MATHEMATICAL GAMES

The capture of the monster: a mathematical group with a ridiculous number of elements

by Martin Gardner

What's purple and commutes?
An Abelian grape.
—Anonymous mathematical riddle, ca. 1965

For the past six years all over the world experts in the branch of abstract algebra called group theory have been struggling to capture a group known as the monster. The name of the group is related to its size: the number of elements in the monster is 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000, or $2^{46} \times 3^{20} \times 5^{9} \times 7^{6} \times 11^{2} \times 13^{3} \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$. Robert L. Griess, Jr., a mathematician at the University of Michigan who is currently on sabbatical leave at the Institute for Advanced Study in Princeton, N.J., prefers to identify the monster with its proper mathematical symbol, F_1. In January, Griess (his name rhymes with rice) proved that the monster exists by actually constructing it. The news is enormously exciting to group theorists because it brings them closer to completing a task that has occupied them for more than a century: the classification of all groups.

The colorful story of this undertaking begins with a bang. In 1832 Évariste Galois, a French mathematical genius and student radical, was killed by a pistol shot in an idiotic duel over a woman. He was not yet 21. Some early, fragmentary work had already been done on groups, but it was Galois who laid the foundations of modern group theory and even named it, all in a long, sad letter that he wrote to a friend the night before he died.

What is a group? Roughly speaking, it is a set of operations performed on something, with the property that if any operation in the set is followed by any operation in the set, the outcome can also be reached by a single operation in the set. The operations are called the elements of the group, and their number is called the order of the group.

Before going on to a more precise definition let us consider an example. You are standing at attention and must carry out any of four commands: "Do nothing," "Left face," "About face" and "Right face." Now suppose you execute a left face followed by an about face. A sequence of this kind will be called a multiplication of the two operations. Note that the "product" of this particular multiplication can be reached by the single operation right face. This set of four operations is a group because it meets the following axioms.

1. Closure: The product of any pair of operations is equivalent to a single operation in the set.

2. Associativity: If the product of any two operations is followed by any operation, the result is the same as following the first operation with the product of the second and the third.

3. Identity: There is just one operation that has no effect, in this case doing nothing.

4. Inverse: For every operation there is an inverse operation such that executing an operation and then its inverse is equivalent to executing the identity operation. In this example left face and right face are inverses of each other, whereas do nothing (the identity) and about face are each their own inverse.

Any set of operations that satisfies these four axioms is a group, and the group of four commands I have just described is called the cyclic 4-group because it can also be modeled by the cyclic permutations of four objects in a row. (In a cyclic permutation of a set of ordered elements the first element moves into the second position, the second element moves into the third position and so on, with the last element moving into the first position.) Label the four objects 1, 2, 3 and 4 and assume that they are lined up in numerical order: 1234. The identity operation, which I shall call *I*, leaves the order of the objects unaltered. Operation *A* permutes them to 4123, *B* to 3412 and *C* to 2341 This group can be completely characterized by the "multiplication" table at the top right in the illustration at the left. Each cell in the table gives the operation that is equivalent to performing the operation indicated at the left end of its row followed by the operation indicated at the top of its column. If a similar construction is carried out for the first model, letting *I, A, B* and *C* stand for the four commands ("Do nothing," "Left face" and so on) in their listed order, the same table results, proving that the cyclic 4-group and the group of four commands are isomorphic, or equivalent.

Note that the table for the cyclic 4-group is symmetrical about one of its

IDENTITY GROUP

	I
I	I

CYCLIC 2-GROUP

	I	A
I	I	A
A	A	I

CYCLIC 3-GROUP

	I	A	B
I	I	A	B
A	A	B	I
B	B	I	A

CYCLIC 4-GROUP

	I	A	B	C
I	I	A	B	C
A	A	B	C	I
B	B	C	I	A
C	C	I	A	B

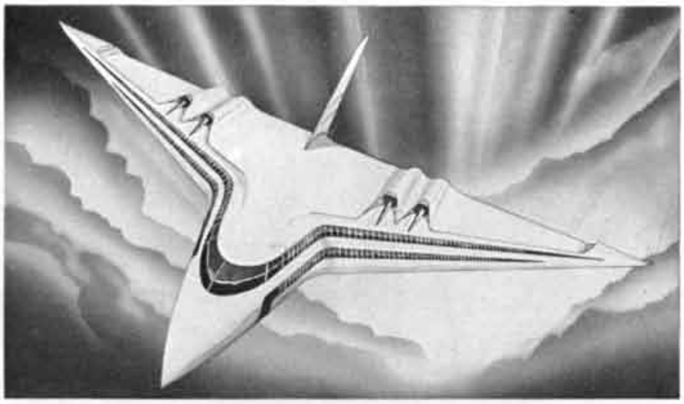
KLEIN 4-GROUP

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

The only groups of order 1 through order 4

Technology:

Mongols on horseback, planes without jets, and a plant that grows gas.



For almost all of man's history, human transportation was made possible by the energy of human or animal muscle, or by the inanimate and uncontrolled motion of water or air.

Like many other animals, man carried or dragged himself and the objects he needed moved. By taming the horse around 4000 B.C., he arranged to move himself much more efficiently—so well, in fact, that by the 13th century, Mongol armies swept across much of the known world using nothing but shaggy ponies.

Still, for over 5,000 years, little progress was made in the way man transported himself.

A century of progress.

Suddenly, in the 19th century, inanimate energy under the complete control of human beings came into being. Fulton's steamship, Stephenson's steam locomotive and Otto's internal combustion engine are but three examples. In the 20th century, the jet principle carried transportation beyond the atmosphere. Yet, despite our progress, today's declining oil supplies have placed us in the same position in which previous centuries would have been if their horses started dying out.

