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A Model of Decision-Making Involving Two Information Processors

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Abstract. Two assumptions which are generally made in decision theory are: (a) all the information necessary for decision-making is available, and thus the decision problem reduces to utilizing the information to make the best decision possible, and (b) the decision-maker is human, and if he is assisted in decision-making, the assistance is provided by humans. In this paper, we propose a model which does not make either of these assumptions. In the model, we consider information acquisition to be an integral part of the decision problem. Moreover, we consider the issue of the human and computer working jointly towards solving decision problems. Thus, the model explicitly considers the issue of more than one agent's performing information-gathering actions. The model also considers the impact of a time constraint on decision-making.

Key words. Decision problem, information-gathering, efficient strategy, uniformed action principle.

1. Introduction

Decision-making has been an important part of research in psychology and economics for over 40 years now [von Neumann and Morgenstern (1944); see Hogarth (1987) for a survey of recent research on the various aspects of decision-making]. Generally speaking, however, normative decision theories have not dealt formally with the *process* of information-gathering. Of course, every good textbook in decision theory has some discussion of how to go about solving decision problems when there are alternative information sources; but very little work has been attempted in the direction of characterizing good, let alone optimal, information-gathering strategies for general classes of decision problems.

A beginning has been made in the analysis of optimal information-gathering strategies in Hall, Moore and Whinston (1985, 1986), and Moore and Whinston (1986, 1987); but in these works two important issues have been ignored: the possibility that more than one agent may be performing information-gathering actions, and the explicit consideration of a time constraint on information-gathering activity. Our principal focus in this paper will be on extending these models (more specifically, Model 1 of Moore and Whinston (1986, 1987) to take these two issues into account.

In fact, these two issues are related to a considerable extent: the fact that decisions generally have to be made within a limited period of time often means that one may wish to delegate part of the information-gathering activity to another agent.

When considering a situation in which more than one agent (or information processor) carries out information-gathering activities, one can visualize several scenarios:

(a) Both information processors are human; with the human being who is originally faced with the decision problem being called the principal, and the person to whom he delegates part of the information-gathering (or more generally, problem-solving) activity being called the agent. (A generalization of this situation is an organization, where the principal is the superior, and the agents are subordinates.) We shall not be dealing with the two-human-information-processor situation here, however, in that the incentive and misrepresentation issues which generally arise in such situations are not addressed.

(b) One of the information processors is human, while the other is a computer. In this situation, it seems natural to suppose that the problem originates with the human being, and that it is he or she who must make the final decision as well. On the other hand, as expert systems are further refined, one may at some point want to consider the situation in which the principal is a computer, and the agent is a human being (presumably a semi-professionally trained technician).

(c) Both information processors are computers. The natural generalization of this situation is the more interesting situation in which one has a distributed network of processors.

Our principal interest in this paper will be with scenario (b), above. In considering this situation, two assumptions will be made throughout. First, we shall suppose that the computer has capabilities comparable to that of a decision support system. Secondly, we shall suppose that the human is in direct interaction with the environment; so that the problem is initiated by the human, and the final decision is in his or her hands as well.

These two assumptions do not appear to be at variance with the current view of problem-solving combining human beings and machines. Thus, Rubinstein (1985) has stated the following:

> To form an effective human-computer symbiosis, we must focus on non-programmable activities that include rich and imaginative perceptions of context. We must acquire the wisdom to ask appropriate questions, identify reasonable goals in the context of human values and represent problems from complementary as well as conflicting points of view . . . A productive symbiosis of humans and computers will assign the non-programmable tasks to humans and the programmable ones to computers.

Before proceeding further, we should mention that an issue which is often discussed when considering multiple information processors is that of trying to reconcile the probability assessments of the various processors (if they are viewed as being experts; see Winkler (1986), Lindley (1986), Clemen (1986), Morris (1983, 1986). However, in this paper we shall only consider the case of a single prior probability distribution, which

is known to the human decision-maker and may or may not be communicated to the computer. Thus, in our context, the issue of combining probability assessments of various experts does not arise (or, alternatively, can be viewed as having somehow been solved prior to the beginning point of our analysis).

This paper is structured as follows. In Section 2, we summarize as much of the Moore and Whinston model as is needed for our present purposes, and extend the model to include the introduction of time as a constraint, and two information processors. In Section 3, we consider efficient dual processor strategies for solving decision problems; and begin the process of characterizing optimal solutions for such problems. In Section 4, we discuss what we call the 'uninformed action principle'. The conclusions of the paper are presented in Section 5, where we shall briefly touch on some areas in which we believe further research is both needed and likely to be fruitful.

2. A Model of Decision Making

In this section, as much of the Moore and Whinston (1986, 1987) basic model of decision-making is presented as will be needed for our purposes. The model is then generalized to include time as a constraining factor, and is then generalized a second time to allow for two information processors in the decision process.

2.1. MODEL I

In the Moore and Whinston framework (this model is developed in full detail in Moore and Whinston (1986, 1987)), a decision problem, **D**, is defined by eight elements

$$\mathbf{D} = \langle X, \phi, D, \omega^*, A, \{\mathbf{M}_a \mid a \in A\}, c, r \rangle,$$

where X is the set of possible (mutually exclusive) states. The generic notation 'x' is used to denote elements of X.

ϕ: $X \to [0, 1]$ is the probability density function. ϕ defines the probability distribution function π: $P(X) \to [0, 1]$, where '$P(X)$' denotes the power set of X, by $\pi(Y) = \Sigma_{x \in Y} \phi(x)$, for $Y \subseteq X$.

D is the set of available (final) decisions.

ω^*: $X \times D \times \mathbb{R} \to \mathbb{R}$ is the payoff function (the inclusion of the third variable allows for the effect of the cost of information-gathering on payoffs).

A is the set of 'initial' (information-gathering) actions, or experiments, available.

$\mathbf{M}_a$ is the information structure associated with action $a \in A$. (Each $\mathbf{M}_a$ is a partition of X, as will be explained in more detail below.)

c: $a \to \mathbb{R}_+$ is the cost function; $c(a)$ is the cost of utilizing action $a \in A$.

r is a positive integer representing the number of information-gathering actions which can be taken before a final decision is made.

ASSUMPTIONS

(a) X, D, and A are all finite, and $(\forall x \in X)$: $\phi(x) > 0$.
(b) The true state $\hat{x} \in X$ does not change while the decision problem is being solved.
(c) A has $n + 1$ elements, where $n \geqslant 1$, and is written as $A = \{0, 1, \ldots, n\}$.

Associated with each $a \in A$ is a set of information signals, Y_a, and a function η_a: $X \to Y_a$. Each Y_a is assumed to contain a finite number, $n(a)$, of different signals so that, without loss of generality, Y_a can be written as $Y_a = \{1, 2, \ldots, n(a)\}$.

It is also assumed that

(a) for each $a \in \forall$, η_a is onto Y_a, and
(b) $n(0) = 1$ (so that the $a = 0$ action is the null information action).

For a given $x \in X$, there is a single signal receivable for each of the n information signal sets; so that information is viewed to be obtained deterministically (i.e., information is noiseless). Thus, if the information-gathering action (or 'experiment') $a \in A$ is performed, and the signal $y \in Y_a$ is received, it is known that the true state, $\hat{x}$, is an element of the set M_{ay} defined by

$$M_{ay} = \{x \in X \mid \eta_a(x) = y\} = \eta_a^{-1}(\{y\})$$

for $a = 0, 1, \ldots, n$, $y = 1, \ldots, n(a)$.

Notice that the family of subsets of X, $\mathbf{M}_a$, defined by

$$\mathbf{M}_a = \{M_{a1}, \ldots, M_{a,n(a)}\}, \quad \text{for } a = 0, 1, \ldots, n,$$

will be a partition of X and can be regarded as providing a summary of the information obtainable from action (or 'experiment') a; in the sense that, after a is performed, one will know to which of the sets M_{ay} the true state belongs. We formalize and extend this idea as follows.

DEFINITION 2.1.1. Let $B \subseteq X$ be non-empty. A family of subsets of X, $\mathbf{B}$, is an information structure on B iff:

(i) $(\forall B' \in \mathbf{B})$: $B' \neq \varnothing$.
(ii) $\mathbf{B}$ is a partition of B (that is, the sets in $\mathbf{B}$ are pairwise disjoint, and their union equals B).

Notice, that for $a \in A$, $\mathbf{M}_a$ is an information structure on X (by Definition 2.1). $\mathbf{M}_a$ is referred to as the *information structure associated with* (or induced by) a.

DEFINITION 2.1.2. Let $B \subseteq X$ be non-empty, and let $a \in A$. The information structure induced on B by a, $\imath(B, a)$, is defined as

$$\imath(B, a) = \{B \cap M_{a1}, B \cap M_{a2}, \ldots, B \cap M_{a,n(a)}\} \backslash \{\varnothing\}.$$

Note that if $B \subseteq X$ is non-empty, and $a \in A$, then $\imath(B, a)$ is an information structure on B.

DEFINITION 2.1.3. Let $B \subseteq X$ be non-empty, and let $\mathbf{B}$ be an information structure on B. An *action function on* $\mathbf{B}$ is a function α: $\mathbf{B} \rightarrow A$.

DEFINITION 2.1.4. Let $B \subseteq X$ be non-empty, let $\mathbf{B} = \{B_1, \ldots, B_k\}$ be an information structure on B, and let α: $\mathbf{B} \rightarrow A$ be an action function on $\mathbf{B}$. The *refinement of* $\mathbf{B}$ *by* α, $R(\mathbf{B}, \alpha)$, is defined by

$$R(\mathbf{B}, \alpha) = \bigcup_{j=1}^{k} \imath[B_j, \alpha(B_j)].$$

DEFINITION 2.1.5. Let $B \subseteq X$ be non-empty, and let $\mathbf{B}_1$ and $\mathbf{B}_2$ be information structures on B. We will say that B_1 *is as fine as* B_2 (or that B_1 *is a refinement of* B_2), and write $B_1 \geqslant B_2$ iff:

$$(\forall B' \in \mathbf{B}_1)(\exists B'' \in \mathbf{B}_2) : B' \subseteq B''.$$

Note that if B is a non-empty subset of X, $\mathbf{B}$ is an information structure on B, and α is an action function on $\mathbf{B}$, then $R(\mathbf{B}, \alpha)$ is (an information structure on B and is) a refinement of $\mathbf{B}$.

ASSUMPTION. The decision-maker can take up to r information-gathering actions. where $1 \leqslant r \leqslant n$. Since the null information action is included in A (and its associated coast is assumed to be zero), one can assume without loss of generality that the decision-maker takes exactly r information-gathering actions.

DEFINITION 2.1.6. A *feasible strategy for* $\mathbf{D}$, σ, is a sequence of $r + 1$ pairs

$$\sigma = \langle(\mathbf{B}_1, \alpha_1), (\mathbf{B}_2, \alpha_2), \ldots, (\mathbf{B}_r, \alpha_r), (\mathbf{B}_{r+1}, \delta)\rangle$$

satisfying

(1) $\mathbf{B}_1 = \{X\}$,
(2) (a) $\alpha_t : \mathbf{B}_t \rightarrow A$, for $t = 1, 2, \ldots, r$.
(b) $\mathbf{B}_{t+1} = R(\mathbf{B}_t, \alpha_t)$, for $t = 1, 2, \ldots, r$.
(3) δ: $\mathbf{B}_{r+1} \rightarrow D$.

