DECISION MAKING WITH FUZZY SETS

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ABSTRACT

When goals and constraints are stated imprecisely, decision problems grow in importance, particularly in the investigation of complex and social systems. In this paper the methodology of Zadeh's fuzzy set theory is summarized and applied to fuzzy decision making.

INTRODUCTION

One of the most difficult problems of decision making arises when either the constraints or the goals are fuzzy or vague. The following statements exemplify fuzziness: the sales outlook is *optimistic; x* is *much larger* than *y;* the forecasted sales are *substantially greater* than 500; and, select the *most available* men. In these sentences the underlined term is responsible for the fuzziness. Conventional quantitative techniques are unsuited for dealing with decision problems involving fuzziness. As decision sciences become more and more involved in both humanistic and complex systems, fuzziness becomes a prevalent phenomena in describing these systems. The basis for this contention is what Zadeh [8] calls the "Principal of Incompatibility," which he informally defines by stating: "as the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance become almost mutually exclusive characteristics."

The human mind is capable of handling fuzzy statements and decisions. For example, the command: "*tall* people in the back and *short* people in the front," where short and tall are fuzzy concepts, usually can be resolved by the people involved. However, computers and conventional quantitative techniques are not equipped to handle such decisions. As the elements involved in a decision become too large for the mind to handle, researchers should develop some rigorous techniques which enable them to handle large scale fuzzy decisions in an orderly manner and with the aid of the computers. Starting in 1965 with his article "Fuzzy Sets", Zadeh laid the initial foundations for a theory which hopefully will lead to a methodology to solve complex fuzzy problems.

In this article, we will review some of the concepts and applications of fuzzy theory as it applies to the decision-making situation. We also will point out several areas in which the application of this theory falls short. It must be kept in mind that this theory is still in the development stage, and this article should be viewed as a report on the progress of this theory and not a final version of fuzzy theory. The most significant applications of this theory to date have occurred in the field of automata and learning systems [4]; however, this theory has immense applicability in the area of decision making.

DEFINITION OF FUZZY SETS AND SOME ELEMENTARY OPERATIONS

Def. 1 Assume there is some universe of discussion consisting of a set of elements[1]

$$\overline{X} = \{ X_1, X_2, \ldots, X_n \}.$$

A fuzzy subset A of $\overline{X}$ is characterized by a membership function

$$U_A(\overline{X}) \text{ where } U_A : \overline{X} \to [0, 1].$$

This membership function associates with each member X_i of $\overline{X}$ a number $U_A(X_i)$ in the interval [0, 1] which represents the grade of membership of X_i in A. A can be written as

$$A = \left\{ \frac{U_A(X_1)}{X_1}, \frac{U_A(X_2)}{X_2}, \ldots, \frac{U_A(X_n)}{X_n} \right\}$$

which is a set of two-tuples.

The larger $U_A(X_i)$, the stronger the degree of membership of X_i in A.

For example, if

$$\overline{X} = \{ 1, 2, 3, 4, 5 \}$$

and if A is the fuzzy subset: X is approximately 2.5, then, A could subjectively be given by

$$A = \{ \frac{0.6}{1}, \frac{0.8}{2}, \frac{0.8}{3}, \frac{0.6}{4}, \frac{0.2}{5} \}.$$

Another example would be

$$B = \{ \frac{0.5}{\text{John}}, \frac{0.7}{\text{Bill}}, \frac{0.2}{\text{Joe}}, \frac{0.9}{\text{Tom}} \},$$

where the grade of membership depends upon the height of each of the boys.

In assigning the grade of membership of a variable to a fuzzy set, one usually has to assign it in some subjective manner. However, one must realize that subjectivity is the essence of fuzziness, and we are trying to discover a methodology to handle fuzzy problems. Furthermore, two of the most useful tools in decision-making – Bayesian Theory and Utility Theory – also involve subjectivity. This subjectivity in the assignment of membership is one of the most important drawbacks of the fuzzy set approach to decision-making. As will be seen later in the article, solutions are very sensitive to this membership function. Accordingly, one of the areas on which

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[1]For explicative purposes we shall deal only with finite sets; the extension to continuous sets is straight forward.

particular research effort should be spent if this theory is to become very useful is in investigating the assignment of membership grades.