World, where now?

It may appear that the future of transportation looks bleak. Energy shortages, escalating fuel prices and the pollution from the internal combustion engine all seem to suggest new limitations on our ability to transport ourselves.

We have three basic choices: to abandon our ease of travel, endure the cost, shortages and pollution, or advance to new levels of technology that solve the problem.

Electrification of the car is one solution already near at hand. These noiseless and exhaust-free vehicles will be powered by nickel-zinc batteries which can store three times as much electrical energy, pound for pound, as the traditional lead-acid battery.

There are, of course, other alternatives besides electrification. We already have nuclear submarines, and nuclear surface vessels have been built. There has even been considerable talk again of nuclear-powered airplanes. Unfortunately, the public's apprehension of nuclear fission makes this seem unlikely in the near-term. But what about fusion or solar vehicles? Not airplanes bearing solar cells, but rather, airplanes depending on fuels generated by fusion or sunlight.

The plant world's role.

By using the energy of sunlight, plants can combine carbon dioxide and water to form foodstuffs and oxygen. Plants could be harvested for fuel-worthy alcohols and hydrocarbons. Given sufficient energy (and either fusion or the sun can supply that), human beings can synthetically combine carbon dioxide and water to form liquid fuel and oxygen in any desired quantity.

A big lift for transportation.

Today, Vertical Take-Off and Landing (VTOL) planes and helicopters are relieving some of the transportation pressure on earth's surface. But it might also be possible to lift automobiles above the surface of the earth by making them travel on compressed air jets. This type of car could travel over water as easily as over land.

Another solution might be magnetic levitation. For instance, a central magnetized guide-rail would repel a train that is similarly magnetized. If the magnetic field intensities are great enough, the train will lift a tiny distance above the rail. The train could move forward under the impulse of an electromagnetic field. Since there would be virtually no friction without solid contact, vibration-free speeds of three hundred miles an hour could be maintained.

A world without the horse?

Just a hundred years ago, most people could not conceive of a world in which the horse was not a major form of transportation. But a few people had the vision to develop new and better ideas.

We believe the answer to today's transportation problems will be the creative use of both existing and yet-to-be discovered technologies.

Science and Technology can solve many problems? If they don't, what else will?

Gould is an electrical/electronics company committed to growth through technology. This message is a condensation of a 12-page White Paper, "Technology and the Future of Transportation." For the complete text, write Gould Inc., Dept. S11, 10 Gould Center, Rolling Meadows, IL 60008.

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COGNAC
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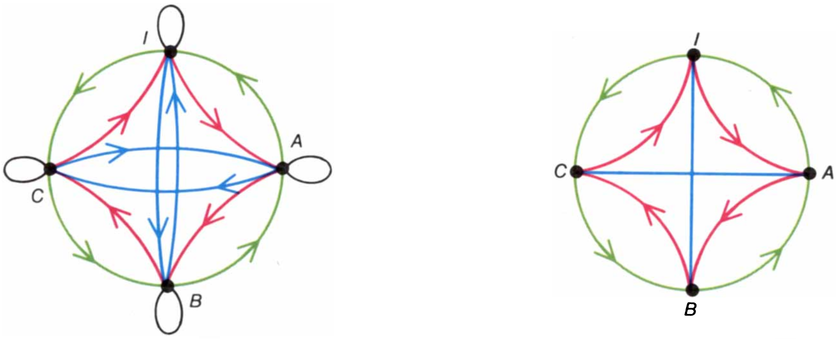
Courvoisier, Jarnac, France
The Brandy of Napoleon

	I	A	B	C
I	I	A	B	C
A	A	B	C	I
B	B	C	I	A
C	C	I	A	B



A Cayley color graph for the cyclic 4-group **(lower left)** *and a simplified version* **(lower right)**

diagonal axes, which is shown in color. This characteristic of the table indicates that the group also obeys the commutative axiom, meaning that the product of any two operations is the same regardless of which one is performed first. Groups that display this property are called Abelian groups after the Norwegian mathematician Niels Henrik Abel. Any cyclic permutation of *n* objects generates an Abelian group, which is equivalent to the group of the orientation-preserving rotations of a regular polygon of *n* sides. (A rotation preserves a figure's orientation if the figure ends up in exactly the same position in which it began.) Thus the cyclic 4-group can be modeled by the orientation-preserving rotations of a square.

There is just one group of order 1: the trivial group consisting of the identity operation. It is not hard to see that this operation meets all four of the criteria defining a group. For example, if you do nothing to something twice in succession, it is the same as doing nothing, and so the closure axiom is satisfied. The only order-2 group is almost as trivial. This group, the table for which is shown in the illustration on page 20, can be modeled with two operations to be performed on a penny: doing nothing to the penny (*I*) and turning the penny over (*A*). The only group of order 3 is the cyclic 3-group, which is equivalent to the set of cyclic permutations of three objects and to the set of orientation-preserving rotations of an equilateral triangle. There are just two groups of order 4: the cyclic 4-group and another group known as the Klein 4-group.

The Klein 4-group can be easily modeled with the following operations on two pennies placed side by side: doing nothing (*I*), turning over the left penny (*A*), turning over the right penny (*B*) and turning over both pennies (*C*). The table for the group, shown in the illustration on page 20, reveals that this group too is Abelian.

The simplest example of a non-Abelian group is the set of six symmetry operations on the equilateral triangle: the identity, rotating the triangle 120 degrees clockwise, rotating it 120 degrees counterclockwise, and flipping it over about any of its three altitudes. To prove that the elements of this group do not commute label the corners of a cardboard triangle, rotate the triangle 120 degrees in either direction and turn it over about any altitude; then perform the same two operations in the reverse order and compare the results. If each vertex of the triangle is identified with a different object, the resulting 6-group is equivalent to the group of all permutations on three objects.

