# GENTZEN'S CUT ELIMINATION THEOREM FOR NON-LOGICIANS

LARRY W. MILLER
Tulane University

The most important theorem of constructive mathematical logic, Gentzen's cut elimination theorem, is largely unknown even among those acquainted with mathematical logic. Usual treatments of mathematical logic, oriented to semantics, use formalization only to find a syntactic property (theoremhood) to correspond to the semantic property (validity). What is needed for a more meaningful use of formalization (and so of mathematical logic) is an effective way of studying formal proofs. The importance of Gentzen's cut elimination theorem lies in the fact that it gives a normal form for formal proofs, so they can be looked at meaningfully. I.e., one can read off certain results about a theorem simply by looking at its normal form proof.[1] The purpose of this paper is to present Gentzen's cut elimination theorem for mathematically minded philosophers.[2] Since our interest is in understanding, we shall give up the usual rigor of logical treatments of this theorem and present it in the spirit of mathematics.

If logic is the study of valid inference among statements, then to make this study mathematical (and so get mathematical logic) we must idealize the inferences and statements so they can be treated mathematically. So we must formalize. Natural language becomes a precisely defined formal language

[1] To show a formal system consistent, it suffices to show that f is not a theorem where f is a false statement. It usually will not be evident that there is no proof of f, but we may be able to see directly that there is no normal form proof of f.

[2] To understand everything perhaps one needs a bit of mathematical sophistication, but the main drift should be clear in any case. Some of the tricks of exposition which in my opinion make this treatment of Gentzen's cut elimination theorem more readable than usual treatments were inspired in part by W. W. Tait's treatment of infinitary propositional calculus. See his "Normal derivability in classical logic" in *The Syntax and Semantics of Infinitary Languages*, Springer Lecture Notes in Mathematics, vol. 72 (1968), 204–236.

– nouns and sentences respectively become terms and well formed formulas (wffs), etc. Some of the wffs are logically simple (do not contain any logical symbols (other than $=$)) – we call these atoms. Other wffs (molecules?) are built up from atoms by negation, conjunction, disjunction, implication, equivalence, universal quantification, and existential quantification – we use $\sim$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\forall$, $\exists$ for these respectively.

In usual treatments of mathematical logic (after one has specified what it means for a wff to be true in an interpretation of the formal system), one syntactically singles out certain wffs (the theorems or provable wffs) by giving axioms and rules of inference. The set of theorems is the least set of wffs containing the axioms and closed under the rules of inference. The idea is to make the set of theorems equal the set of wffs which are valid (true in all interpretations).
E.g., one can take as axioms:

$$\mathrm{A} \supset .\, \mathrm{B} \supset \mathrm{A}$$
$$\mathrm{A} \supset (\mathrm{B} \supset \mathrm{C}) \,.\!\supset\!.\, (\mathrm{A} \supset \mathrm{B}) \supset (\mathrm{A} \supset \mathrm{C})$$
$$\sim\sim\mathrm{A} \supset \mathrm{A}$$
$$\forall \mathrm{x}\mathrm{A}(\mathrm{x}) \supset \mathrm{A}(\mathrm{t})$$
$$\forall \mathrm{x}(\mathrm{A} \supset \mathrm{B}) \supset .\, \mathrm{A} \supset \forall \mathrm{x}\mathrm{B} \qquad \text{if x not free in A}$$
$$\mathrm{t} = \mathrm{t}$$
$$\mathrm{t} = \mathrm{t}' \supset .\, \mathrm{A} \equiv \mathrm{A}' \qquad \text{where A}' \text{ results from A by replacing some occurrences of t in A by t}'.$$

FOR THE LOGICIAN. In the fourth axiom t must be free for x in A(x). In the last axiom t must not occur in A in the scope of a quantifier on a variable free in t or in t′.
One can take as rules of inference:

$$\frac{\mathrm{A} \quad \mathrm{A} \supset \mathrm{B}}{\mathrm{B}} \qquad \text{(modus ponens)}$$

$$\frac{\mathrm{A}}{\forall \mathrm{x}\mathrm{A}} \qquad \text{(generalization)}$$