In assigning the membership value of a particular X_i to a fuzzy set, the closer X_i is to satisfying the requirements of the set, the larger $U_A(X_i)$ is. In order to obtain additional insight into the assessment problem, we will consider one very important fuzzy set in decision-making.

Assume that there is a set of n alternatives with their associated profits and that a person is interested in obtaining the fuzzy set: a good profit. He can use any membership function which keeps the linear order of profit. In other words, the higher the profit of an alternative, the stronger its membership in this fuzzy set.

As an example, assume that there are five alternatives X_1, X_2, X_3, X_4, X_5, and the associated profits 4, 8, 10, 3, 7. One can then define the universe of alternatives as

$$\bar{X} = \{X_1, X_2, X_3, X_4, X_5\}.$$

The fuzzy set of the alternatives with a good profit could be

$$A = \{\frac{0.4}{X_1}, \frac{0.8}{X_2}, \frac{1}{X_3}, \frac{0.3}{X_4}, \frac{0.7}{X_5}\} \quad \text{or } A = \{\frac{0,}{X_1} \frac{0.5,}{X_2} \frac{0.8,}{X_3} \frac{0,}{X_4} \frac{0.3}{X_5}\}$$

or any other set which preserves the linear order,[2] depending upon the subjective definition of a reasonable profit. This example indicates one general rule in assigning membership grades. If with respect to a given characteristic the alternatives have quantitative values, then, any fuzzy set involving this characteristic must not disturb the ordering relationship of the alternatives. A person may use any nonlinear function to obtain the grade of membership as long as he does not change the ordinal relationship of the alternatives.

A method for obtaining the membership function when the characteristic is not given in quantified terms is described below. Assume the same set of n alternative plants, but the fuzzy set of interest is: a good availability of office help. In this case the prime characteristic is not quantifiable in the same sense as profit. Therefore, the analyst may ask the personnel director of the company to rate the n plants on an absolute scale of 0 to 1. This procedure can be used even if the characteristic is initially in quantitative terms.

Non-fuzzy sets can be handled as fuzzy sets when the grade of membership is either 1 or 0. As an example, let

$$\bar{X} = \{1, 2, 3, 4, 5\}$$

be the universe, and let A be the subset: $\{X > 2\}$.This could be written as a fuzzy set

$$A = \{\frac{0}{1}, \frac{0}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}.$$

This set enables a person to handle both fuzzy and non-fuzzy statements using the same techniques.

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[2] In the second case, even though X_1 and X_4 are both assigned the value zero, the order is still kept.

Def. 2 Two fuzzy sets A and B over the same universe $\bar{X}$ are equal if

$$U_A(X) = U_B(X) \quad \text{for all } X \in \bar{X}$$

and A is said to be a subset of B, denoted $A \subset B$, if

$$U_A(X) \leqslant U_B(X) \quad \text{for all } X \in \bar{X}.$$

Def. 3 Let A and B be two fuzzy sets over $\bar{X}$. The intersection of A and B, denoted $A \cap B$, is a fuzzy set C over the universe $\bar{X}$ where the membership function for C is

$$U_{A \cap B}(X) = U_C(X) = \text{Min}\ (U_A(X), U_B(X)) \quad \text{for all } X \in \bar{X}.$$

In fuzzy set theory the notation $U_A(X) \wedge U_B(X)$ is used for Min $(U_A(X), U_B(X))$. The notation of intersection bears a close relation to the connective "and." $A \cap B$ can be read "both A and B."

As an example, let $\bar{X}$ = {Bill, Joe, Tom} ;

A = set of tall men = $\{\frac{0.7}{\text{Bill}}, \frac{0.2}{\text{Joe}}, \frac{0.5}{\text{Tom}}\}$;

and, B = set of fat men = $\{\frac{0.3}{\text{Bill}}, \frac{0.4}{\text{Joe}}, \frac{0.9}{\text{Tom}}\}$.