To test your understanding of a group you might pause to consider the following three models.

1. With a deck consisting of four face-down playing cards the following operations are defined: the identity operation (*I*), transposing the top two cards in the deck (*A*), transposing the bottom two cards (*B*) and removing the middle two cards and putting the lower one on the bottom of the deck and the other one on the top (*C*).

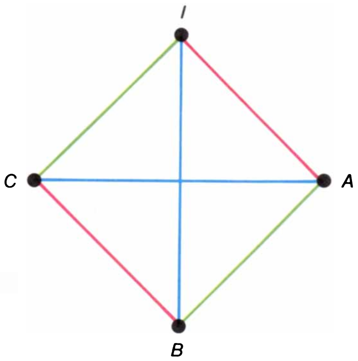
2. A dollar bill is placed either face up or face down and either right-side up or upside down. The operations are the identity (*I*), rotating the bill 180 degrees (*A*), turning it over about its vertical axis (*B*) and turning it over about its horizontal axis (*C*).

3. A sock is on either the left foot or the right foot in one of two states, right-side out or inside out. The operations are the identity (*I*), taking off the sock, reversing it and putting it back on the same foot (*A*), moving the sock to the other foot without reversing it (*B*) and taking off the sock, reversing it and moving it to the other foot (*C*).

For each of these groups make a multiplication table and determine whether the group is equivalent to the cyclic 4-group or the Klein 4-group. The answers will be given next month.

The multiplication table of a group can be represented graphically by a diagram called a Cayley color graph after the mathematician Arthur Cayley. For example, the graph at the lower left in the top illustration on this page is a Cayley color graph for the cyclic 4-group, the table for which is at the top of the illustration. The four points of the graph correspond to the four operations of the group. Every pair of points have been joined by a pair of lines going in opposite directions, with the direction of each line indicated by an arrowhead, and a

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I



A simplified Cayley color graph for the Klein 4-group

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color has been assigned to each operation in accordance with the key shown at the top of the table. To understand how the graph reproduces the information in the table consider the line from *B* to *A*. The color of the line is determined by starting at *B* on the left side of the table, moving to the right to the cell containing *A* and then using the color assigned to the letter, *C*, at the top of the column the cell is in. The same procedure yields the colors of all the other lines.

When two points in a Cayley color graph are joined by two lines of differing color, the operations represented by the two colors are the inverse of each other. If both lines are the same color, the operation associated with that color is its own inverse. In this case the graph can be simplified by replacing the two directed lines of like color with a single undirected line of that color. In addition the identity operation is represented by a loop that joins each point to itself, and so because one of these loops is at each corner of the graph they can all be omitted. The simplified version of the graph is at the lower right in the top illustration on page 24.

A simplified Cayley graph for the Klein 4-group is shown at the bottom of page 24 and one for the non-Abelian permutation 6-group is shown below. For graphs of higher order it is more convenient to stop using colors and instead label each line with the symbol corresponding to the operation it represents.

It should be obvious that given the color graph for any group, the table for the group can be constructed. The converse is also true. The graphs are valuable aids, however, because they often reveal properties that are not easily seen in a multiplication table. For example, it is not difficult to see that if on the 6-graph the lines corresponding to operations *A, B* and *C* are omitted, leaving only the lines corresponding to *D* and *E*, two disconnected graphs are obtained. Each is a color graph for a cyclic 3-group, but only the set of operations *I, D, E* actually forms such a group because this set alone contains the identity operation. Any subset of the elements of a group that itself forms a group is called a subgroup, and so inspection of the color graph has revealed that the cyclic 3-group is a subgroup of the permutation 6-group.

So far only groups with a finite number of operations, or elements, have been discussed. There are also infinite groups, and they fall into two classes: discrete groups, which have a countable infinity of elements, and continuous groups, which have an uncountable infinity of elements. An infinite set is said to be countable if its members can be matched up one for one with the positive integers 0, 1, 2. . . . Hence the integers themselves are an example of a countably infinite set, whereas the points on the real-number line are an example of an uncountably infinite set. In fact, the integers form a discrete Abelian group under the operation of addition, with 0 as the identity element and $-a$ as the inverse of any element a. The real numbers, on the other hand, form a continuous group with respect to addition, and if 0 is excluded, they form a continuous group with respect to multiplication as well. (In the latter case 1 is the identity element, and $1/a$ is the inverse of a.) Continuous groups are called Lie (pronounced Lee) groups after the Norwegian mathematician Marius Sophus Lie. A trivial geometrical example of a Lie group is the group of symmetry rotations of a circle (or a sphere or a hypersphere). The degree of rotation can be as small as one likes.

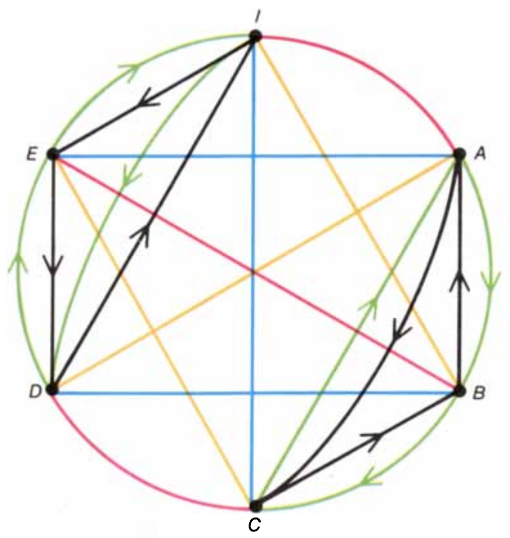
The group is one of the most powerful and unifying of all concepts in mathematics. Moreover, in addition to turning up in every branch of mathematics, groups have endless applications in science. Wherever there is symmetry there is a group. The Lorentz transformations of relativity theory form a Lie group based on the continuous rotation of an object in space-time. Finite groups underlie the structure of all crystals and are indispensable in chemistry, quantum mechanics and particle physics. The famous eightfold way, which classifies the family of subatomic particles known as hadrons, is a Lie group. Every geometry can be defined as the study of the properties of a figure that are left invariant by a group of transformations.