The set of all feasible strategies for $\mathbf{D}$ is denoted by '$\Sigma(\mathbf{D})$'.

Thus a feasible strategy is viewed as being composed of two parts:

(i) the *information-gathering strategy*:

$$\alpha = \langle(\mathbf{B}_1, \alpha_1), \ldots, (\mathbf{B}_r, \alpha_r)\rangle,$$

(ii) the *decision strategy*, $(\mathbf{B}_{+1}, \delta)$.

Notice that our definition of a feasible strategy formally expresses two basic ideas.

(1) At each stage in the information-gathering process, a fully-specified information-gathering strategy must state, for each possible outcome to that point, which

action is to be undertaken next. After $q - 1$ actions or experiments have been performed, it will have been learned which of the sets in $\mathbf{B}_q$ contains the true state. The statement that $\alpha_q: \mathbf{B}_q \to A$ then formally expresses the idea that, for each possible outcome after $q - 1$ steps, a next (qth) action has been specified.

(2) After the full set of r information-gathering actions has been performed, what will have been learned is which of the sets in $\mathbf{B}_{r+1}$ contains the true state. For each possible outcome; that is, for each $B \in \mathbf{B}_{r+1}$, a final decision must be specified. Formally, this can be expressed by saying that our (final) decision strategy is a function with domain $\mathbf{B}_{r+1}$ and range D.

The definitions given below characterize the cost of an information-gathering strategy.

DEFINITION 2.1.7. If $\sigma = \langle(\mathbf{B}_1, \alpha_1), \ldots, (\mathbf{B}_r, \alpha_r), (\mathbf{B}_{r+1}, \delta)\rangle$ is a feasible strategy for $\mathbf{D}$, and $q \in \{1, \ldots, r + 1\}$, the sequences $\langle\beta_t(B)\rangle$ and $\langle a(t, B)\rangle$ ($t = 1, \ldots, q$) for each $B \in \mathbf{B}_q$ are defined by

$$\beta_t(B) \quad = \quad \text{that } B' \in \mathbf{B}_t \text{ such that } B \subseteq B'.$$

and

$$a(t, B) \quad = \quad \alpha_t[\beta_t(B)];$$

that is, $a(t, B)$ is the action taken at the tth stage along the path leading to B. $\beta_t(B)$ is referred to as the *predecessor of b at t*. (Notice that, since B is non-empty, and $\mathbf{B}_q$ is a refinement of the partition $\mathbf{B}_t$, there will be *exactly one* such set, B').

ASSUMPTION. With each $a \in A$ is associated a nonnegative cost, $c(a)$; the *cost of employing a*; and we assume: $c(0) = 0$.

In a given realization of the decision problem, the application of a feasible strategy, $\sigma = \langle(\mathbf{B}_1, \alpha_1), \ldots, (\mathbf{B}_r, \alpha_r), (\mathbf{B}_{r+1}, \delta)\rangle$ will result in the determination that $\hat{x}$, the true state, is an element of some $B \in \mathbf{B}_{r+1}$. The cost of determining that $\hat{x} \in B$ will be the sum of the costs of all the actions taken along the path yielding B, and will therefore be given by

$$C(B) \quad = \quad \sum_{t=1}^{r} c[a(t, B)].$$

The expected informational cost of strategy σ is therefore given by:

$$\Gamma(\sigma) \quad = \quad \sum_{B \in \mathbf{B}_{r+1}} \pi(B)C(B).$$

Given a feasible strategy for $\mathbf{D}$, $\sigma = (\alpha, \mathbf{B}, \delta)$, the expected payoff for σ is denoted by '$\Omega^*(\sigma)$'; that is

$$\Omega^*(\sigma) \quad = \quad \sum_{B \in \mathbf{B}} \sum_{x \in B} \phi(x)\omega^*[x, \delta(B), C(B)].$$

Formally the goal of the decision problem can be stated as follows:

OBJECTIVE. Choose $\sigma^* = (\alpha^*, \mathbf{B}_{r+1}^*, \delta^*) \in \Sigma(\mathbf{D})$ such that for all $\sigma = \langle \alpha, \mathbf{B}, \delta \rangle \in \Sigma(\mathbf{D})$, we have $\Omega^*(\sigma^*) \geqslant \Omega^*(\sigma)$.

In order to simplify our analysis somewhat, we shall assume throughout this paper that the payoff function is linearly separable in monetary outcomes; so that we can write

$$\omega^*(x, d, c) = \omega(x, d) - c,$$

where $\omega(x, d)$ is the gross payoff obtained when the true state of the world is x and the decision d is chosen, and c is the cost incurred in gathering information (for a detailed development of this representation, see Moore and Whinston (1968), pp. 292–294). With this assumption the expected net payoff of σ can be expressed as

$$\Omega^*(\sigma) = \Omega(\sigma) - \Gamma(\sigma);$$

where $\Omega(\sigma)$ is the expected gross payoff from σ and is given by

$$\Omega(\sigma) = \sum_{B \in \mathbf{B}_{r+1}} \sum_{x \in B} \phi(x)\omega[x, \delta(B)].$$

DEFINITION 2.1.8. The *finest information structure obtainable from* A, $\mathbf{B}^A$, is defined by

$$\mathbf{B}^A = \left\{ \bigcap_{a=1}^{n} M_{a1}, \bigcap_{a=1}^{n-1} M_{a1} \cap M_{n2}, \ldots, \bigcap_{a=1}^{n-1} M_{a1} \cap M_{n,n(a)}, \right.$$
$$\left. \bigcap_{a=1}^{n-2} M_{a1} \cap M_{n-1,2} \cap M_{n1}, \ldots, \bigcap_{a=1}^{n} M_{a,n(a)} \right\} \backslash \{\varnothing\}.$$

The above definition essentially specifies the best information (i.e., finest information structure) one can obtain, given the set of information-gathering actions.

DEFINITION 2.1.9. If $B \subseteq X$ is non-empty, the potential gross payoff associated with B, $v(B)$, and the conditionally optimal decision set for B, $D^*(B)$, are defined by

$$v(B) = \max_{d \in D} \left[\sum_{x \in D} \phi(x|B)\omega(x, d) \right]$$

and

$$D^*(B) = \left\{ d \in D | \sum_{x \in B} \phi(x|B)\omega(x, d) = v(B) \right\},$$

respectively.

For a given set B, $v(B)$ is the maximum conditional expected gross payoff which can be obtained if one knows only that the true state is an element of B (and not *which* element it is). $D^*(B)$ is the subset of D for which this maximal conditional expected payoff can be obtained.

DEFINITION 2.1.10. For each $d \in D$, $\mathbf{B}^A(d) \subseteq \mathbf{B}^A$ is defined by

$$\mathbf{B}^A(d) = \{B \in \mathbf{B}^A \mid d \in D^*(B)\}$$

and for each $d \in D$, X_d is defined by

$$X_d = \bigcup_{B \in \mathbf{B}^A(d)} B.$$

As is established in Moore and Whinston (1986) p. 299: if it has been determined that the true state, $\hat{x}$, is an element of X_d, for some $d \in D$; *nothing can be gained by further information-gathering*, given the available actions, A.

DEFINITION 2.1.11. If $\mathbf{A}$ and $\mathbf{B}$ are two finite families of sets, say

$$\mathbf{A} = \{A_1, \ldots, A_m\} \quad \text{and} \quad \mathbf{B} = \{B_1, \ldots, B_n\},$$

we define

$$\bigcap(\mathbf{A}, \mathbf{B}) = \{A_1 \cap B_1, A_1 \cap B_2, \ldots, A_1 \cap B_n, A_2 \cap B_1, \ldots, A_m \cap B_n\} \setminus \{\varnothing\};$$

that is $\bigcap(\mathbf{A}, \mathbf{B})$ is the collection of all *non-empty* sets which can be obtained as an intersection of a set from $\mathbf{A}$ and a set from $\mathbf{B}$.

If, say, $\mathbf{A} = \{A\}$, we shall sometimes simplify the notation of 2.1.11 to write '$\bigcap(A, \mathbf{B})$' in place of '$\bigcap(\{A\}, \mathbf{B})$', but this should cause no confusion. The following proposition, the proof of which will be left to the interested reader, sets out some basic properties of this operation.

PROPOSITION 2.1.1. *Let* $\mathbf{A} = \{A_1, \ldots, A_m\}$, $\mathbf{B} = \{B_1, \ldots, B_n\}$, *and* $\mathbf{C} = \{C_1, \ldots, C_p\}$ *be families of sets. Then we have*

$$(1) \quad \bigcap(\mathbf{A}, \mathbf{B}) = \bigcap(\mathbf{B}, \mathbf{A}) = \bigcup_{i=1}^{m} [\bigcap(A_i, \mathbf{B})] = \bigcup_{j=1}^{m} [\bigcap(A, B_j)],$$

$$(2) \quad \bigcap[\bigcap(\mathbf{A}, \mathbf{B}), \mathbf{C}] = \bigcap[\mathbf{A}, \bigcap(\mathbf{B}, \mathbf{C})].$$

Because of the second conclusion in the above proposition we shall sometimes write '$\bigcap(\mathbf{A}, \mathbf{B}, \mathbf{C})$' in place of '$\bigcap[\bigcap(\mathbf{A}, \mathbf{B}), \mathbf{C}]$' or '$\bigcap[\mathbf{A}, \bigcap(\mathbf{B}, \mathbf{C})]$'. Moveover, if $\mathbf{A}_i$ is a finite family of sets, $i = 1, \ldots, q$; we can unambiguously extend our notation to write $\bigcap(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_q)$ to denote the collection of all non-empty sets which can be obtained as an intersection of exactly q sets, one from each of the $\mathbf{A}_i$. Thus, for example, in the decision problem we have been discussing in this section, we denote the set of non-null experiments by $A^* = \{1, \ldots, n\}$. The finest information structure obtainable from A, $\mathbf{B}^A$, is then given by $\mathbf{B}^A = \bigcap(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n\}$.

We conclude this section with the following proposition, the proof of which should pose no problems for the interested reader.

PROPOSITION 2.1.2. *Let $B \subseteq X$, let* **B** *be an information structure on X, $a \subseteq A$, and α*: $\mathbf{B} \to A$ be an action function on **B** (see 2.1.3, above). Then we have

(1) $\iota(B, a) = \bigcap(B, \mathbf{M}_a)$,

(2) if $\alpha(\iota)$ satisfies

$$(\forall B \in \mathbf{B})\colon \alpha(B) = a^*,$$

for some $a^* \in A$, then

$$R(\mathbf{B}, \alpha) = \bigcap(\mathbf{B}, \mathbf{M}_{a^*}).$$

2.2. AN EXTENDED MODEL OF DECISION MAKING

In this section, the issue of time is dealt with explicitly. When considering time, two issues arise. First, the time spent on an individual information-gathering activity will vary; and if there is a time constraint, the total time taken to perform an information-gathering strategy has to satisfy the constraint. Secondly, the payoff could be a function of time, and as time progresses, the payoff could drastically decrease. This implies that the results of an information-gathering action should provide information to improve the payoff by at least as much as the decrease that occurs in the payoff as a result of the time taken to perform the action. Thus, in defining strategy to solve a decision problem in this framework, one should take both the above two factors into account; however, the following analysis considers only the issue of time as a constraint.