Then, the set of both tall and fat men is

$$C = A \cap B = \{\frac{0.3}{\text{Bill}}, \frac{0.2}{\text{Joe}}, \frac{0.5}{\text{Tom}}\}.$$

Def. 4 Let A and B be two fuzzy sets over $\bar{X}$. The union of A and B, denoted $A \cup B$, is a fuzzy set C over $\bar{X}$ where the membership function for C is

$$U_{A \cup B}(X) = U_C(X) = \text{Max}\ (U_A(X), U_B(X)) \quad \text{for all } X \in \bar{X}.$$

In fuzzy set theory the notation $U_A(X) \vee U_B(X)$ is used for Max $(U_A(X), U_B(X))$. The notation of union bears a close relation to the connective "or." $A \cup B$ can be read "A or B or both." From our previous example, the set of tall or fat men is

$$C = A \cup B = \{\frac{0.7}{\text{Bill}}, \frac{0.4}{\text{Joe}}, \frac{0.9}{\text{Tom}}\}.$$

Def. 5 Let A be a fuzzy set over $\bar{X}$. The compliment of A, denoted A', is a fuzzy set over $\bar{X}$ where the membership function for A' is

$$U_{A'}(X) = 1 - U_A(X) \quad \text{for all } X \in \bar{X}.$$

The notation A' bears a close relation to the "not A" statement.

From our previous example, the set "not tall men" is

$$A' = \{\frac{0.3}{\text{Bill}}, \frac{0.8}{\text{Joe}}, \frac{0.5}{\text{Tom}}\}.$$

It should be noted that the operations union and intersection are associative and distributive over one another and that De Morgan's law holds.

Def. 6 The algebraic sum of two fuzzy sets A and B, denoted $A \oplus B$, is the fuzzy set C over $\bar{X}$ with membership function

$$U_{A \oplus B}(X) = U_C(X) = U_A(X) + U_B(X) - U_A(X)\, U_B(X) \quad \text{for all } X \in \bar{X}.$$

The algebraic sum can be used to represent the soft "or" situations when one wants to make statements like "A or B but not both."

Def. 7 Let A be a fuzzy set over $\bar{X}$, and let $\alpha > 0$ be a scalar. The operation A^{α} is defined as a fuzzy set over $\bar{X}$ with membership function

$$U_{A^{\alpha}}(X) = (U_A(X))^{\alpha} \text{ for all men } X \in \bar{X}.$$

If $\alpha > 1$, the effect of raising A to the α power results in a fuzzy subset B in which the degree of membership for those X that are large is reduced much less than for those that are small. If $\alpha < 1$, the effect is the opposite.

SIMPLE DECISION MAKING USING FUZZY SETS

As Bellman and Zadeh [1] state, one can look at a decision as a group of goals (or objectives) and constraints with a symmetry between these two entities. For instance, let $X = \{X_1, X_2, \ldots, X_n\}$ be a set of alternatives and G be a fuzzy goal – a good profit, a minimal investment cost, or minimal disruption of the environment. One can assign to each of these goals a fuzzy set over the universe of alternatives $\bar{X}$ where the membership function is determined by how well each of the alternatives satisfies that goal. In addition, let C be a fuzzy constraint. A decision problem could be to select a plant which gives a good profit, which has a minimal investment, and which minimally disrupts the environment.

In the format of the above decision problem, the constraints and the goals are connected in a symmetric manner by "and" statements. The goals and constraints intersect to form the decision space–the fuzzy set whose membership function is the degree to which each alternative is a solution.

Def. 8 Assume we are given a set of alternatives $\bar{X} = \{X_1, X_2, \ldots, X_n\}$ and a set of goals, $G_1, G_2, \ldots, G_P$ which can be expressed as fuzzy sets on the space of alternatives and a set of constraints, $C_1, \ldots, C_m$ which can also be expressed as fuzzy sets on the space of alternatives. The goals and constraints then combine to form a decision D, which is a fuzzy set resulting from the intersection of the G's and C's. Thus,

$$D = C_1 \cap C_2 \cap \ldots \cap C_m \cap G_1 \cap G_2 \cap \ldots \cap G_P,$$

and the membership function

$$U_D(X) = \text{Min}\,(U_{C_1}(X), \;.\;.\;, U_{C_m}(X), U_{G_1}(X), \;\ldots, U_{G_p}(X))$$

$$= U_{C_1}(X) \wedge \ldots \wedge U_{C_m}(X) \wedge U_{G_1}(X) \wedge \ldots \wedge U_{G_p}(X) \text{ for all } X \in \bar{X}.$$

The decision membership function $U_D(X)$ can be interpreted as the degree to which each of the alternatives satisfies the constraints and goals.