Even recreational mathematics sometimes involves groups. Since every finite group can be modeled by a set of permutations on *n* objects, it is hardly surprising to find groups related to card shuffling, ball juggling, campanology (bell ringing), sliding-block puzzles and all kinds of combinatorial puzzles. In an earlier column (reprinted in my *New Mathematical Diversions from Scientific American*) I explained how groups apply to braiding theory and so underlie numerous magic tricks involving ropes and twisted handkerchiefs.

In view of the great elegance and utility of groups it is understandable that mathematicians would like to be able to classify them. The Lie groups have been classified, but there are other infinite

I	IDENTITY
A	1 2 3 → 132
B	1 2 3 → 321
C	1 2 3 → 213
D	1 2 3 → 312
E	1 2 3 → 231

	I	*A*	*B*	*C*	*D*	*E*
I	*I*	*A*	*B*	*C*	*D*	*E*
A	*A*	*I*	*D*	*E*	*B*	*C*
B	*B*	*E*	*I*	*D*	*C*	*A*
C	*C*	*D*	*E*	*I*	*A*	*B*
D	*D*	*C*	*A*	*B*	*E*	*I*
E	*E*	*B*	*C*	*A*	*I*	*D*



A simplified Cayley color graph for the permutation 6-group

groups that still elude classification. What about finite groups? One might suppose they would be easier to classify than Lie groups, but that has not proved to be the case. It is this difficult task that is now on the brink of being completed.

All finite groups are constructed from building blocks called simple groups in much the same way that chemical compounds are constructed from elements, proteins from amino acids and composite numbers from primes. A simple group is one that has no "normal" subgroups other than itself and the trivial identity subgroup. Remember that a subgroup is defined as any subset of the elements of a group that is itself a group. What is meant by "normal" is best explained as follows. Consider a group *G* with a subgroup *S*. The set of products obtained by multiplying each element of

A Simple Ballad

(To be sung to the tune of "Sweet Betsy from Pike")

What are the orders of all simple groups?
I speak of the honest ones, not of the loops.
It seems that old Burnside their orders has guessed
Except for the cyclic ones, even the rest.

Groups made up with permutes will produce some more:
For A_n is simple, if n exceeds 4.
Then, there was Sir Matthew who came into view
Exhibiting groups of an order quite new.

Still others have come on to study this thing.
Of Artin and Chevalley now we shall sing.
With matrices finite they made quite a list.
The question is: Could there be others they've missed?

Suzuki and Ree then maintained it's the case
That these methods had not reached the end of the chase.
They wrote down some matrices, just four by four,
That made up a simple group. Why not make more?

And then came the opus of Thompson and Feit
Which shed on the problem remarkable light.
A group, when the order won't factor by two,
Is cyclic or solvable. That's what is true.

Suzuki and Ree had caused eyebrows to raise,
But the theoreticians they just couldn't faze.
Their groups were not new: if you added a twist,
You could get them from old ones with a flick of the wrist.

Still, some hardy souls felt a thorn in their side.
For the five groups of Mathieu all reason defied;
Not A_n, not twisted, and not Chevalley,
They called them sporadic and filed them away.

Are Mathieu groups creatures of heaven or hell?
Zvonimir Janko determined to tell.
He found out what nobody wanted to know:
The masters had missed 1 7 5 5 6 0.

The floodgates were opened! New groups were the rage!
(And twelve or more sprouted, to greet the new age.)
By Janko and Conway and Fischer and Held,
McLaughlin, Suzuki, and Higman, and Sims.

No doubt you noted the last lines don't rhyme.
Well, that is, quite simply, a sign of the time.
There's chaos, not order, among simple groups;
And maybe we'd better go back to the loops.

the subgroup by each element of the group is called the right coset of the group, *GS*. Conversely, the set of products obtained by multiplying each element of the group by each element of the subgroup is called the left coset, *SG*. If the two cosets are equivalent, the subgroup is called normal. For example, the cyclic 3-group is a normal subgroup of the permutation 6-group. Hence the 6-group is not simple. Simple groups are the building blocks of all groups, and so to classify the finite groups it is necessary to classify all the finite simple groups.

Almost all finite simple groups belong to families with an infinity of members. Families of this type provide a quite satisfactory system of classification, since there are procedures for constructing any individual member, or group. For example, the cyclic permutation groups of prime order (which are modeled by rotations of regular polygons with a prime number of sides) are finite simple groups. In fact, they are the only finite simple groups that are Abelian as well as the only ones that are cyclic. A famous result in mathematics called Lagrange's theorem states that the order, or number of elements, of any subgroup must be a divisor of the order of the group in which it is contained. Since a prime number has no divisor (other than 1 and itself), this theorem implies that any group of prime order has no subgroups (other than the identity and itself). If a group has no other subgroups, however, then it certainly has no normal subgroups, and so it follows that any group of prime order is simple.