The decision problem in this case is defined by nine elements:

$$\mathbf{D} = \langle X, \phi, D, \omega, A, \{\mathbf{M}_a | a \in A\}, c, t, T\rangle,$$

where, t: $A \to \mathbb{R}_+$ is the time function; $t(a)$ is the time taken to perform action $a \in A$. T is the total time available in which to perform the information-gathering actions. (T could represent the time beyond which the state changes. It could also represent the time after which the payoff is negligible or zero; an example of such a situation is the time period in which a bid has to be submitted in a sealed bid auction.) The definitions of the other seven elements, and the assumptions stated in Section 2 will be maintained throughout the remainder of the paper.

DEFINITION 2.2.1. Let $a^* \in A'$, where A' indicates the set of actions other than the null information action, be such that $(\forall a \in A')\colon t(a) \geqslant t(a^*)$. If we then define

$$r = \left\lfloor \frac{T}{t(a^*)} \right\rfloor. \tag{1}$$

it is readily seen that r is the maximum number of information-gathering actions possible in a strategy.

It is obvious that the number of actual information-gathering actions performed along any path in a strategy will generally be some $r' < r$. However, since both the

cost and time taken to perform a null information-gathering action are assumed to be zero (i.e., $c(0) = 0$ and $t(0) = 0$), it can and will be assumed that exactly r actions are performed along any given path, with $r - r'$ of them being the null information-gathering action. (The introduction of a second information processor will necessitate a re-evaluation of our definition of r. This will be done in Section 3.)

DEFINITION 2.2.2. A feasible strategy for **D**, σ, is a sequence of $r + 1$ pairs

$$\sigma = \langle(\mathbf{B}_1, \alpha_1), (\mathbf{B}_2, \alpha_2), \ldots, (\mathbf{B}_r, \alpha_r), (\mathbf{B}_{r+1}, \delta)\rangle$$

satisfying

(1) $\mathbf{B}_1 = \{X\}$.
(2) (a) $\alpha_j\colon \mathbf{B}_j \to A$, for $j = 1, 2, \ldots, r$.
 (b) $\mathbf{B}_{j+1} = R(\mathbf{B}_j, \alpha_j)$, for $j = 1, 2, \ldots, r$.
 (c) $(\forall B \in \mathbf{B}_{r+1})$: $\Sigma_{j=1}^{r}\ t[a(j, B)] \leqslant T$ (Recall that $a(j, B) = \alpha_j[\beta_j(B)]$ for $j = 1, \ldots, r$; and that $\beta_j(B)$ is the predecessor of B at j. See Definition 2.1.7.).
(3) $\delta\colon \mathbf{B}_{r+1} \to D$.

The expected time, $\bar{t}$, taken by a strategy σ, is defined by

$$\bar{t}(\sigma) = \sum_{B \in \mathbf{B}_{r+1}} \pi(B) \sum_{j=1}^{r} t(a(j, B)).$$

Now that time has been incorporated into the model, we next consider the issue of dual information processor solving a decision problem.

2.3. DELEGATION

Delegation occurs when a decision-maker, generally referred to as the principal, assigns or makes a contract with another individual (or organization) referred to as the agent, to solve a problem with which the principal is faced. It is generally assumed within the above definition of delegation that the principal has some control over the agent; and is able to monitor his actions either perfectly (Townsend, 1979), or imperfectly (Harris and Raviv, 1978, 1979; Holmstrom, 1979). Thus, the primary focus of research in this area has been on developing a contract between principal and agent which is based on the monitoring assumptions, and is such that the agent, in maximizing his payoff, maximizes the principal's payoff as well.

The computer, by its nature, is assumed not to have a payoff independent of the human. Hence, these contract, monitoring and control issues do not arise here. It should be noted that, besides the case of the computer as agent, monitoring and control are of no significance in cases where the principal completely relinquishes the problem to the agent for analysis. An example of such a situation is a physician's referring a patient to a specialist.

Delegation may be a viable alternative for verification or validation of a solution. However, here we deal only with the issue of delegation within the context of solving a decision problem, and the issue of delegation for purposes of validation is not considered.

In a decision problem solving situation, delegation is considered a viable alternative if one or more of the following situation arises:

(a) The principal has less knowledge of a portion of the problem domain than does some agent (specialist), in the sense that the agent can further partition the finest partition obtainable by the principal. That is, suppose we let the set of information-gathering actions (experiments) available to the principal be denoted by 'A_1' and that available to the agent (specialist) by 'A_2', and let the finest information structure obtainable from A_1 be $\mathbf{B}^{A_1}$ (see Definition 2.1.8). We can formally represent the situation just described verbally as: there exists $B \in \mathbf{B}^{A_1}$ and $a \in A_2$ such that $\# \imath(B, a) \geqslant 2$, where for $S \subseteq X$, '$\# S$' denotes the number of elements in S.

(b) The expected cost of solving the problem by the principal is unambiguously greater than that of assigning it to an agent. That is, if σ^* denotes the optimal non-delegation strategy, and there exists a delegation strategy, σ, satisfying $\Gamma(\sigma^*) > \Gamma(\sigma)$ and $\Omega(\sigma) \geqslant \Omega(\sigma^*)$, then delegation is certainly a viable alternative.

(c) Delegation would be important when time plays a critical role in the problem. If the payoff is time dependent and if there exists a delegated strategy, σ, such that $\bar{t}(\sigma) < \bar{t}(\sigma^*)$ and $\Omega^*(\sigma) \geqslant \Omega^*(\sigma^*)$, where σ^* is the optimal non-delegated strategy, then delegation may be an appropriate step to take. A second situation in which the time factor might dictate delegation is when T is small, since solving the problem in parallel would increase the number of information-gathering steps one could take within the specified time constraint.

We shall suppose that both the agent and principal can perform information-gathering tasks. Thus, if the principal and agent are called information processors 1 and 2 (denoted by IP1 and IP2), with the sets of available information-gathering actions A_1 and A_2, respectively; then the set of available information-gathering actions is defined as $A = A_1 \cup A_2$.

Since our framework utilizes two information processors, a crucial issue is that of communication: A_1 and A_2 can be considered to be made up of a set of information-gathering actions and a set of interaction actions or communication actions as follows:

$$A_1 = A_1^e \cup A_1^c, \qquad A_2 = A_2^e \cup A_2^c,$$

where A_1^e is the set of information-gathering actions (experiments) available to IP1. A_1^c is the set of interactions performed by IP1 to communicate information to IP2; A_2^e is the set of information-gathering actions available to IP2. A_2^c is the set of interactions available to IP2 to communicate with IP1.

We assume that the interaction actions have a finite cost, as well as taking finite time to perform, but that they do not partition the state space as do the information-gathering actions.

ASSUMPTION. If $a \in A_1^c \cup A_2^c$, then $\mathbf{M}_a = \{X\}$, but $t(a) > 0$ and $c(a) > 0$.

The following features will be utilized in specifying the strategies available with human-computer interaction:

(a) Since the human is in contact with the environment, the decision problem originates with the human and the final decision is with the human, who is therefore viewed as the principal (IP1 in the analysis).
(b) The fact that the computer is assumed to have the capabilities of a DSS allows one to view the communication actions by the humans as non-programming actions (i.e., actions not involving writing programs) and, additionally, the computer can perform several information-gathering actions between interaction points.

Thus, a dual processor strategy from this point on will be written as

$$\sigma = \langle[(\mathbf{B}_1, \alpha_1^1)], [(\mathbf{B}_2^1, \alpha_2^2)(\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1)(\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)]\rangle$$

where $\mathbf{B}_1 = \{X\}$

$$\alpha_j^s : \mathbf{B}_j^s \to A_s: \quad j \in \{2, \ldots, r\};\ s = 1, 2;$$

1 indicates IP1 and 2 indicates IP2.

$$\alpha_1^1 : \mathbf{B}_1 \to A_1, \qquad \delta: \mathbf{B}_{r+1}^1 \to D.$$

The implication of assuming the final information structure belongs to IP1 is that the final decision is made by the human, and the computer is utilized as a decision support system. However, if one were considering expert systems, one might require the final information structure to belong to IP2, and assume the role of IP1 to be one of gathering environmental information which IP2 is not capable of accessing directly.

In order to simplify the analysis we shall make an apparently restrictive assumption regarding communication (we shall discuss this assumption in Section 3); namely that if at some j, and for some $s \in \{1, 2\}$, there exists $B \in \mathbf{B}_j^s$ such that $\alpha(B) \in A_s^c$; then we assume that every action taken by s at j is a communication action; i.e., $\alpha_j^s : \mathbf{B}_j^s \to A_s^c$. Such a point j will be called a communication point.

DEFINITION 2.3.1. If

$$\sigma = \langle[(B_1, \alpha_1^1)], [(\mathbf{B}_2^1, \alpha_2^1), (\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1), (\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)]\rangle$$

is a dual processor strategy, the set of communication points for σ, $Q(\sigma) \subseteq \{1, \ldots, r\}$, is defined by

$$Q(\sigma) = \{j \in \{1, \ldots, r\} \mid (\exists s \in \{1, 2\} \text{ and } B \in \mathbf{B}_j^s) : \alpha_j^s(B) \in A_s^c\}.$$

If $Q(\sigma)$ is not empty, we shall denote it by $Q(\sigma) = \{q_1, q_2, \ldots, q_p\}$, where $1 \leqslant q_1 < q_2 < \cdots < q_p \leqslant r$.

Partly because of the assumed framework of the decision problem, and partly just to simplfy our analysis, we shall asume that any feasible information-gathering strategy has the following features.

(1) At the instant IP1 is communicating with IP2 (or vice versa), IP2 (IP1) is neither performing an information-gathering action nor sending information back to IP1 (IP2). In other words, the human cannot instruct the computer about the problem being solved when the computer is executing a program, or when it is providing output (results) to the human; and similar restrictions apply to the computer.
(2) A communication action does not partition the communicator's information structure, but does partition the information structure of the agent to whom the information was communicated.
(3) The first communication action is performed by IP1, and until that occurs IP2 does not perform any information-gathering actions.
(4) After IP2 communicates back to IP1, it is assumed IP2 does not perform information-gathering actions until requested to do so by IP1.

The assumptions which have been discussed thus far, with the exception of the time constrains, are incorporated in the definition of a viable strategy, which follows.