Given that a person is able to obtain the fuzzy decision, how does he select the alternative to be implemented? Let K be the non-fuzzy set consisting of all the alternatives for which $U_D(X)$ reaches its optimal value. Then the set K will be called the optimizing set, and any alternative in K is an optimizing decision. Simply, he may select as his best alternative any alternative which has the maximum value of membership in D.

Consider the following example. A person has to choose to locate a new plant in one of three locations, X_1, X_2, X_3. He wants to select a location that minimizes real estate cost, G_1, is located near supplies, C_1, and is located near markets, C_2. Let

$$\bar{X} = \{X_1, X_2, X_3\}.$$

By some subjective or objective means he has calculated the membership functions

$$G_1 = \{\frac{0.5}{X_1}, \frac{0.8}{X_2}, \frac{0.3}{X_3}\}$$

$$C_1 = \{\frac{0.7}{X_1}, \frac{0.9}{X_2}, \frac{0.5}{X_3}\}$$

$$C_2 = \{\frac{0.4}{X_1}, \frac{0.2}{X_2}, \frac{0.9}{X_3}\}.$$

Then,

$$D = G_1 \cap C_1 \cap C_2$$

$$D = \{\frac{0.4}{X_1}, \frac{0.2}{X_2}, \frac{0.3}{X_3}\}.$$

The optimal decision would be X_1.

The above decision-making is basically a maximin decision making procedure. For each alternative the minimum possible value on all the constraints and goals was calculated to give D. Then, the maximum value over the alternatives in D was calculated. Certainly the optimal decision is highly sensitive to the assessment values. If, for example,

$$C_1 = \{\frac{0.07}{X_1}, \frac{0.09}{X_2}, \frac{0.05}{X_3}\},$$

then, the optimal decision would be X_3. Thus, great care must be used in assigning the membership grades.

In the above procedure it was assumed that all the goals and constraints are of equal importance in the decision and that the fuzzy set grade assignments are commensurate. However, it is necessary to introduce some methodology to enable a person to handle situations where the different goals and constraints have differing degrees of importance in the decision. In order to achieve this aim, let us review the decision procedure.

(1) Construct a decision set, D, by selecting for each alternative, X_i, the smallest membership value in any of the constraints and goals for X_i. $U_D(X_i) = \text{Min } U(X_i)$ over all C's and G's.

(2) Then, select as the optimal decision, the alternative with the highest membership in D. If there is a tie, one may select either of the tied values.

If a particular goal or constraint is of great importance, one should avoid selecting any alternative as the solution that has a small membership value in this goal. This aim can be accomplished by making those alternatives which are low in important goals and constraints have a low membership in decision set D. (See statement (2) above.) From statement (1), one can conclude that the membership function of D is determined for each alternative by the lowest membership in all the goals and constraints. Therefore, if a person makes even lower the grade of membership of weak alternatives in important goals and constraints, he will be less likely to select a weak alternative. Therefore, each goal and constraint should be assigned a number $\alpha \geqslant 0$, called the power of the goal or constraint. The more important a goal or constraint is, the larger α is. Then, each goal and constraint can be raised to its power, an operation described in the definition section, to reflect its importance in the decision.