Another important family of finite simple groups is the set of alternating groups, which are modeled by the even permutations on n objects for all integers n greater than 4. An even permutation is one that can be obtained in an even number of steps, where each step consists in switching two objects. For example, the cyclic 3-group is also an alternating group because 231 can be produced from 123 in two steps (transpose first 1 and 2 and then 1 and 3) and the same is true for any other pair of the three cyclic permutations of three objects. Exactly half of all permutations are even, and because n objects can be permuted in $n!$ ways every alternating group has an order of $n!/2$. The odd permutations do not form groups because any odd permutation followed by another odd permutation is equivalent to an even permutation, so that the closure axiom is not satisfied.

There are 16 other infinite families of finite simple groups, all of them non-Abelian and noncyclic. The orders of the simple groups (excluding the cyclics) form an infinite sequence that starts with 60, the order of the alternating group on five objects. (This group is equivalent to the group of rotations of a regular dodecahedron or icosahedron.) The sequence begins 60, 168, 360, 504, 660, 1092, 2448, 2520, 3420, 4080, 5616, 6048, 6072, 7800, 7920.... If 1 and all the prime numbers are inserted into this infinite sequence, the resulting sequence gives the orders of all finite simple groups.

Unfortunately the list includes a small number of groups (starting with the group of order 7920) that cannot be fitted into any infinite family. These are the non-Abelian anomalies, the jokers that defy all classification. Mathematicians know them as the sporadic simple groups, but they are quite complicated. If there is an infinite number of these sporadics, and if there is no pattern ordering them, then the task of classifying all finite simple groups, and therefore all finite groups, is hopeless. There are, however, compelling reasons for thinking there are no sporadics other than the 26 already identified. (A classic history of sporadic groups appears in the box on page 28. First published in *The American Mathematical Monthly* in November, 1973, the tale is said to have been "found scrawled on a library table in Eckhart Library at the University of Chicago; author unknown or in hiding." The "loops" referred to therein are the simple cyclic groups, and A_n is the symbol for the alternating group for n objects.)

The search for the sporadic simple groups began in the 1860's when the French mathematician Émile Léonard Mathieu discovered the first five. The smallest of them, known as M_{11}, has 7,920 operations and is modeled by certain permutations on 11 objects. A century slipped by before the sixth sporadic, of order 175,560, was found in 1965 by Zvonimir Janko of the University of Heidelberg. Three years later John Horton Conway of the University of Cambridge surprised everyone by finding

NAME OF GROUP	NUMBER OF ELEMENTS	DISCOVERED BY
M_{11}	$2^4 \times 3^2 \times 5 \times 11$	
M_{12}	$2^6 \times 3^3 \times 5 \times 11$	
M_{22}	$2^7 \times 3^2 \times 5 \times 7 \times 11$	Mathieu
M_{23}	$2^7 \times 3^2 \times 5 \times 7 \times 11 \times 23$	
M_{24}	$2^{10} \times 3^3 \times 5 \times 7 \times 11 \times 23$	
J_1	$2^3 \times 3 \times 5 \times 7 \times 11 \times 19$	Janko
J_2	$2^7 \times 3^3 \times 5^2 \times 7$	Hall, Wales
J_3	$2^7 \times 3^5 \times 5 \times 17 \times 19$	Higman, McKay
J_4	$2^{21} \times 3^3 \times 5 \times 7 \times 11^3 \times 23 \times 29 \times 31 \times 37 \times 43$	Benson, Conway, Janko, Norton, Parker, Thackray
HS	$2^9 \times 3^2 \times 5^3 \times 7 \times 11$	Higman, Sims
MC	$2^7 \times 3^6 \times 5^3 \times 7 \times 11$	McLaughlin
Sz	$2^{13} \times 3^7 \times 5^2 \times 7 \times 11 \times 13$	Suzuki
C_1	$2^{21} \times 3^9 \times 5^4 \times 7^2 \times 11 \times 13 \times 23$	
C_2	$2^{18} \times 3^6 \times 5^3 \times 7 \times 11 \times 23$	Conway
C_3	$2^{10} \times 3^7 \times 5^3 \times 7 \times 11 \times 23$	
He	$2^{10} \times 3^3 \times 5^2 \times 7^3 \times 17$	Held, Higman, McKay
F_{22}	$2^{17} \times 3^9 \times 5^2 \times 7 \times 11 \times 13$	
F_{23}	$2^{18} \times 3^{13} \times 5^2 \times 7 \times 11 \times 13 \times 17 \times 23$	Fischer
F_{24}	$2^{21} \times 3^{16} \times 5^2 \times 7^3 \times 11 \times 13 \times 17 \times 23 \times 29$	
Ly	$2^8 \times 3^7 \times 5^6 \times 7 \times 11 \times 31 \times 37 \times 67$	Lyons, Sims
O	$2^9 \times 3^4 \times 5 \times 7^3 \times 11 \times 19 \times 31$	O'Nan, Sims
R	$2_{14} \times 3^3 \times 5^3 \times 7 \times 13 \times 29$	Conway, Rudvalis, Wales
F_5	$2^{14} \times 3^6 \times 5^6 \times 7 \times 11 \times 19$	Conway, Fischer, Harada, Norton, Smith
F_3	$2^{15} \times 3^{10} \times 5^3 \times 7^2 \times 13 \times 19 \times 31$	Smith, Thompson
F_2	$2^{41} \times 3^{13} \times 5^6 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 47$	Fischer, Leon, Sims
F_1	$2^{46} \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$	Fischer, Griess

The list of 26 sporadic finite simple groups

Heart Computer

Your heart can tell you three things that can help you live longer and stay healthier. The rest is up to you.