DEFINITION 2.3.2. A dual processor strategy,

$$\sigma = \langle[(\mathbf{B}_1, \alpha_1^1)], [(\mathbf{B}_2^1, \alpha_2^1)(\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1)(\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)]\rangle$$

is called *viable* iff either

(A) $Q(\sigma) = \varnothing; \sigma^1 = \langle(\mathbf{B}^1, \alpha_1^1), (\mathbf{B}_2^1, \alpha_2^1), \ldots, (\mathbf{B}_r^1, \alpha_r^1), (\mathbf{B}_{r+1}^1, \delta)\rangle$ is a feasible strategy in the sense of Definition 2.2.2 and

$$\mathbf{B}_j^2 = \{X\} \quad \text{and} \quad \alpha_j^2 : \mathbf{B}_j^2 \rightarrow \{0\}, \quad \text{for } j = 2, 3, \ldots, r.$$

Or

(B) There exists a positive integer k such that $\# Q(\sigma) = p = 2k$, and
  (1) For each $i \in \{1, \ldots, p\}$ one has
    (a) If i is odd then

$$\alpha_{q_i}^1 : \mathbf{B}_{q_i}^1 \rightarrow A_1^c, \quad \mathbf{B}_{q_i+1}^1 = \mathbf{B}_{q_i}^1, \alpha_{q_i}^2 : \mathbf{B}_{q_i}^2 \rightarrow \{0\}$$

and

$$\mathbf{B}_{q_i+1}^2 = \bigcap(\mathbf{B}_{q_i}^1, \mathbf{B}_{q_i}^2)$$

(see Definition 2.1.11).

(b) If i is even, then

$$\alpha^1_{q_i}\colon \mathbf{B}^1_{q_i} \to \{0\}, \quad \mathbf{B}^1_{q_i+1} = \bigcap(\mathbf{B}^1_{q_i}, \mathbf{B}^2_{q_i}), \quad \alpha^2_{q_i}\colon \mathbf{B}^2_{q_i} \to A^c_2 \quad \text{and} \quad \mathbf{B}^2_{q_i+1} = \mathbf{B}^2_{q_i}.$$

(c) If $q_1 > 1$ then $\mathbf{B}^2_j = \{X\}$ and $\alpha^2_j\colon \mathbf{B}^2_j \to \{0\}$, for $j = 2, \ldots, q_1$.

(d) If i is even, $i < p$ and $q_i + 1 < q_{i+1}$, then

$$\mathbf{B}^2_j = \mathbf{B}^2_{q_i} \quad \text{and} \quad \alpha^2_j\colon \mathbf{B}^2_j \to \{0\}, \quad \text{for } j = q_i + 1, \ldots, q_{i+1}.$$

(e) If $q_p < r$, then

$$\mathbf{B}^2_j = \mathbf{B}^2_{q_p} \quad \text{and} \quad \alpha^2_j\colon \mathbf{B}^2_j \to \{0\}, \quad \text{for } j = q_p + 1, \ldots, r.$$

(2) For each $j \in \{1, \ldots, r\}\backslash Q(\sigma)$, one has,

$$\alpha^s_j\colon \mathbf{B}^s_j \to A^e_s \quad \text{and} \quad \mathbf{B}^s_{j+1} = R(\mathbf{B}^s_j, \alpha^s_j), \quad \text{for } s = 1, 2.$$

In order to incorporate a time constraint into this problem, we need to allow for the fact that the information-gathering actions taken by the two procesors between communication points may require different amounts of time. In order to allow for this, we look at the time used up between successive points of communication by the human along each path, as follows.

DEFINITION 2.3.3. For each $h \in \{1, \ldots, p/2\}$ and each $B \in \mathbf{B}^1_{r+1}$, $\tau_h(B)$, is defined as:

$$\tau_h(B) = \max\left\{\sum_{j=q_{2h-1}+1}^{q_{2h}-1} t[a^1(j, B)], \sum_{j=q_{2h-1}+1}^{q_{2h}-1} t[a^2(j, B)]\right\} +$$
$$+ t[a^1(q_{2h-1}, B)] + t[a^2(q_{2h}, B)] + \sum_{j=q_{2h}+1}^{q_{2h+1}-1} t[a^1(j, B)],$$

where $a^s(j, B) = \alpha^s_j[\beta^s_j(B)]$, $s = 1, 2$; we define $q_{p+1} = r + 1$, and

$$\sum_{j=q_{2h}+1}^{q_{2h+1}-1} t[a^1(j, B)] = 0, \quad \text{if } q_{2h} + 1 = q_{2h+1}.$$

We can now define a feasible strategy for the dual processor case, as follows:

DEFINITION 2.3.4. A dual processor strategy

$$\sigma = \langle[(\mathbf{B}_1, \alpha^1_1)], [(\mathbf{B}^1_2, \alpha^1_2)(\mathbf{B}^2_2, \alpha^2_2)], \ldots, [(\mathbf{B}^1_r, \alpha^1_r)(\mathbf{B}^2_r, \alpha^2_r)], [(\mathbf{B}^1_{r+1}, \delta)]\rangle$$

is *feasible* iff,

(a) σ is viable, and

(b) $(\forall B \in \mathbf{B}^1_{r+1})\colon t(B; \sigma) \equiv \sum_{j=1}^{q_1-1} t[a^1(j, B)] + \sum_{h=1}^{p/2} \tau_h(B) \leqslant T.$

The set of feasible processor strategies will be denoted by $\Sigma(\mathbf{D})$. Any $\sigma \in \Sigma(\mathbf{D})$ can be classified into one of four categories, namely: independent, sequential, con-current or mixed strategies.

DEFINITION 2.3.5. A feasible strategy σ is called:

(1) *independent* iff it satisfies condition A of the viability definition (Definition 2.3.2.);

(2) *sequential* iff $Q(\sigma) \neq \varnothing$ and for each odd integer i such that $1 \leqslant i \leqslant p - 1$:

$$\alpha_j^1 \colon \mathbf{B}_j^1 \rightarrow \{0\}, \quad \text{for } j = q_i + 1, \ldots, q_{i+1};$$

(3) *concurrent* iff $Q(\sigma) = \{1, r\}$ and for $j = 2, 3, \ldots, r - 1$

$$\alpha_j^1 \colon \mathbf{B}_j^1 \rightarrow A_1^e$$

and

$$\alpha_j^2 \colon \mathbf{B}_j^2 \rightarrow A_2^e;$$

(4) *mixed* iff σ is feasible, but is in none of the preceding three categories.

Notice that a 'mixed strategy' may be a mixture of concurrent and sequential strategies: in that

(a) for i odd, there may be situations in which between q_i and q_{i+1}, IP1 is performing information-gathering actions as well as IP2, while

(b) In other situations only IP2 may be performing information-gathering actions, while IP1 waits for the results.

Notice also that if σ is a sequential strategy, then each processor is idle at any time that the other processor is performing an action, whether the action is information-gathering or a communication action. Notice that a concurrent strategy in which

$$\alpha_j^1 \colon \mathbf{B}_j^1 \rightarrow \{0\}, \quad \text{for } j = 2, 3, \ldots, r$$

could almost be viewed as an independent strategy also; since in this case the computer performs all the information-gathering activities.

3. Efficient Dual Processor Strategies

In this section, we shall investigate some conditions under which a strategy is dominated by another feasible strategy; where we shall say that a strategy, σ, (*weakly*) *dominates* (respectively, *strictly dominates*) a second strategy, σ', iff

$$\Omega(\sigma) - \Gamma(\sigma) \geqslant \Omega(\sigma') - \Gamma(\sigma')$$

[respectively, $\Omega(\sigma) - \Gamma(\sigma) > \Omega(\sigma') - \Gamma(\sigma')$]. One of the uses to which we shall put these definitions is to develop the notion of an efficient strategy, which will be our main topic in this section.

In this paper, we shall not formally consider the impact of time upon the payoff received in a decision problem. However, a strategy which requires less time to implement than a second strategy, while obtaining as much information and costing no more, would generally be considered better than the second strategy. This is the motivation for the following definition.

DEFINITION 3.1. A feasible strategy

$$\sigma = \langle [(\mathbf{B}_1, \alpha_1^1)], [(\mathbf{B}_2^1, \alpha_2^1)(\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1)(\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)] \rangle$$

is *time dominated* by a feasible strategy

$$\sigma^* = \langle [(\mathbf{B}_1, \alpha_1^1)], [(\mathbf{B}_2^1{}^*\alpha_2^1)(\mathbf{B}_2^2{}^*, \alpha_2^2)], \ldots, [(\mathbf{B}_2^1{}^*, \alpha_r^1), (\mathbf{B}_r^2{}^*, \alpha_r^2)], [(\mathbf{B}_{r+1}^1{}^*, \delta)] \rangle$$

if

(a) $\Omega(\sigma^*) - \Gamma(\sigma^*) \geqslant \Omega(\sigma) - \Gamma(\sigma)$,
(b) $\bar{t}(\sigma) > \bar{t}(\sigma^*)$.

We shall say that a feasible strategy, σ, is *time efficient* iff there exists no feasible strategy, σ', such that σ' time dominates σ.

Our next two results enumerate two simple conditions under which a strategy will be strictly dominated, and thus provide (by the negation of these conditions) a pair of necessary conditions for a strategy to be optimal. The propositions themselves will probably strike the reader as being intuitively obvious: however, as such they provide an important check on the appropriateness of our theoretical model. Moreover, the conditions to be developed are readily verifiable and, because of this, will be useful in our definition of an efficient strategy.

PROPOSITION 3.1. *If*

$$\sigma = \langle [(\mathbf{B}_1, \sigma_1^1)], [(\mathbf{B}_2^1, \alpha_2^1)(\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1)(\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)] \rangle$$

is a feasible strategy and q_i *and* $q_{i+1} \in Q(\sigma)$ *are such that*

$$\alpha_{q_i}^1 \colon \mathbf{B}_{q_i}^1 \to A_1^c, \qquad \alpha_{q_{i+1}}^2 \colon \mathbf{B}_{q_{i+1}}^2 \to A_2^c \tag{1}$$

and

$$\mathbf{B}_{q_{i+1}}^1 \geqslant \mathbf{B}_{q_{i+1}}^2, \tag{2}$$

then σ *is a (strictly) dominated strategy.*

Proof. Since (2) holds, we see that, for each $B \in \mathbf{B}_{q_{i+1}}^1$, there exists $B' \in \mathbf{B}_{q_{i+1}}^2$ such that $B \subseteq B'$. Therefore

$$\mathbf{B}_{q_{i+1}}^1 + 1 = \bigcap(\mathbf{B}_{q_{i+1}}^1, \mathbf{B}_{q_{i+1}}^2) = \mathbf{B}_{q_{i+1}}^1. \tag{3}$$

In order to construct a strategy, $\hat{\sigma}$, which strictly dominates σ, we distinguish two cases.

(1) $q_{i+1} = q_p$. In this case we construct $\hat{\sigma}$ from σ by setting

$$\hat{\alpha}^1_{q_i}(B) = 0, \quad \text{for all } B \in \mathbf{B}^1_{q_i},$$

and

$$\hat{\alpha}^2_j(B) = 0, \quad \text{for } j = q_i, q_i + 1, \ldots, r;$$

while leaving everything else as in σ. It is then clear, using (3) that $\hat{\sigma}$ is feasible, that $\Omega(\sigma) = \Omega(\hat{\sigma})$, and that $\Gamma(\hat{\sigma}) < \Gamma(\sigma)$.