Here $\dfrac{\mathrm{A}_1 \ldots \mathrm{A}_\mathrm{n}}{\mathrm{B}}$ means if $\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{n}$ are all theorems so is B.
This axiom system is designed to make the Deduction Theorem (which makes theorem proving easy) easy to prove. We

call this axiomatization of first order logic with identity L. $L_i$ will be L minus the identity axioms and $L_q$ will be L minus the quantifier axioms and the generalization rule (so $L_i$ and $L_q$ axiomatize first order logic and propositional calculus with free variables and identity axioms, i.e., elementary calculus). We write $\vdash A$, $\vdash_i A$, or $\vdash_q A$ if A is a theorem of L, $L_i$, or $L_q$. By Gödel's Completeness Theorem $\vdash A$ if and only if A is true in all normal interpretations (where $=$ is interpreted as identity).

As we have mentioned, to make more use of a formal system than just to match the syntactic property of theoremhood with the semantic property of (normal) validity – Gödel's Completeness Theorem – what one needs is a way to study formal proofs effectively. Ideally, one needs to be able to find a simplest proof or a proof in some normal form for any theorem. To facilitate the search for a normal form for proofs, we reformulate our formal system L so we derive not wffs but finite lists of wffs. In this way and by making axioms trivial and putting the meat of the system in the rules of inference, we can isolate the role of each logical connective. We call the lists of wffs *sequents* and denote them by capital Greek letters. (Semantically, one can think of the sequent $A_1, \ldots, A_n$ as $A_1 \vee \ldots \vee A_n$, i.e., the sequent $A_1, \ldots, A_n$ is to be true if $A_i$ is true for some $i = 1, \ldots, n$.)

*Sequent calculus*

Axioms: $A, \sim A$ where A is an atom

$t = t$ where t is a term

Rules of inference:

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \quad (\wedge\text{-rule})$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad (\vee\text{-rule})$$

$$\frac{\Gamma, A}{\Gamma, \forall x A} \quad x \text{ not in } \Gamma \qquad (\forall\text{-rule})$$

$$\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \quad t \text{ free for } x \text{ in } A(x) \qquad (\exists\text{-rule})$$

$$\frac{\Gamma, A \quad \Gamma, \sim A}{\Gamma} \quad \text{(cut-rule)}$$

$$\frac{\Gamma}{\Delta} \quad \text{if each wff in } \Gamma \text{ occurs in } \Delta \quad \text{(S-rule)}$$

E.g. of S-rule:

$$\frac{A, B}{A, A, B}, \quad \frac{A, A, B}{A, B}, \quad \frac{A, B}{B, A}, \quad \frac{A, B}{A, B, C}, \text{ and } \frac{A, B}{A, B}.$$

Given the S-rule our $\wedge$- and cut-rules are equivalent to

$$\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \wedge B} \text{ and } \frac{\Gamma, A \quad \Delta, \sim A}{\Gamma, \Delta}.$$

We have one rule for each logical connective except negation. What about negation and the second identity axiom for L? We find it convenient to assume that for each atom A there is an atom $\sim A$ such that $\sim\sim A = A$. We take as further syntactic identities:

$\sim(A \vee B) = \sim A \wedge \sim B$
$\sim(A \wedge B) = \sim A \vee \sim B$
$\sim\forall x A = \exists x \sim A$
$\sim\exists x A = \forall x \sim A$
$A = A'$ if $A'$ results from A by replacing some occurrences of a term t in A by a term $t'$ where $t = t'$ (with the previously mentioned restriction).

$\Vdash \Gamma$ means $\Gamma$ is a theorem of sequent calculus. $\Vdash^{cf} \Gamma$ means $\Gamma$ is a theorem of sequent calculus without use of the cut-rule. We can now state Gentzen's cut elimination theorem: If $\Vdash \Gamma$ then $\Vdash^{cf} \Gamma$, i.e., the cut-rule is superfluous and can be eliminated.

LEMMA For all wffs A we have $\Vdash^{cf} A, \sim A$.