The Pulsetach will shortly become the number one selling system of its type in the nation.

JS&A has never offered a pulse meter. And for good reason.

If you've ever used one, you'll quickly discover that your heart does not beat like a clock. It's irregular. It might beat at 40 beats per minute for one instant and at 120 the next. Since most pulse meters measure each beat as it occurs, you never feel confident that you're getting a very good reading.

We also considered size. Each pulse meter we examined was large or cumbersome and awkward to carry or store.

WE WAITED

We waited a few years. In the meantime, we discovered three ways your heart (through your pulse) helps you monitor your health.

Pulse Rate Your pulse rate can tell you if you are getting enough oxygen throughout your body. A high pulse rate indicates that your heart must pump faster to supply that oxygen and may indicate poor physical condition.

Target Zone Your pulse can tell you if your heart is beating fast enough during exercise. There's an area called the "Target Zone." Below this level, you're not exercising hard enough to do your heart or respiratory system any good. Above this level, you can be dangerously over-exerting yourself.

Cardiac Recovery Time The time it takes for your pulse rate to return to normal after you've exercised is the real measure of whether or not your exercise program is doing you any good. This time can be as healthy as one minute or as poor as several minutes.

The three things we learned convinced us that the ideal pulse meter must have the following features:

1. It must measure a series of heart beats and simultaneously compute the average to avoid the strange readings from irregular heart beats.

2. It must be small enough to use while exercising.

3. It should have a timing capability to determine the Cardiac Recovery Time.

It wasn't until a small Utah medical electronic instrument company created what we feel not only provides the capabilities listed above, but excels in other areas too.

FITS ON FINGER

The unit is called the Pulsetach, and it fits right over your finger. It weighs less than an ounce and can be worn easily during most exercise programs.

The large liquid crystal display can easily be seen in normal room lighting or in bright sunlight, and because liquid crystal displays consume very little power, the readily-available watch batteries will last for years. The Pulsetach automatically turns itself off in five minutes if you forget.

The heart of the system is a powerful microcomputer CMOS semi-conductor integrated circuit that will take up to 4 pulse beats, compute an average pulse rate, and then flash that rate on the liquid crystal display.

FINGERTIP SCANNER

The sensor consists of a Gallium Arsenide infrared light-emitting diode which scans your fingertip hundreds of times a second to determine your pulse rate. This new system is one of the most accurate and is also used in sophisticated hospital systems.

The unit also contains a quartz-controlled timing circuit which will accurately time either your exercise period or your Cardiac Recovery Time. And you can switch back and forth between the pulse and chronograph mode while you are exercising.

We realize that the Pulsetach sounds like a very sophisticated unit. And it is. But as sophisticated as it is internally, it's an extremely easy unit to operate. There are just two buttons to press which operate the pulse reading and the chronograph timing circuit. A third button engages the audio circuit.

The Pulsetach system fits comfortably on your finger while it monitors your heart and determines your Cardiac Recovery Time.

HEAR YOUR PULSE

The audio circuit simply beeps every time your pulse beeps. This feature lets you monitor your pulse by hearing it as you run or exercise and it can be shut off by pressing the button a second time. The timing circuit is quartz-controlled and extremely accurate.

The Pulsetach not only has combined all of the most advanced technology in an extremely small size, but it costs less than many other systems lacking its advanced features.

The Pulsetach can be used for joggers, athletes, all forms of exercise and even cardiac recovery patients, as it operates quite effectively with pacemakers.

REAL WORKOUT

We suggest you order a Pulsetach for our 30-day no-obligation trial. When you receive your unit, give it a real workout. Notice how simple it is to operate and how easily you can read your pulse rate. Use it to stay in your Target Zone and to determine and then improve your Cardiac Recovery Time.

Monitor your Cardiac Recovery Time. Determine your Target Zone and see if you're really exercising in that area. Then use the Pulsetach to watch those important signs slowly improve thanks to the accuracy and information you get from the unit.

By knowing the important factors that help you monitor your health, you'll feel better, exercise more effectively, and many doctors feel you'll live longer.

TWO UNITS AVAILABLE

To order your Pulsetach pulse meter, send your check for **$119.95** plus $2.50 postage and handling (Illinois residents add 6% sales tax) to the address below. (Allow 20 days for personal checks to clear.) Credit card buyers may call our toll-free number below.

You can also order the more expensive hospital unit that averages 16 beats and has all the features including the small size of the previous unit. It costs **$169.95**.

We'll send your Pulsetach pulse meter complete with 90-day limited warranty and instructions which include information on determining your Target Zone, Cardiac Recovery Time and other helpful information.

Then after your test, if you're not fully convinced that the Pulsetach is the best unit of its kind, the most convenient, and the greatest value, return it within 30 days for a prompt and courteous refund including the $2.50 charge for postage and handling. You can't lose.

Your Pulsetach is totally solid-state so service should never be required, but if it is, the manufacturer has a national service-by-mail facility backing each unit. JS&A is America's largest single source of space-age products–further assurance that your Pulsetach is backed by a substantial company.

We've waited an awful long time to jump into the pulse monitoring field. But what a great entry. Order your Pulsetach at no obligation today.