(2) $q_{i+1} < q_p$. In this case, it follows from our definition of a feasible strategy σ we have

$$(\forall B \in \mathbf{B}^1_{q_i+2}){:}\ \alpha^1_{q_i+2}(B) \in A^c_1,$$

and thus, once again using the definition of a feasible strategy, $\mathbf{B}^2_{q_i+2} = \mathbf{B}^2_{q_i+1}$, so that

$$\mathbf{B}^2_{q_i+2+1} = \bigcap(\mathbf{B}^1_{q_i+2}, \mathbf{B}^2_{q_i+2}) = \mathbf{B}^1_{q_i+2}.$$

Consequently, it follows that the strategy, $\hat{\sigma}$, having

$$\hat{\alpha}^1_{q_i}(B) = 0 \quad \text{for each } B \in \mathbf{B}^1_{q_i}$$

and

$$\hat{\alpha}^2_j(B) = 0 \quad \text{for each } B \in \mathbf{B}^2_j; \quad j = q_i, \ldots, q_{i+2}$$

with everything else left exactly as in σ, will (a) be feasible [since σ is], (b) have $\Omega(\hat{\sigma}) = \Omega(\sigma)$, and (c) $\Gamma(\hat{\sigma}) < \Gamma(\sigma)$. □

Proposition 3.1, and its proof, expresses the intuitively obvious fact that if at q_{i+1} IP2 will provide no new information to IP1, then a communication should not be planned at that point. Moreover, under these conditions, the program instructions planned for the point q_i, as well as any information-gathering actions planned for IP2 between the points q_i and q_{i+1}, were wasted as well. It is of some importance that two things be noticed about the above result.

In the first place, a special case of conditions (1) and (2) occurs when we have:

$$\alpha^2_j(B) = 0, \quad \text{for all } B \in \mathbf{B}^2_j; \quad j = q_i, \ldots, q_{i+1} - 1.$$

Secondly, the strategy $\hat{\sigma}$ constructed from σ in the proof of Proposition 3.1 may well *not* represent the best way to eliminate the inefficiency expressed by the fact that σ satisfies (1) and (2). Two other possibilities are that (a) we should modify σ by setting

$$\alpha^1_j(B) = 0, \quad \text{for all } B \in \mathbf{B}^1_j, \quad j = q_i + 1, \ldots, q_{i+1};$$

or that we should change the action functions for IP2 between $q_i + 1$ and q_{i+1} (or both). In other words, the only certain information provided by just the fact that σ satisfies (1) and (2) is that σ will be strictly dominated by the strategy $\hat{\sigma}$ constructed in the proof, however, $\hat{\sigma}$ may itself be dominated by other strategies.

Notice also that if in a feasible strategy,

$$\sigma = \langle[(\mathbf{B}_1, \alpha^1_1)], [(\mathbf{B}^1_2, \alpha^1_2)(\mathbf{B}^2_2, \alpha^2_2)], \ldots, [(\mathbf{B}^1_r, \alpha^1_r), (\mathbf{B}^2_r, \alpha^2_r)], [(\mathbf{B}^1_{r+1}, \delta)]\rangle$$

we have for some $q_i \in Q(\sigma)$ that

$$\alpha^1_{q_i}(B) \in A^c_1 \text{ for each } B \in \mathbf{B}^1_{q_i} \text{ and } \mathbf{B}^2_{q_i} \geqslant \mathbf{B}^1_{q_i}, \tag{4}$$

σ may nonetheless be efficient. Condition (4) will hold, for example, if $q_i > 1$,

$$\mathbf{B}^2_{q_i-1} \geqslant \mathbf{B}^1_{q_i-1},$$

and $q_i = q_{i-1} + 1$; all of which may well be perfectly reasonable. However, we do have the following proposition.

PROPOSITION 3.2. *Suppose*

$$\sigma = \langle[(\mathbf{B}_1, \alpha^1_1)], [(\mathbf{B}^1_2, \alpha^1_2)(\mathbf{B}^2_2, \alpha^2_2)], \ldots, [(\mathbf{B}^1_r, \alpha^1_r)(\mathbf{B}^2_r, \alpha^2_r)], [(\mathbf{B}^1_{r+1}\delta)]\rangle$$

is a feasible strategy such that $Q(\sigma) \neq \varnothing$, and that for some q_i and $q_{i+1} \in Q(\sigma)$, we have

$$\alpha^1_{q_i}\colon \mathbf{B}^1_{q_i} \to A^c_1, \qquad \alpha^2_{q_{i+1}}\colon \mathbf{B}^2_{q_{i+1}} \to A^c_2, \tag{5}$$

and

$$\mathbf{B}^2_{q_{i+1}} \geqslant \mathbf{B}^1_{q_{i+1}}. \tag{6}$$

Then if there exists some $j \in \{q_i + 1, \ldots, q_{i+1} - 1\}$ and $B \in \mathbf{B}^1_j$ such that

$$\alpha^1_j(B) \neq 0, \tag{7}$$

σ is a strictly dominated strategy.

Proof. Since $\mathbf{B}^2_{q_{i+1}} \geqslant \mathbf{B}^1_{q_{i+1}}$, we recall that for each $B \in \mathbf{B}^2_{q_{i+1}}$, there exists $B' \in \mathbf{B}^1_{q_{i+1}}$ such that

$$B \subseteq B';$$

and it follows that

$$\mathbf{B}^1_{q_{i+1}} + 1 = \bigcap\{\mathbf{B}^1_{q_{i+1}}, \mathbf{B}^2_{q_{i+1}}\}.$$

Therefore, we see that the strategy σ' obtained from σ by setting

$$\alpha^1_g(B) = 0 \text{ for each } B \in \mathbf{B}^1_g \text{ and each } g \in \{q_i + 1, \ldots, q_{i+1} - 1\},$$

while leaving everything else as in σ, is well-defined and feasible, and has $\Omega(\sigma) = \Omega(\sigma')$, while $\Gamma(\sigma) > \Gamma(\sigma')$.

DEFINITION 3.2. We shall say that a feasible strategy, σ, is *communication efficient* iff there exists no $q_i, q_{i+1} \in Q(\sigma)$ satisfy either (1) and (2) of Proposition 3.1, or (5)–(7) of Proposition 3.2.

We can now state our definition of an efficient stratgey as follows.

DEFINITION 3.3. A feasible strategy for **D**,

$$\sigma = \langle[(\mathbf{B}_1, \alpha_1^1)], [(\mathbf{B}_2^1, \alpha_2^1)(\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1)(\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)]\rangle$$

is *efficient* iff

(1) For each $j \in \{1, 2, \ldots, r\}\backslash Q(\sigma)$, each $s \in \{1, 2\}$, and each $B \in \mathbf{B}_j^s$, if $\alpha_j^s(B) = \hat{a} \neq 0$ then $\# \imath(B, \hat{a}) \geqslant 2$

(2) For each $j \in \{1, 2, \ldots, r\}$, and each $B \in \mathbf{B}_j^s$; we have the following: if there exists $d \in D$ such that $B \subseteq X_d$, then either:

(a) $\alpha_j^s(B) = 0$,

or

(b) $s = 2$, and there exists $h \in \{1, \ldots, p/2\}$ such that

$$j = q_{2h} \quad \text{and} \quad \alpha_j^2(B) \in A_2^c.$$

(3) σ is time efficient.

(4) σ is communication efficient.

(5) For each $B \in \mathbf{B}_{r+1}^1$, $\delta(B) \in D^*(B)$ (see definition 2.1.9).

We shall denote the set of efficient strategies for **D** by $\Sigma^e(\mathbf{D})$.

We can show that $\Sigma^e(\mathbf{D})$ is a *dominating set* for **D**; that is, $\Sigma^e(\mathbf{D})$ is non-empty; and given any feasible strategy for **D**, σ, there exists $\sigma^* \in \Sigma^e(\mathbf{D})$ such that σ^* weakly dominates σ. In fact, we have shown in Propositions 3.1 and 3.2 that if σ is not communication efficient, then there exists a strategy, $\hat{\sigma}$, which strictly dominates σ; and it is an immediate consequence of the definition that if σ is not time efficient, then there exists a feasible strategy, σ^*, which (weakly) dominates σ.

Similarly, if σ does not satisfy condition 1, then there exists $j \in \{1, \ldots, r\}\backslash Q(\sigma)$, $s \in \{1, 2\}$, and $B \in \mathbf{B}_j^s$ such that

$$\alpha_j^s(B) = \hat{a} \neq 0 \quad \text{and} \quad \# \imath(B, \hat{a}) < 2.$$

But then we must have

$$\# \imath(B, \hat{a}) = 1;$$

that is, the information structure defined on B by $\hat{a}$ is simply $\{B\}$; and thus performing experiment $\hat{a}$ provides no new information at all at this point. Therefore, the strategy σ^* obtained from σ by changing $\alpha_j^s(B)$ from

$$\alpha^s(B) \neq \hat{a} \quad \text{to} \quad \alpha^s(B) = 0,$$

while leaving everything else unchanged; will also be feasible, obtain the same expected gross payoff and will have a lower expected cost than σ.

Next, we note that if σ fails to satisfy condition 2 of Definition 3.3, then a costly experiment (or communication by IP1 to IP2) is performed at a point where there is no information which can improve the final decision. Clearly, this is wasteful, and

we leave to the interested reader the details of the intuitively obvious argument that if σ does not satisfy condition 2, then there exists a feasible σ^* which strictly dominates σ.

Finally, we note that if σ does not satisfy condition 5, then it is obvious that one can obtain a feasible σ^* which strictly dominates it; one simply re-defines δ to obtain δ^* satisfying

$$\text{for each } B \in \mathbf{B}_{r+1}^{1}, \quad \delta^*(B) \in D^*(B).$$

Notice that in the preceding paragraphs, we showed that if σ is a feasible strategy which violates one of the conditions of Definition 3.3, then we can use it to define a new feasible strategy, σ^*, which corrects the initial inefficiency of σ; and *violates no conditions of Definition 3.3 which are not violated by* σ. If σ^* still violates one of the conditions of Definition 3.3, we can use it to define a new feasible strategy, σ^{**}, which violates one fewer of the conditions of Definition 3.3. Since all feasible strategies are finite, this process must terminate with an efficient, dominating strategy after a finite number of steps.

Finally, we note that the strategy, $\hat{\sigma}$, defined by

$$\hat{\alpha}_j^s(B) = 0, \quad \text{for } j = 1, \ldots, r;\ s = 1, 2;$$

and which takes $\hat{\delta}(X)$ to be any $\hat{d} \in D$ satisfying

$$\sum_{x \in X} \phi(x)\omega(x, \hat{d}) \geqslant \sum_{x \in X} \phi(x)\omega(x, d), \quad \text{for all } d \in D$$

(notice that we will have $\mathbf{B}_{r+1}^{1} = \{X\}$ for this strategy); is efficient, that is, $\hat{\sigma} \in \Sigma^e(\mathbf{D})$. Therefore, we have proved Theorem 3.1.