PROOF By induction on the logical complexity of A. If A is an atom then A, $\sim A$ is an axiom. If $A = B \vee C$ then by the induction hypothesis $\Vdash^{cf} B, \sim B$ and $\Vdash^{cf} C, \sim C$ so we can put

$$\cfrac{\cfrac{\begin{array}{c}|\\ B, \sim B\end{array} \qquad \begin{array}{c}|\\ C, \sim C\end{array}}{B, C, \sim B \wedge \sim C}\ \wedge}{B \vee C, \sim(B \vee C)}\ \vee$$

The proof for $A = B \wedge C$ is dual to this (Interchange wffs with their negations). If $A = \forall xB$ then by the induction hypothesis $\Vdash^{cf} B, \sim B$ so we can put

$$\dfrac{\dfrac{B, \sim B}{B, \exists x \sim B}\ \exists}{\forall xB, \sim \forall xB}\ \forall$$

The proof for $A = \exists xB$ is dual to this.

THEOREM $\Vdash A_1, \ldots, A_n$ if and only if $\vdash A_1 \vee \ldots \vee A_n$.
In particular, $\Vdash A$ if and only if $\vdash A$.

PROOF (only if) We must show $\vdash A \vee \sim A$ and $\vdash t = t$ and that the theorems of L are closed under (translations of) the rules of inference of the sequent calculus. E.g., (for the $\wedge$-rule) if D is a disjunction of wffs of L then we must have if $\vdash D \vee A$ and $\vdash D \vee B$ then $\vdash D \vee (A \wedge B)$. All this is well known.

(if) We must show the translations of the axioms of L are theorems of sequent calculus and that the theorems of sequent calculus are closed under (translations of) modus ponens and generalization. The latter is trivial since the translations are the cut- and $\forall$-rules.

Axiom 1 translates to $\sim A, \sim B, A$ which is a theorem of the sequent calculus by the Lemma.

Axiom 2 translates to $A \wedge B \wedge \sim C, A \wedge \sim B, \sim A, C$ and we have by the Lemma

$$\dfrac{\dfrac{\dfrac{A, \sim A \qquad B, \sim B}{A \wedge B, \sim A, \sim B}\ \wedge \qquad C, \sim C}{A \wedge B \wedge \sim C, \sim A, \sim B, C}\ \wedge \qquad A, \sim A}{A \wedge B \wedge \sim C, A \wedge \sim B, \sim A, C}\ \wedge$$

Axiom 3 goes to $\sim A, A$. Axiom 4 goes to $\exists x \sim A(x), A(t)$ and we have

$$\dfrac{\begin{array}{c} \Big| \\ \sim A(t), A(t) \end{array}}{\exists x \sim A(x), A(t)}\ \exists$$

Axiom 5 translates to $\exists x(A \wedge \sim B(x)), \sim A, \forall x B(x)$ and we have

$$\dfrac{\dfrac{\dfrac{A, \sim A \qquad \sim B(x), B(x)}{A \wedge \sim B(x), \sim A, B(x)}\ \wedge}{\exists x(A \wedge \sim B(x)), \sim A, B(x)}\ \exists}{\exists x(A \wedge \sim B(x)), \sim A, \forall x B(x)}\ \forall$$

The identity axioms are trivial so the theorem is proved.

Our reformulation of L as a sequent calculus is now justified. We observe that in the cut-rule A disappears so if we look at proofs using the cut-rule we do not know what wffs have been "cut" out of the proof. However, in each rule of inference other than the cut-rule, each wff that occurs in a premise also occurs (in some sense) in the conclusion. Hence, each wff in a cut-free proof of $\Gamma$ occurs (in some sense) in $\Gamma$. We want to make this precise.

DEFINITION A is a subformula of A and of $\sim A$.
A and B are subformulas of $A \vee B$ and of $A \wedge B$.
For any term t $A(t)$ is a subformula of $\forall x A(x)$ and of $\exists x A(x)$.
If A is a subformula of B and B a subformula of C then A is a subformula of C.
Nothing else is a subformula

Then each wff in a premise of any rule of inference other than the cut-rule is a subformula of a wff in the conclusion.