JS&A PRODUCTS THAT THINK®

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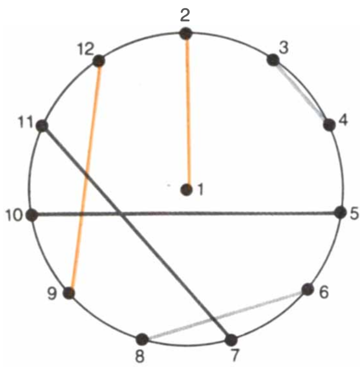
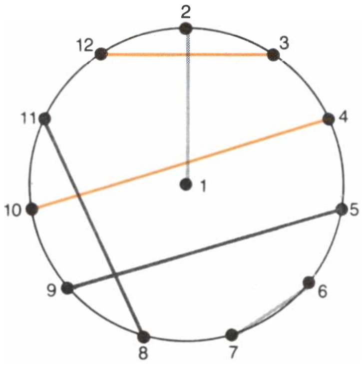




Two solutions to last month's bridge-tournament problem

three more sporadics. His work was based on Leech's lattice, a scheme devised by John Leech, a British mathematician, for packing unit hyperspheres densely in a space of 24 dimensions. (In Leech's lattice each hypersphere touches exactly 196,560 others.)

Leech discovered his lattice while working on error-correcting codes. It turns out that there is a close connection between certain sporadic groups and codes employed in reconstructing a message distorted by noise. Two of Mathieu's sporadic groups, M_{23} and M_{24}, are related to the Golay error-correcting code that is often used for military purposes. Roughly speaking, a good error-correcting code is based on a subset of unit hyperspheres placed as far apart from one another as is possible in a dense packing.

An amusing episode involving M_{24} was called to my attention by John McKay, a codiscoverer of two sporadics who is a computer scientist at Concordia University in Montreal. In 1898 George Abram Miller, an American group theorist, published a paper in *Messenger of Mathematics* (Vol. 27, pages 187–190) proving that M_{24} did not exist. As one might expect, this publication does not appear in Miller's collected papers. McKay believes there may be some connection between Miller's ill-fated proof and the fact that although Eric Temple Bell in the index to his *Development of Mathematics* (second edition, 1945) lists page 445 after Miller's name, no mention of Miller can be found on that page. There is, however, a vitriolic attack on group theorists "little advanced beyond mathematical illiteracy," who keep discovering new finite groups by "obstinate grubbing" in narrow categories that they themselves define "apparently with the express purpose of dignifying their calculations with an air of pseudo generality."

At the start of this year two dozen sporadic groups had been proved to exist, and two more, J_4 and F_1, were believed to be authentic. (A complete list of these 26 sporadic groups is shown in the illustration on page 30.) J_4, which was proposed by Janko in 1975, was finally constructed in February by David Benson, Conway, Simon P. Norton, Richard Parker and Jonathan Thackray, a group of mathematicians at Cambridge. F_1 (the monster), which is by far the largest sporadic, was conjectured independently by Griess and by Bernd Fischer of the University of Bielefeld in 1973 and constructed in January by Griess, as I have mentioned. Several much smaller sporadics, the construction of which required long computer calculations, are embedded in F_1 in such a way that their existence follows almost trivially from the existence of F_1. Yet to everyone's astonishment Griess's construction of F_1 was carried out entirely by hand. F_1 is said to be based on a group of symmetry rotations in a space of 196,883 dimensions!

Is the list of 26 sporadics complete? Most group theorists are convinced it is, but the task of proving the conjecture could be formidable. Indeed, the final published proof is likely to require as many as 10,000 printed pages. It should be noted, however, that proofs in group theory tend to be unusually long. A famous proof by John Thompson and Walter Feit, which among other things established William Burnside's conjecture that all non-Abelian finite simple groups are of even order, covered more than 250 pages: an entire issue of *The Pacific Journal of Mathematics* (Vol. 13, pages 775–1029; 1963).

In 1972 Daniel Gorenstein of Rutgers University outlined a 16-step program for completing the classification of the finite simple groups. This guide to a final proof was soon improved and greatly "speeded up" by Michael Aschbacher of the California Institute of Technology. Both men are world experts on groups. (In January, Aschbacher won the much coveted Cole Prize in algebra.) In May, 1977, Gorenstein told *The New York Times* that he had been working on the classification problem five hours a day, seven days a week, 52 weeks a year since 1959. "I want to solve it," he said, "because I want to solve it, not because it will benefit mankind." Like most other group theorists, Gorenstein is convinced that no new sporadic groups will be found, and that a proof that the list of 26 groups is complete is only a year or two away.

There is, of course, no way to predict whether a practical application will or will not be found for any mathematical result whose discovery was not motivated by practical considerations. We do know that groups lie at the very heart of the structure of the universe. Nature seems to prefer small, uncomplicated groups, but this could be an illusion created by the fact that applications of small groups are the easiest to find, particularly in a world limited to three spatial dimensions. Who can say that at some far distant date, if the human race survives, even the monster will not turn out to have some remarkable but at present unimaginable application?

The illustration on this page gives two answers to last month's problem of designing a bridge tournament for 12 players so that they meet at three tables for 11 days and each player is a partner of every other player just once and an opponent of every other player just twice. The first day's distribution is given by a disk on which partners are connected by colored lines and tables are denoted by lines of matching color. Rotating the disk clockwise one step at a time generates the cyclic design for the remaining 10 days. Disk patterns other than the two shown generate additional solutions.

January's column on checkers showed a final position for a blocked game of 24 moves, the shortest game possible if there are no captures. I stated that this pattern was thought to be unique, and Robert F. Holt was the first reader to prove it is not. Alan Beckerson found all the possible final positions, a total of 28. Only two of these patterns (including the one I gave) are symmetrical, and in 16 of them if it were not Black's turn, White would be able to make one more move.

Many variations on last December's "impossible problem" have been received that eliminate the need for an upper bound on the two numbers to guarantee a unique solution. One of the most elegant variants comes from Barry Wolk of the University of Manitoba. As in the original version, mathematicians S and P are discussing two unknown integers, both of them greater than 1. S knows only the sum of the numbers, whereas P knows only their product.

