the passing from an arithmetic specification for the sign sequence to the E-index be made effective (for the reals or in general)? Is there a nice characterization of the sign seqeuences of E-indexed surreals (or reals)?

• We have shown that every hyperarithmetic surreal has a C-index. However, a hereditarily hyperarithmetic game which is a number need not have an isomorphic hereditarily recursively enumerable game. Does a similar situation exist with general games? Since addition is well defined on the equivalence classes of combinatorial games ([3]), we formulate the question as follows: is it true that for any hyperarithmetic game g there exists a recursive game g' such that $g + g'$ is a zero game?

• The concept of a G-good function looks a bit tamer on the countable impartial games than on the countable numbers. Is there a nice characterization of all functions on ω_1 induced by R-good or G-good functions? Can it be shown that not every function from ω_1 to ω_1 is induced (a cardinality argument shows that this must be so if one assumes the continuum hypothesis)? In particular, is successorship induced by a G-good function? Reciprocation (with the $\mathbf{On}_2$ field structure)?

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DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
CAMBRIDGE, MA 02138, USA
E-mail: lurie@husc.harvard.edu