SUBFORMULA PROPERTY OF CUT-FREE PROOFS: Each wff in a cut-free proof of $\Gamma$ is a subformula of a wff in $\Gamma$.

If we could prove Gentzen's cut elimination theorem (If $\Vdash \Gamma$ then $\Vdash^{cf} \Gamma$) then we would have a normal form for proofs which would be meaningful in the sense that no wff could occur which was not a subformula of a wff in the conclusion.

E.g. 1. If Gentzen's cut elimination theorem holds then

L, $L_i$, and $L_q$ are consistent since f (where f is a false wff) cannot be a theorem of sequent calculus by the subformula property of cut-free proofs – f cannot have a cut-free proof.

2. Note that if $\Gamma$ is $=$-free then any cut-free proof of $\Gamma$ is $=$-free by the subformula property of cut-free proofs. Hence, if Gentzen's cut elimination theorem holds and A is $=$-free then $\vdash$ A if and only if $\vdash_i$ A (eliminability of $=$). In other words, L, first order logic with identity, is a conservative extension of $L_i$, first order logic.

3. Again note that if $\Gamma$ is quantifier-free then any cut-free proof of $\Gamma$ is quantifier-free by the subformula property of cut-free proofs. So if Gentzen's cut elimination theorem holds and A is quantifier-free then $\vdash$ A if and only if $\vdash_q$ A. In other words, L is a conservative extension of $L_q$, propositional calculus with free variables and identity axioms. Exactly the same argument shows $L_i$ is a conservative extension of propositional calculus with free variables.

REMARK Examples 2 and 3 show that with respect to $=$-free or quantifier-free wffs Gentzen's cut elimination theorem would enable one to show that $=$ or quantifiers are ideal in the Kantian sense, i.e., their use (to prove $=$- or quantifier-free wffs) is non-essential. Hilbert wanted to justify the use of infinity in mathematics by showing it is ideal. Example 3 shows this exactly for infinity-as-quantifiers.

We want to turn now to the proof of Gentzen's cut elimination theorem. But first we need a result which says roughly that the $\wedge$-, $\vee$-, $\forall$-, and $\exists$-rules can be inverted; i.e., given a (cut-free) proof of the conclusion, one gets a (cut-free) proof of each premise.

INVERTIBILITY LEMMA

(1) If $\Vdash^{cf} \Gamma, A \wedge B$ then $\Vdash^{cf} \Gamma, A$ and $\Vdash^{cf} \Gamma, B$.

(2) If $\Vdash^{cf} \Gamma, A \vee B$ then $\Vdash^{cf} \Gamma, A, B$.

(3) If $\Vdash^{cf} \Gamma, \forall x A(x)$ then for any term t $\Vdash^{cf} \Gamma, A(t)$.

(4) If $\Vdash^{cf} \Gamma, \exists x A(x)$ then $\Vdash^{cf} \Gamma, A(t_1), \ldots, A(t_n)$ where $t_1, \ldots, t_n$ are all the terms such that the $\exists$-rule is used in the cut-free proof of $\Gamma, \exists x A(x)$ in the form

$$\frac{\Delta_i, A(t_i)}{\Delta_i, \exists x A(x)}\ \exists.$$

PROOF Replace each occurrence of $\left|\begin{array}{l} A \wedge B \\ A \vee B \\ \forall x A(x) \\ \exists x A(x) \end{array}\right|$ which leads to the occurrence of $\left|\begin{array}{l} A \wedge B \\ A \vee B \\ \forall x A(x) \\ \exists x A(x) \end{array}\right|$ in the conclusion by $\left|\begin{array}{l} A \text{ or by } B \\ A, B \\ A(t) \\ A(t_1), \ldots, A(t_n) \end{array}\right|$ to get a cut-free proof of $\left|\begin{array}{l} \Gamma, A \text{ or of } \Gamma, B \\ \Gamma, A, B \\ \Gamma, A(t) \\ \Gamma, A(t_1), \ldots, A(t_n) \end{array}\right|$. (In the case of (3) one must further substitute t for x in the proper places.) Each (affected) instance of a $\left|\begin{array}{c} \wedge \\ \vee \\ \forall \\ \exists \end{array}\right|$-rule becomes an instance of the S-rule so the proof is complete.