THEOREM 3.1. *The set of efficient strategies for* $\mathbf{D}$, $\Sigma^e(\mathbf{D})$, *is a dominating set for* $\mathbf{D}$.

The point of establishing Theorem 3.1 is that is shows that in seeking an optimal strategy for $\mathbf{D}$, we can confine our search to those strategies which are in $\Sigma^e(\mathbf{D})$. That is, as the reader can easily see, if σ^* is an efficient strategy for $\mathbf{D}$ which satisfies

$$(\forall \sigma \in \Sigma^e(\mathbf{D})) : \Omega(\sigma^*) - \Gamma(\sigma^*) \geqslant \Omega(\sigma) - \Gamma(\sigma),$$

then σ^* is optimal for $\mathbf{D}$. Moreover, in our proof of Theorem 3.1, we have actually also established the following.

COROLLARY. *If* σ^* *is optimal for* $\mathbf{D}$, *then* σ^* *satisfies conditions* 1, 2, 4 *and* 5 *of Definition* 3.3.

It is worthwhile digressing briefly at this point to reconsider the effect of some conventions we used in our definition of viable strategies (Definition 2.3.2). Specifically, our definition required that a viable strategy (a) takes exactly r steps, and (b) along each branch of the strategy each communication point occurs after exactly the same number of information-gathering steps as occur between communication points

along any other branch. In the context of a dual information-processor frame-work, either or both of these conventions may appear to be restrictive; that is, it may appear that the best dual-processor strategies may satisfy the time constraints, etc., but violate one or both of these conventions. However, both of these conventions can be adopted without loss of generality; more precisely, the collection of all the strategies which satisfy both of these conventions is a dominating set. We can see this as follows. Suppose IPi ($i = 1, 2$) has experiment and communication sets as follows:

$$A_i^e = \{e_{i0}, e_{i1}, \ldots, e_{im_i}\} \quad \text{and} \quad A_i^c = \{a_{i1}, a_{i2}, \ldots, a_{in_i}\}, \quad \text{for } i = 1, 2$$

and suppose, for the sake of convenience, that $m_1 \geqslant m_2$ (the reader will have no trouble filling in the details of a similar argument for the case where $m_2 > m_1$). Consider the sequential strategy which begins with IP1 taking all n_1 communication actions, after which IP1 does nothing while IP2 takes action e_{21}, and then takes all n_2 communication actions. After this is done, IP1 takes action e_{11}, then takes all n_1 communication actions; IP2 then takes action e_{22}, followed by all n_2 communication actions, and so on until IP2 has performed all m_2 information-gathering actions. IP2 then performs his remaining $m_1 - m_2$ information-gathering actions. For purposes of this discussion, we shall call this our *model strategy*. Notice that it takes exactly $r \equiv m_2(n_1 + n_2 + 2) + (m_1 - m_2)$ steps; and clearly, since the final information structure for this strategy is the finest attainable, and the strategy uses all forms of communication available at each step, any strategy which involves more steps (not counting the null information-gathering actions, e_{i0}) must be strictly dominated (of course, both strategies may violate the time constraint, but we shall ignore this for the moment).

Now let us add the communication actions a_{i0} ($i = 1, 2$) to the sets A_i^c, where we suppose $c(a_{i0}) = 0$, and that a_{i0} leaves both information structures (for both IP1 and IP2, that is) unchanged; in other words, a_{i0} is the null communication action, for each i. Without going into the nast details of a formal argument, it is clear that any efficient strategy can be regarded as having been obtained by substituting alternative actions, or the null action for the values given by the action functions in our model strategy. (Notice that we can view things in this way because no efficient strategy would use the same experimental action twice along a given path. Therefore, if we have changed q experimental actions along a given path in our model strategy, there will be as many more points where changes can still be made as there are experimental actions which have not yet been taken.) Moreover, by replacing the communication actions in our model strategy by a_{i0}'s where appropriate, we can maintain the formality of treating communication actions as taking place after each information-gathering action is taken, while allowing actual communication points to occur at different points along different paths. Notice also that this can be done while maintaining the convention of treating each strategy as if it takes exactly r steps. Thus, we can see that the family of strategies which involve exactly r information-gathering actions, and symmetric communication points is a dominating set.

We conclude this section by looking at some properties which characterize undominated strategies; obtaining in the process what amounts to some working formulas for the construction of a 'good' dual processor strategy. The list which we shall develop here is by no means exhaustive, and in fact, additional results of this nature are presented in Jacob (1986).

Our first result makes note of the common-sense consideration that if two actions are available at a given point, and one action provides at least as much information, while costing less than the second action, then a strategy which takes the second action at that point will necessarily be strictly dominated. The result is presented here without proof, but it can be proved by an argument very similar to the proof of Proposition 3.2.1 in Moore and Whinston (1986).

PROPOSITION 3.3. *Suppose*

$$\sigma = \langle[(\mathbf{B}_1^1, \alpha_1^1)], [(\mathbf{B}_2^1, \alpha_2^1), (\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1), (\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)]\rangle$$

is a feasible strategy for $\mathbf{D}$, *and that there exists* $j \in \{1, \ldots, r\}$, $s \in \{1, 2\}$ $B^* \in \mathbf{B}_j^s$, *and* $a^* \in A_s^e$ *such that*

$$\iota(B^*, a^*) \geqslant \iota(B^*, \alpha_j^s(B^*)),$$

$$c(a^*) < c(\alpha_j^1(B^*)) \quad \text{and} \quad t(a^*) \leqslant t(\alpha_j^s(B^*))$$

then σ *is dominated.*

Proposition 3.3 can be generalized, in that it suffices that the strategy σ^*, which differs from σ (essentially) only in that $\alpha_j^{*s}(B^*)$ is taken to be a^*, rather than $\alpha_j^s(B^*)$, satisfies the time constraint. However, the exact statement of the generalized result is too complicated to be worthwhile. Similar considerations apply to our next result.

PROPOSITION 3.4. *Suppose*

$$\sigma = \langle[(\mathbf{B}_1^1, \alpha_1^1)], [(\mathbf{B}_2^1, \alpha_2^1), (\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1), (\mathbf{B}_r^1, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)]\rangle$$

is a feasible strategy for $\mathbf{D}$, *and for some* $B^* \in \mathbf{B}_{r+1}^1$ *and* $j \in \{1, \ldots, r\}\backslash Q(\sigma)$, *we have*

$$\beta_j^1(B^*) = B^* \quad \text{and} \quad a^1(j, B^*) = 0 \quad \text{(see Definition 2.1.7)},$$

and suppose there exists $\bar{a} \in A_1^e$ *satisfying*

$$\text{(a)} \quad \sum_{B' \in \iota(B^*, \bar{a})} \pi(B')v(B') - \sum_{x \in B^*} \phi(x)\omega[x, \delta B^*)] > \pi(B^*)c(\bar{a}),$$

and

$$\text{(b)} \quad \sum_{j=1}^{q_i - 1} t[a^1(j, B^*)] + \sum_{h=1}^{p/2} \tau_h(B^*) + t(\bar{a}) \leqslant T$$

Then σ *is strictly dominated.*

Proof. The proof for the dual processor case is identical to that for the single processor; and since the latter result is proved in Moore and Whinston (1986), Proposition 3.2.3, we shall not repeat the argument here.

4. The 'Uninformed Action Principle'

In the present context, the basic idea of what we shall call the *uninformed action principle* is simply that information-gathering actions which do not make use of the results of prior information-gathering actions should always be done as early as possible in a strategy. In order to explain more exactly what we mean, we begin with the following definition.

DEFINITION 4.1. Let

$$\sigma = \langle[(\mathbf{B}_1^1, \alpha_1^1)], [(\mathbf{B}_2^1, \alpha_2^1), (\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1), (\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)]\rangle$$

be a feasible strategy for $\mathbf{D}$, let $j \in \{1, \ldots, r\}\backslash Q(\sigma)$, and let $s \in \{1, 2\}$. We shall say that α_j^s is an *uninformed action* (function) iff there exists a non-null action, $a^* \in A_s^e$, satisfying

$$(\forall B \in \mathbf{B}_j^s): \alpha_j^s(B) = a^*. \tag{1}$$

[Hereafter, we shall abbreviate the statement in (1) by '$\alpha_j^s \equiv a^*$'.] ∎

We call an action function satisfying (1) 'uninformed' because the same action, a^*, is performed at the jth step regardless of the outcomes of any actions performed prior to that point. If σ is a feasible strategy for $\mathbf{D}$, and $\alpha_1^1 \notin A_1^c$, then α_1^1 is necessarily an uninformed action. In fact, it may be the case that the optimal strategy involves taking several uninformed actions. The principle which we want to make note of is simply this: under the present assumptions, one can never lose, and may gain a great deal, by moving uninformed actions to the beginning of a strategy. Consider first the following result.

PROPOSITION 4.1. *Suppose*

$$\sigma = \langle(\mathbf{B}_1, \alpha_1), (\mathbf{B}_2, \alpha_2), \ldots, (\mathbf{B}_r, \alpha_r), (\mathbf{B}_{r+1}, \delta)\rangle$$

is a feasible strategy for a single information-processor (so that σ satisfies Definition 2.2.2); *and suppose that $j, k \in \{1, \ldots, r\}$ and $a^* \in A$ are such that a^* is non-null; $k > j \geqslant 2$, and $\alpha_j \equiv a^*$. Then σ is always weakly dominated, and may be strictly dominated, by a strategy, $\hat{\sigma}$, in which we have*

$$(\forall B \in \hat{\mathbf{B}}_j): \hat{\alpha}_j(B) = a^*.$$

Proof. 1. Suppose initially that $k = j + 1 \geqslant 3$. Under the given assumptions, we shall define an alternative feasible strategy, $\hat{\sigma}$, by keeping everything the same as in σ except the action functions α_j and α_k; which we change as follows. We first define

$\hat{\alpha}_j$ on $\hat{\mathbf{B}}_j = \mathbf{B}_j$ by

$$\hat{\alpha}_j(B) \;=\; a^*, \quad \text{for each } B \in \mathbf{B}_j. \tag{1}$$

We then define $\hat{\alpha}_{j+1} = \hat{\alpha}_k$ by

$$\hat{\alpha}_{j+1}(B') \;=\; \alpha_j[\beta_j(B')], \quad \text{for } B' \in \hat{\mathbf{B}}_{j+1} \tag{2}$$

(see Definition 2.1.7); noting that (2) yields a well-defined action function, since $\hat{\mathbf{B}}_{j+1} \geqslant \mathbf{B}_j$.