For instance, in our previous example, if the goal, G_1, was very important ($\alpha = 2$) and the constraint C_2 was not very important ($\alpha = ½$), one would have

$$D = G_1^2 \cap C_1 \cap C_2^{1/2}$$

$$G_1 = \{\frac{0.5}{X_1}, \frac{0.8}{X_2}, \frac{0.3}{X_3}\}$$

$$G_1^2 = \{\frac{0.25}{X_1}, \frac{0.64}{X_2}, \frac{0.09}{X_3}\}$$

$$C_1 = \{\frac{0.7}{X_1}, \frac{0.9}{X_2}, \frac{0.5}{X_3}\}$$

$$C_2 = \{\frac{0.4}{X_1}, \frac{0.2}{X_2}, \frac{0.9}{X_3}\}$$

$$C_2^{1/2} = \{\frac{0.63}{X_1}, \frac{0.45}{X_2}, \frac{0.95}{X_3}\}.$$

We have increased the membership function of the constraint that was not important, thereby reducing its effect in determining the D set. Likewise, we have decreased the membership function of the important goal and made it more important in the determination of the D. As can be seen from G_1 and G_1^2, the lower membership grades are reduced more significantly then the higher,

$$D = \{\frac{0.25}{X_1}, \frac{0.45}{X_2}, \frac{0.09}{X_3}\},$$

and the optimal decision is X_2, which reflects the fact that G_1 is important and that X_2 is high in G_1.

CONSTRAINTS OVER DIFFERENT SPACES

In a fuzzy decision, the goals and constraints must be fuzzy sets over the alternatives. We would like to develop some techniques to enable a person to handle situations in which some of the goals and constraints are complex relations not necessarily directly related to the alternative space. Such situations would occur, for instance, when the cost of production would depend upon the type of material used rather than the plant selected or when the disruption of the environment depends on the type of waste disposal method used rather than the plant selected. Before we can work an example of such situations, we must introduce some more structure on the fuzzy sets.

Def. 9 Let $\bar{X}$ and $\bar{Y}$ be two spaces. Assume there exists a mapping from $\bar{Y}$ to $\bar{X}$

$$f: \bar{Y} \rightarrow \bar{X}$$
$$f(\bar{Y}) = \bar{X}.$$

Let A be a fuzzy set on $\bar{Y}$ with membership function $U_A(\bar{Y})$.

Then the mapping f induces a fuzzy set $\bar{A}$ in $\bar{X}$ whose membership function is given by

$$U_{\bar{A}}(\bar{X}) = \text{Sup}_{Y \varepsilon f^{-1}(\bar{X})} U_A(\bar{Y})$$

where the supremum is taken over the set of points $f^{-1}(\bar{X})$ in $\bar{Y}$ which are mapped by f into $\bar{X}$.

Thus, the case in which the goals and the constraints are defined as fuzzy sets in different spaces can be reduced to the case in which they are defined in the same space, the alternative space.

As an application, consider our previous example in which X is a set of alternatives and Y is a set of potential managers.

$$\bar{Y} = \{\text{John, Bill, Tom, Al}\}.$$

Assume the following information is available:

John is willing to manage plant X_1

Bill is willing to manage plant X_1

Tom is willing to manage plant X_2

Al is willing to manage plant X_2 or X_3.

Then, we have the mapping

$$f : \bar{Y} \rightarrow \bar{X}$$

defined by

$$f(J) = X_1$$

$$f(B) = X_1$$

$$f(T) = X_2$$

$$f(A) = X_2 \text{ or } X_3 .$$

Now we add an additional fuzzy constraint: we want a competent manager, C_3. Let us have as the fuzzy set C_3 (over $\bar{Y}$)

$$U_{C_3}(\bar{Y}) = \{ \frac{0.4}{J}, \frac{0.7}{B}, \frac{0.8}{T}, \frac{0.6}{A} \}$$

which measures the competence of each of the managers.