GENTZEN'S CUT ELIMINATION THEOREM If $\Vdash \Gamma$ then $\Vdash^{cf} \Gamma$.

PROOF Since there are only finitely many cuts in any proof it suffices to show

If $\Vdash^{cf} \Delta, A$ and $\Vdash^{cf} \Delta, \sim A$ then $\Vdash^{cf} \Delta$.

This enables us to eliminate one cut at a time until none are left. We show this by induction on the logical complexity of A. In other words, the procedure is:

1. We show how to replace any cut on a logically complex wff by at most a finite number of cuts (all on logically simpler wffs).
2. We show how to eliminate any cut on an atom (a logically simple wff).

Case 1. $A = B \wedge C$. By the Invertibility Lemma $\Vdash^{cf} \Delta, B$ and $\Vdash^{cf} \Delta, C$ and $\Vdash^{cf} \Delta, \sim B, \sim C$ so we can put

$$\cfrac{\Delta, B \qquad \cfrac{\Delta, C \qquad \Delta, \sim B, \sim C}{\Delta, \sim B}\ \text{cut on C}}{\Delta}\ \text{cut on B}$$

We have replaced one cut on $B \wedge C$ by two cuts on B and C which by the induction hypothesis can be eliminated.
Case 2. $A = B \vee C$. Dual to Case 1.
Case 3. $A = \exists x B(x)$. By the Invertibility Lemma $\Vdash^{cf} \Delta, B(t_1), \ldots, B(t_n)$ and $\Vdash^{cf} \Delta, \sim B(t_i)$ for each $i = 1, \ldots, n$ so we can put

$$\dfrac{\dfrac{\overset{\vdots}{\Delta, \sim B(t_n)} \quad \overset{\vdots}{\Delta, B(t_1), \ldots, B(t_n)}}{\Delta, B(t_1), \ldots, B(t_{n-1})}\text{cut on } B(t_n) \qquad \overset{\vdots}{\Delta, \sim B(t_{n-1})}}{\Delta, B(t_1), \ldots, B(t_{n-2})}\text{cut on } B(t_{n-1})$$

$$\vdots$$

$$\dfrac{\overset{\vdots}{\Delta, \sim B(t_1)} \qquad \overline{\Delta, B(t_1)}}{\Delta}\text{cut on } B(t_1)$$

We have replaced one cut on $\exists x B(x)$ by n cuts on $B(t_i)$ for $i = 1, \ldots, n$ each of which can be eliminated by the induction hypothesis.
Case 4. $A = \forall x B(x)$. Dual to Case 3.
Case 5. A is an atom. Take the cut-free proof of $\Delta, A$ and replace each occurrence of A which leads to the occurrence of A in the conclusion by $\Delta$. The result is a cut-free proof of $\Delta, \Delta$ (and so by the S-rule of $\Delta$) unless $A, \sim A$ was used as an axiom. After the replacement this becomes $\Delta, \sim A$ but we have a cut-free proof of $\Delta, \sim A$ so we tack it on wherever needed and we're done.

COROLLARY 1 L, $L_1$, and $L_q$ are consistent.

COROLLARY 2 If A is =-free then $\vdash A$ if and only if $\vdash_1 A$. So L is a conservative extension of $L_1$.

PROOF If $\vdash A$ then $\Vdash^{cf} A$ and since A is =-free by the subformula property of cut-free proofs, the identity axioms are never used.

COROLLARY 3 If A is quantifier-free then $\vdash A$ if and only if $\vdash_q A$. So L is a conservative extension of $L_q$.

PROOF If $\vdash A$ then $\Vdash^{cf} A$ and since A is quantifier-free the quantifier axioms are never used.

Before we state the next Corollary, a generalization of Corollary 3, we must explain what the Herbrand resolution of a wff is. With each wff A in L we associate a prenex normal form[1] wff H(A) – the Herbrand resolution of A – in an extension of L such that $\forall$ does not occur in H(A).