Having defined $\hat{\alpha}_j$ and $\hat{\alpha}_{j+1}$, it remains only to prove (for the present case, where $k = j + 1$), that we can take

$$(\hat{\mathbf{B}}_h, \hat{\alpha}_h) \;=\; (\mathbf{B}_h, \alpha_h), \quad \text{for } h = 1, \ldots, r;\, h \neq j, h \neq j + 1 \tag{3}$$

and

$$(\hat{\mathbf{B}}_{r+1}, \hat{\delta}) \;=\; (\mathbf{B}_{r+1}, \delta). \tag{4}$$

To do this, it obviously suffices to establish that

$$\hat{\mathbf{B}}_{j+2} \;=\; \mathbf{B}_{j+2}. \tag{5}$$

To prove that (5) holds, we first note that

$$\hat{\mathbf{B}}_{j+2} \;=\; \bigcup_{B \in \mathbf{B}_j} \hat{\mathbf{B}}_{j+2}(B); \tag{6}$$

where $\hat{\mathbf{B}}_{j+2}(B)$ is the information structure induced on B after the $(j + 1)$st step (see Moore and Whinston (1986), Section 2). Thus, for $B \in \hat{\mathbf{B}}_j = \mathbf{B}_j$:

$$\hat{\mathbf{B}}_{j+1}(B) \;=\; \iota(B, \hat{\alpha}_j(B)] \;=\; \iota(B, a^*) \;=\; \bigcap(B, \mathbf{M}_{a^*}) \tag{7}$$

and

$$\hat{\mathbf{B}}_{j+2}(B) \;=\; \bigcup_{B' \in \mathbf{B}_{j+1}(B)} \iota(B', \hat{\alpha}_{j+1}(B')]. \tag{8}$$

Now, let $B \in \mathbf{B}_j$ be fixed for the moment. From our definition of $\hat{\alpha}_{j+1}$, we see that, for each $B' \in \hat{\mathbf{B}}_{j+1}(B)$, $\hat{\alpha}_{j+1}(B') = \alpha_j(B)$ and thus, defining $\alpha_j(B) = \bar{a}$, we have from (8) and Proposition 2.1.1 that

$$\hat{\mathbf{B}}_{j+2}(B) \;=\; \bigcup_{B' \in \hat{\mathbf{B}}_{j+1}(B)} \bigcap(B', \mathbf{M}_{\bar{a}}) \;=\; \bigcap[\hat{\mathbf{B}}_{j+1}(B), \mathbf{M}_{\bar{a}}], \tag{9}$$

Substituting (7) into (9), we then obtain

$$\begin{aligned} \hat{B}_{j+2} \;&=\; \bigcap[\bigcap(B, \mathbf{M}_{a^*}), \mathbf{M}_{\bar{a}}] \\ &=\; \bigcap[\mathbf{M}_{a^*}, \bigcap(B, \mathbf{M}_{\bar{a}})] \;=\; \bigcap[\bigcap(B, \mathbf{M}_{\bar{a}}), \mathbf{M}_{a^*}], \end{aligned} \tag{10}$$

where the last two equalities are by Proposition 2.1.1. However, by similar reasoning,

$$\mathbf{B}_{j+2}(B) \;=\; \bigcap[\bigcap(B, \mathbf{M}_{\bar{a}}), \mathbf{M}_{a^*}]$$

and, since B was an arbitrary element of $\mathbf{B}_j$, we see that (5) holds.

Note. In the remainder of this section we shall refer to a strategy $\hat{\sigma}$ obtained from a given feasible strategy, σ, by the formulas (1)–(4) as the *strategy obtained from σ by interchanging the jth and $(j + 1)$st action functions.*

2. Having established our result for the case where $k = j + 1$, we see that the extension to the general case is entirely straightforward; for if $k > j \geqslant 2$, it follows from part 1 of our proof that the strategy σ^* obtained from σ by interchanging the kth and $(k - 1)$st action functions is feasible and (weakly) dominates σ. Similarly, if $k - 1 > j$, the strategy σ^{**}, obtained from σ^* by interchanging the $(k - 1)$st and $(k - 2)$nd actions is feasible and dominates σ^* (and thus σ as well), and so on. □

While the above result establishes only that the strategy which interchanges the jth and kth action functions weakly dominates σ, it will generally be easy to construct an alternative feasible strategy which *strictly* dominates σ. In order to see why this is so, suppose that in the feasible strategy σ, the jth action is informed, but that $\alpha_{j+1} \equiv a^*$, for some nonnull action, a^*. In this case if we modify σ by setting $\alpha_j \equiv a^*$, the sub-problems one needs to solve at the $(j + 1)$st stage (that is, the problem of how to proceed after each possible outcome of the actions up to and including the jth action) are simpler and easier to solve, in principle. As a matter of fact, it may turn out that on a portion of its domain, the action function defined in (2), above, is inefficient; obtaining no new information at all. We can illustrate this point with a very simple example, as follows.

Suppose our information actions can be considered to be subdividing the unit square, with information structures as shown in Figure 1, where the number corresponds to the information-gathering action (or experiment).

Suppose further that each action results in the same cost, $c > 0$, that the probability distribution is uniform, and that $r = 3$. We shall work with the information-gathering strategy whose decision tree representation is given in Figure 2; where we note that the third action is uninformed.

Figure 3 then illustrates the sequence of information structures on X obtained with this strategy. If we modify the strategy σ by interchanging the 2nd and 3rd action functions, we obtain the strategy $\hat{\sigma}$ whose decision tree representation and information structure sequence are illustrated in Figures 4 and 5, respectively.

Figure 4 indicates the strategy $\hat{\sigma}$ which is obtained by interchanging the 2nd and 3rd action functions. However, notice that the actions at the points where question marks

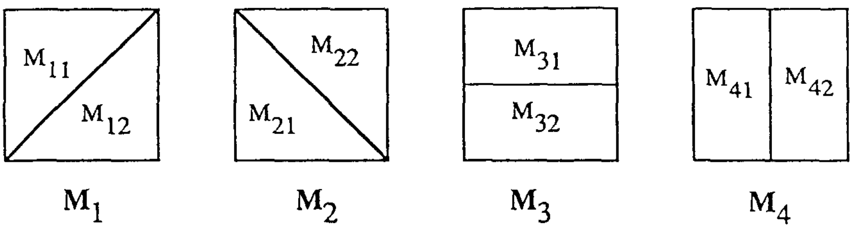

Fig. 1.

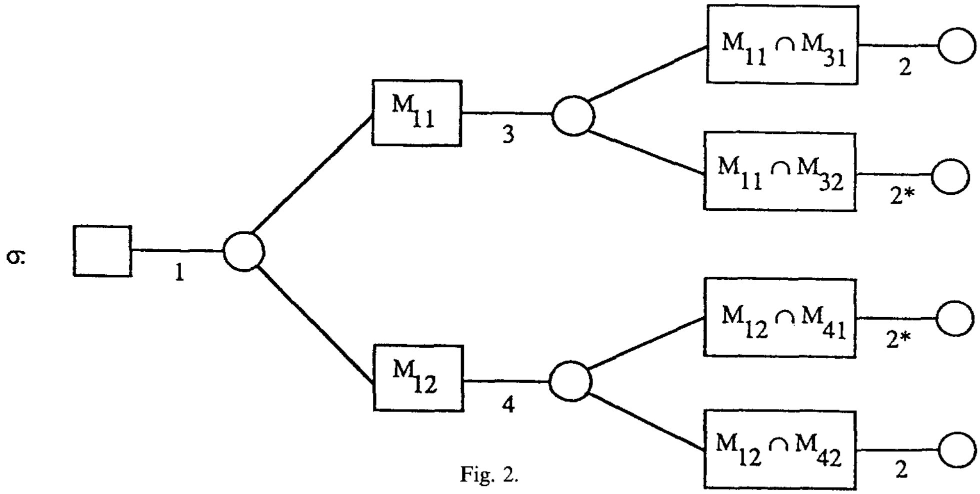

Fig. 2.

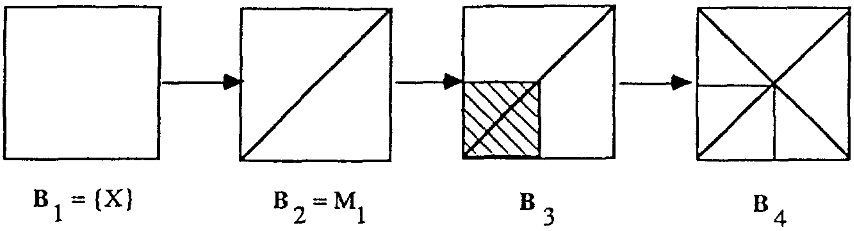

Fig. 3.

appear in Figure 4 are *inefficient*; for example, we have

$$M_{11} \cap M_{22} \subseteq M_{31},$$

and thus no new information can be obtained by taking action 3 at that point. If we modify the strategy $\hat{\sigma}$ by eliminating these inefficient actions (corresponding to taking the null action on the shaded sets in Figure 5), we obtain a new strategy, $\hat{\sigma}^*$, whose expected cost is given by:

$$\Gamma(\hat{\sigma}^*) \;=\; c + c + c/2 \;=\; 5c/2$$

compared with the expected cost of σ, which is

$$\Gamma(\sigma) \;=\; c + c + c \;=\; 3c,$$

despite the fact that σ and $\hat{\sigma}^*$ yield precisely the same final information structure! It must be admitted, however, that this comparison is a bit unfair; the strategy σ also involves taking some inefficient actions. These actions are indicated with the asterisks in Figure 2, and correspond to the fact that had the null action been taken on the sets which are shaded in Figure 3, then the resulting strategy, σ^*, would have resulted in

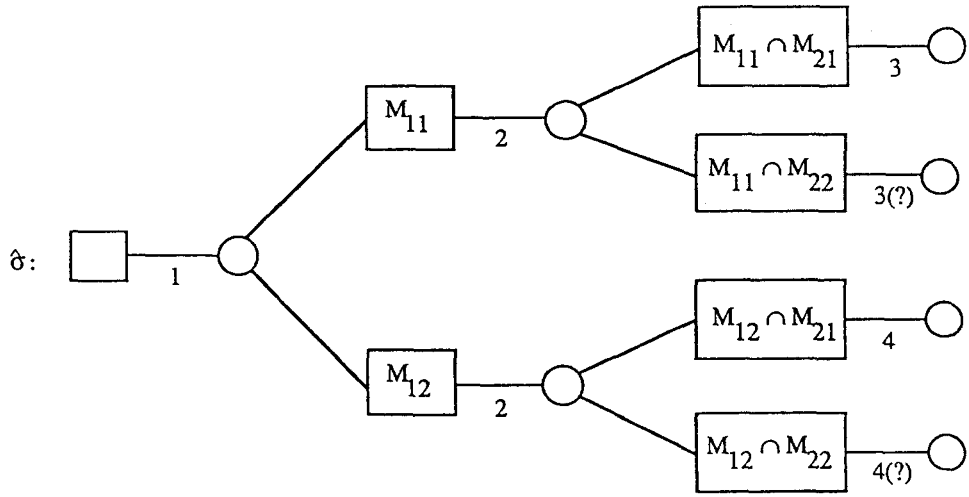

Fig. 4.