In order to apply the fuzzy decision technique, a person must obtain a fuzzy constraint $\bar{C}_3$, a competent manager, over the space of alternatives, $\bar{X}$. In order to achieve this result one should apply the previously defined rule for obtaining fuzzy sets induced by mapping and get

$$U_{\bar{C}_3}(X_1) = \text{Sup}_{Y \varepsilon f^{-1}(X_1)} U_{C_3}(\bar{Y}) .$$

For X_1, The set

$$Y \varepsilon f^{-1}(X_1) = \{ J, B \},$$

$$U_{C_3}(J) = 0.4, \; U_{C_3}(B) = 0.7,$$

therefore $U_{\bar{C}_3}(X_1) = \text{Sup} \{ 0.4, 0.7 \} = 0.7.$

For X_2, the set $Y \varepsilon f^{-1}(X_2) = \{ T, A \}$,

$$U_{C_3}(T) = 0.8, \; U_{C_3}(A) = 0.6$$

Therefore $U_{\bar{C}_3}(X_2) = \text{Sup} \{0.8, 0.6\} = 0.8 .$

For X_3, the set $Y \varepsilon f^{-1}(X_3) = \{A\}$

therefore $U_{\bar{C}_3}(X_3) = \text{Sup}\ \{0.6\} = 0.6$.

Therefore, the constraint of a competent manager induced over the space of alternatives is

$$\bar{C}_3 = \{\frac{0.7}{X_1}, \frac{0.8}{X_2}, \frac{0.6}{X_3}\}.$$

The decision becomes

$$D = G_1 \cap C_1 \cap C_2 \cap \bar{C}_3 .$$

Another type of application a person may want to pursue occurs when some factor in the decision is conditional upon some other space. Before we consider this application, we must define another structure on the fuzzy sets.

Def. 10 A fuzzy set $B(y)$ in $\bar{X} = \{X\}$ is conditional on y if its membership function depends on y as a parameter. This dependence is denoted $U_B(X|y)$.

Suppose y is a member of the space $\bar{Y}$ so that to each $y \varepsilon \bar{Y}$ there exists a fuzzy set $B(y)$ on $\bar{X}$. Thus, there is mapping from $\bar{Y}$ to fuzzy sets in $\bar{X}$. Through this mapping, a given fuzzy set A in $\bar{Y}$ induces a fuzzy set B in $\bar{X}$ defined by

$$U_B(X) = \text{Sup}_y \text{ Min } (U_A(y), U_B X|y))$$

where U_B and U_A are membership functions.

For example, a person may want to minimize the ease of employing workers. All he knows is that the ease of employing workers depends on the proximity to a city.

Let Y = {Near City, Med to City, Far from City} .

The alternative space $\bar{X} = \{X_1, X_2, X_3\}$ are the three alternative plants. The fuzzy set: ease of employing workers, A is given to be

$$U_A(y) = \{\frac{1}{N}, \frac{0.7}{M}, \frac{0.2}{F}\}.$$

The conditional fuzzy set: the location of plant with respect to nearness to a city, is

$$U(X|N) = \{\frac{0.7}{X_1}, \frac{0.5}{X_2}, \frac{0.3}{X_3}\}$$

$$U(X|M) = \{\frac{0.5}{X_1}, \frac{0.5}{X_2}, \frac{0.6}{X_3}\}$$

$$U(X|F) = \{\frac{0.3}{X_1}, \frac{0.5}{X_2}, \frac{0.7}{X_3}\}.$$

These values are obtained from subjective evaluation of the locations of the alternative plants with respect to their proximity to a city. To interpret the above statements, one would say, for example, $U(X|N)$ is the fuzzy set: X is near a city, and $U(X|F)$ is the fuzzy set: X is far from a city. Then, the constraint: ease of employing workers becomes

$$U_{C_4}(X) = \{\frac{0.7}{X_1}, \frac{0.5}{X_2}, \frac{0.6}{X_3}\}.$$

CONCLUSION

We have briefly provided an introduction to the application of fuzzy sets in decision making. The most pressing problem that needs further investigation is the handling of the subjectivity in assigning the grades of membership. Hopefully, a set of evaluation procedures can be developed as was the case for Bayesian theory.

There are many areas where fuzzy set theory has potential applicability. Several interesting areas would be negotiations, consumer purchasing systems, medical diagnosis, and computer dialogue systems. For those readers who are interested in pursuing further information in fuzzy set theory we recommend the reading of [1] and [8]. [1] is particularly recommended in that it provides a format for using fuzzy set theory in dynamic programming. [3] and [7] are also of particular interest in that they provide a probabilistic and entropy structure on fuzzy sets which has application in pattern recognition problems.

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