S: "I see no way you can determine my sum."

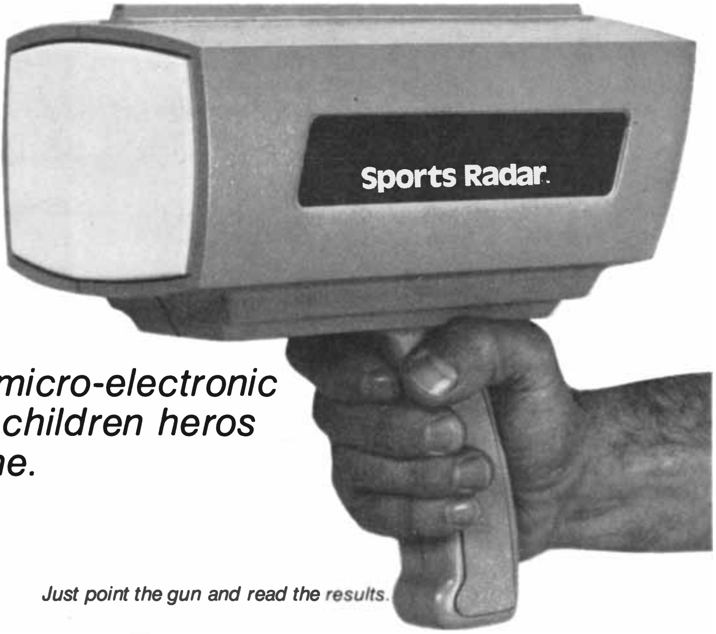
P (after a suitable delay): "That didn't help me. I still don't know your sum."

S (after another delay): "Now I know your product."

What are the two numbers? If Goldbach's conjecture is assumed to be true, one can show that they must be 5 and 6.

NEW PRODUCT

Sports Radar
T.M.



Here's how an exciting new micro-electronic breakthrough will make your children heros and improve your tennis game.

Just point the gun and read the results.

When Roger started pitching for his little league team, he was just another player. And his arm was no better than anybody elses.

Two months later a small miracle took place. Roger was the best pitcher on the team and had a fast ball that was the most powerful in his league—and all thanks to his father.

Roger's success came from a radar gun—the same type device used by police to catch speeding motorists.

SCRAMBLED EGGS

The minute Roger's father was able to clock his son's pitching speed, Roger was subconsciously given a daily challenge of pitching harder and faster to beat his previous speed. The more he practiced, the better he got.

Roger's father paid $2,000 for the radar gun. But in his upper class neighborhood, it wasn't too unusual for a father to spend that kind of money to help his son.

SPEEDING CITATION

A large manufacturer of radar-type security devices saw what Roger's father had done and felt that there was a definite need to produce a low cost radar unit designed exclusively for the sports market.

The company, Solfan Systems, developed the Sports Radar gun—a major breakthrough in projectile speed detection as well as electronic radar circuitry.

Using the doppler effect of radar and phased-lock-loop circuits, Solfan has developed the Sports Radar gun that compares to even the most sophisticated of police radar units that cost $2,000.

OVERLAND EXPRESS

The Sports Radar gun is held in your hand and pointed toward the pitcher. You turn it on, press the ready button, and point the gun. The gun will ignore the moving arm of the pitcher but will lock in on the moving ball. The radar unit would then follow the ball for approximately ten milliseconds and the built in computer measures and computes the speed and flashes the reading on the display. The gun registers the speed to the exact mileage within one-half miles per hour.

The gun can be mounted on a tripod so that the person taking the measurements can also catch the ball.

In tennis, the speed of the serve can be measured by aiming the gun at the person serving. You can also use the unit by yourself by setting the unit on a tripod and measuring the speed from behind.

WORKING AND PLAYING

Aside from its extreme accuracy and advanced electronics, the unit is priced to meet the budget of every sports-minded athlete or parent. It's only $189.95 complete.

You can measure the speed of baseballs, soccer balls, tennis balls, golf balls, hockey pucks, downhill skiers, radio controlled model airplanes or anything that moves—even automobiles.

The speed is flashed on the large LED display and is shown in miles per hour.

The unit accepts two commercially available 6-volt lantern batteries which you can purchase locally or from JS&A for only $2 each. The batteries will last for weeks with normal use.

SUCCESS AND GOOD THINGS

The unit comes in a sports blue color and weighs 38.4 ounces, exclusive of batteries. It's rugged, well built and designed to endure the typical use and abuse it would normally receive.

We urge you to test this exciting new product during our 30-day free trial. Order the Sports Radar gun. When you receive it, measure your child's pitching speed. Test it on your own tennis serve. See how knowing your speed will actually improve it as you try to out perform your previous record fast pitch or serve. Then decide if the Sports Radar gun doesn't make a very exciting addition to your sports equipment.

ONE FOR THE MONEY

If you are not convinced that the Sports Radar gun is something that you'll use constantly to help improve your game, return it for a prompt and courteous refund, including your $3.50 postage and handling. You can't lose—and chances are your son will at least have the most popular new product in the neighborhood.

To order one for your test, simply send your check for **$189.95** plus $3.50 for postage and handling to JS&A Group, Inc., at the address shown below. (Illinois residents please add 5% sales tax.) Credit card buyers may call our toll-free number below. If you wish to buy a set of two six-volt batteries, simply add $4.00 to your order.

We'll then send your unit, the batteries (if you order them from us), a 90-day limited warranty and complete easy-to-understand instructions.

Radar electronics for the sports enthusiast is now a reality. Watch your game improve by ordering your Sports Radar gun at no obligation, today.

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