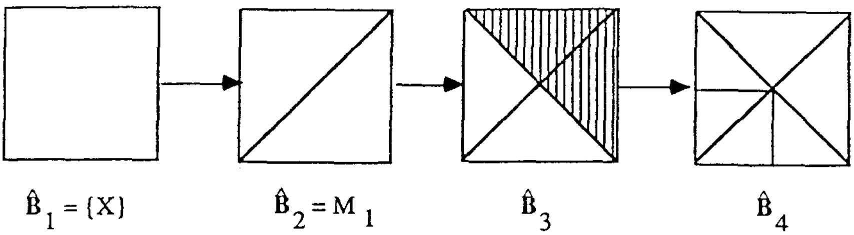

Fig. 5

the same final information structure as σ, but would have had expected costs of

$$\Gamma(\sigma^*) = c + c + (1 - \tfrac{1}{4})c = \frac{11c}{4}.$$

However, $\Gamma(\sigma^*)$ still has strictly higher expected costs than does $\hat{\sigma}$.

In the two-information-processor case, the possibilities for gain in moving an uninformed action forward become even greater, as long as the uninformed action was originally planned to take place after the first communication from IP2 to IP1. The following result sets out the basic relationship.

PROPOSITION 4.2. *Suppose*

$$\sigma = \langle[(\mathbf{B}_1, \alpha_1)], [(\mathbf{B}_2^1, \alpha_2^1), (\mathbf{B}_2^2, \alpha_2^2)], \ldots, [(\mathbf{B}_r^1, \alpha_r^1), (\mathbf{B}_r^2, \alpha_r^2)], [(\mathbf{B}_{r+1}^1, \delta)]\rangle$$

is a feasible dual processor strategy such that for some $a^* \in A_1\backslash\{0\}$ *and some* $j \in \{1, \ldots, r\}\backslash Q(\sigma)$, *where* $Q(\sigma) = \{q_1, q_2, \ldots, q_p\}$, *we have*

$$q_h < j < q_{h+1},\ h \text{ is even, and } \alpha_j^1 \equiv a^*. \tag{11}$$

Then σ is (weakly) dominated, and may be strictly dominated, by a strategy $\hat{\sigma}$ for which, writing

$$Q(\hat{\sigma}) = \{\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_p\},$$

we have

$$\hat{q}_k = q_k, \quad \text{for all } k \neq h, \qquad q_h = q_h + 1 \tag{12a}$$

and

$$(\forall B \in \mathbf{B}^1_{q_{h-1}+1}) : \hat{\alpha}^1_{q_{h-1}+1}(B) = a^*. \tag{12b}$$

Proof. Since the details of a full proof are messy but straightforward, we shall provide a slightly sketchy proof here.

If $h \in \{1, \ldots, p\}$ is such that $q_h < j < q_{h+1}$ (where j is such that $\alpha^1_j \equiv a^*$), it follows from Proposition 4.1 that σ is (weakly) dominated by the strategy σ^* for which $\alpha^{*1}_{q_h+1} \equiv a^*$, with α^{*1}_k being defined from α^1_{k-1}, for $k = q_h + 2, \ldots, j$, in the fashion indicated in Equation (2), above. Obviously we can then interchange the communication, α_{q_h}, with $\alpha^{*1}_{q_h+1}$ to obtain a third strategy, σ^{**}, which (weakly) dominates σ^*, and thus dominates σ as well. Moreover, σ^{**} will also satisfy the time constraint; notice first that, for each $B \in \mathbf{B}^{*1}_{r+1} = \mathbf{B}^1_{r+1}$ [recall the definition of $t(B; \sigma)$ from Definition 2.3.4]:

$$t(B; \sigma^*) \equiv \sum_{j=1}^{q_1-1} t[a^1(j, B)] + \sum_{h=1}^{p/2} \tau_h(B) = t(B; \sigma) \leqslant T. \tag{13}$$

Since h is even, it follows that for each $B \in \mathbf{B}^{*1}_{r+1} = \mathbf{B}^1_{r+1}$ (see Definitions 2.3.3 and 2.3.4):

$$\begin{aligned} t(B; \sigma^{**}) &= \sum_{j=1}^{q_1-1} t[a^1(j, B)] + \sum_{k=1}^{(h-2)/2} \tau_k(B) + \\ &\quad \operatorname{Max}\left\{ \sum_{j=q_{h-1}+1}^{q_h-1} t[a^1(j, B)] + t(a^*), \sum_{j=q_{h-1}+1}^{q_h-1} t[a^2(j, B)] + t(0) \right\} + \\ &\quad t[a^1(q_{h-1}, B)] + t[a^2(q_h, B)] + \sum_{j=q_h+1}^{q_{h+1}-1} t[a^1(j, B)] + \sum_{k=(h+2)/2}^{p/2} \tau_k(B) \\ &\leqslant \sum_{j=1}^{q_1-1} t[a^1(j, B)] + \sum_{k=1}^{(h-2)/2} \tau_k(B) \\ &\quad + \operatorname{Max}\left\{ \sum_{j=q_{h-1}+1}^{q_h-1} t[a^1(j, B)], \sum_{j=q_{h-1}+1}^{q_h-1} t[a^2(j, B)] \right\} + \\ &\quad t[a^1(q_{h-1}, B)] + t[a^2(q_h, B)] + \sum_{j=q_h+1}^{q_{h+1}-1} t[a^1(j, B)] + t(a^*) + \\ &\quad \sum_{k=h+2)/2}^{p/2} \tau_k(B) \\ &= t(B; \sigma^*); \end{aligned} \tag{14}$$

and from (13) and (14) we then obtain:

$$t(B; \sigma^{**}) \leqslant T. \tag{15}$$

Applying Proposition 4.1 once again, we see that σ^{**} is (weakly) dominated by the strategy $\hat{\sigma}$ defined in (12a) and (12b); and clearly for each $B \in \hat{\mathbf{B}}^1_{r+1} = \mathbf{B}^{1**}_{r+1}$:

$$t(B; \hat{\sigma}) = t(B; \sigma^{**}). \qquad \square$$

While the above result only establishes that if σ satisfies (11), a weakly dominating strategy exists which has the form specified in (12); it will generally be easy to construct an alternative strategy, σ^*, which *strictly* dominates σ. The reasons for this include those already discussed in connection with Proposition 4.1; but to these we may add another reason which is almost certainly very important in practice, as follows. Consider the strategies σ^{**} and σ^* constructed in the above proof. As shown there, we must have, for each $B \in \mathbf{B}^{1*}_{r+1} = \mathbf{B}^{1**}_{r+1} = \mathbf{B}^1_{r+1}$:

$$t(B; \sigma^{**}) \leqslant t(B; \sigma^*). \tag{16}$$

Equality can hold in (16) only if (see (14), above):

$$\sum_{j=q_{h-1}+1}^{q_h-1} t[a^1(j, B)] \geqslant \sum_{j=q_{h-1}+1}^{q_h-1} t[a^2(j, B)],$$

in which case any action function $\hat{\alpha}^2_{q_{h-1}+1}$ for which

$$(\forall B \in \mathbf{B}^2_{q_{h-1}+1} : t[\alpha^2_{q_{h-1}+1}(B)] \leqslant t(a^*)$$

will continue to satisfy the time constraint. Thus, if equality holds in (16), we may profitably be able to have IP2 perform more information-gathering actions between q_{h-1} and $\hat{q}_h$. On the other hand, if

$$t(B; \sigma^{**}) < t(B; \sigma^*)$$

IP1 may be able to profitably perform more information-gathering actions between q_h and q_{h+1}. In more common-sense terms, if IP1 will perform the same actions regardless of what IP2 communicates at q_h, then time is necessarily being wasted somewhere, as compared to IP1's undertaking this same fixed action between q_{h-1} and q_h.

5. Conclusion

In this paper, a model was proposed which allows one to investigate the issue of efficient information acquisition when two information processors are present. We developed a class of strategies ('efficient strategies') which we have shown to be a dominating set (Theorem 3.1), and thus this result has the practical advantages of enabling one, when searching for an optimal strategy, to confine one's search to the set of 'efficient strategies' rather than the much larger set of all feasible strategies.

We have also set out further necessary conditions for a strategy to be optimal; as well as developing what we call the 'uninformed action principle', which we feel may be particularly important in studying such decision problems, as well as in designing efficient decision support systems.

The model proposed here has been utilized by Fernandez (1988) to analyze the audit staffing decision. She has also utilized the model to analyze the use of information-gathering actions belonging to two categories, namely compliance and substantive tests. The concepts proposed here were used to analyze the categorization problem in a dual processor framework by Jacob, Moore and Whinston (1988). Although the model has been applied to the above areas further work needs to be done.

To mention just one area in which further work clearly needs to be done, consider for a moment the question of how we should really treat the communication actions in our model. Given the general framework of our model, it seems natural to treat communications from IP1 to IP2 as being based on requests to perform actions (experiments) in A_2^e. However, since one clearly would want to allow for multiple actions between communication points, and since these multiple actions will greatly require contingency instructions, it seems natural to allow strategies to be communicated from IP1 to IP2. Thus, for example, if after the qth step it is known that the true state, x^*, is an element of $B^* \subseteq X$, then any strategy of the form

$$\sigma = \langle(\mathbf{B}_1, \alpha_1), \ldots, (\mathbf{B}_p, \alpha_p)\rangle, \tag{1}$$

where

$$\mathbf{B}_1 = \{B^*\}, \alpha_t: \mathbf{B}_t \to A_2^e, \quad \text{for } t = 1, \ldots, p, \text{ and } p \leqslant r - q, \tag{2}$$

is an allowable communication. We might then base the assumed cost function on strategies of length one, so that, for example, it might be reasonable to assume that for each σ of the form (1) for which $p = 1$ (i.e., for each simple request for IP2 to conduct a single experiment and report back the result), we have

$$c(\sigma) = \gamma_1,$$

where γ_1 is a positive constant. For σ of the form:

$$\sigma = \langle(\mathbf{B}_1, \alpha_1), (\mathbf{B}_2, \alpha_2)\rangle \tag{3}$$

[that is, for s of the form (1), and with $p = 2$], we might then suppose:

$$c(\sigma) = \gamma_1 + (\# \imath[B^*, \alpha_1(B^*)])\gamma_1,$$

and so on. This particular approach assumes a sort of linearity in communications costs, which may not be all that reasonable an assumption. On the other hand, perhaps it is reasonable enough to assume that the monetary costs take this linear form, but that the time costs are non-linear, that is, if we assume that the time required for a simple communication [where σ is of the form (1) with $p = 1$] is some positive constant, τ_1, we may then wish to assume that if σ is of the form (3) that

$$t(\sigma) > \tau_1 + (\# \imath[B^*, \alpha_1(B^*)])\tau_1.$$

Obviously, the question of whether these cost and time-requirement functions are or are not linear is of fundamental importance in determining the form of the optimal information-acquisition strategy; however, this is not a question which we have attempted to address in this paper. On the other hand, the formal model introduced here does provide a natural framework in which to formulate and investigate this sort of problem; and we believe that the investigation of this sort of issue is a logical next step in this area of research.

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