DEFINITION If the prenex normal form of A has the form $\exists x_1 \ldots \exists x_n \forall y B(x_1, \ldots, x_n, y)$ where $B(x_1, \ldots, x_n, y)$ is in prenex normal form then $H(A) = H(\exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n, f(x_1, \ldots, x_n)))$ where f is a new n-ary function constant. We have eliminated one $\forall$-quantifier so if we repeat this procedure we get our Herbrand resolution of A.
E.g. If $A = \forall x \exists y \forall z B(x, y, z)$ then $H(A) = \exists y B(c, y, f(y))$ where c is a new individual constant and f is a new 1-ary function constant.

Since $\vdash \exists x_1 \ldots \exists x_n \forall y B(x_1, \ldots, x_n, y)$ if and only if $\exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$ is a theorem of L with the new function constant added, it is easy to show

> A is a theorem of L if and only if H(A) is a theorem of L* where L* is L plus all new function constants used to form H(A)

COROLLARY 4 (FIRST $\varepsilon$-THEOREM) If A is in prenex normal form and $\vdash A$ then $\vdash_q A_1 \vee \ldots \vee A_n$ where each $A_i$ is a substitution instance of the matrix of A.

PROOF Since A is a theorem of L H(A) is a theorem of L* and so $\Vdash^{cf} H(A)$. By repeated use of the Invertibility Lemma (4) $\Vdash^{cf} A_1, \ldots, A_n$ which is quantifier-free so $\vdash_q A_1 \vee \ldots \vee A_n$.

The First $\varepsilon$-Theorem is a key theorem for proof theory of first order logic. Its importance lies in the facts that (1) it enables one to prove Herbrand's Theorem which gives a constructive necessary and sufficient condition for theoremhood (in L), (2) it enables one to prove easily Gentzen's erweitete Hauptsatz which enables one to permute inferences, (3) it underlies Hilbert's method of proving the consistency of various formalized theories in a constructive manner (see

[1] A wff A of L is in prenex normal form if it is of the form $Q_1x_1 \ldots Q_nx_nB$ where each $Q_i$ is either $\forall$ or $\exists$ and B, the matrix of A, is quantifier-free.

the Consistency Theorem), and (4) it justifies the non-essential use of quantifiers to prove quantifier-free wffs.

REMARK If we would have proved a cut elimination theorem for $\varepsilon$-calculus we could have given a very easy proof of the Second $\varepsilon$-theorem which says $\varepsilon$-calculus is a conservative extension of L.[1]

We turn now to Herbrand's Theorem. The degree of a term t is one greater than the maximum degree of any proper subterm of t (so if t has no proper subterms it is of degree one). If A is in prenex normal let A* be the set of new function constants used to form H(A) plus an individual constant (if not already present). For $0 < m$ the *m-reduction* of A is the disjunction of all wffs got by substituting for the free variables in the matrix of H(A) terms which are of degree $\leq m$ and which are built up from symbols in A*.

HERBRAND'S THEOREM If A is in prenex normal form and contains no free variables then $\vdash A$ if and only if for some m the m-reduction of A is a tautology.

PROOF (only if) If $\vdash A$ then by the First $\varepsilon$-Theorem $\vdash_q A_1 \vee \ldots \vee A_n$ where each $A_i$ is a substitution instance of the matrix of A so $A_1 \vee \ldots \vee A_n$ is a tautology. Replace any terms not made up solely of symbols in A* by an individual constant in A*. If m is the maximum degree of any term in the result then we have a subdisjunction of the m-reduction of A so the latter is a tautology.

(if) If the m-reduction $A_1 \vee \ldots \vee A_p$ is a tautology then $\vdash_q A_1 \vee \ldots \vee A_p$. For each $i = 1, \ldots, p$ we have a proof of H(A) in L* from $A_i$ so H(A) is a theorem of L*. Hence, $\vdash A$.

We want now to give an application of the Herbrand Theorem to illustrate its usefulness. If A is of the form $\forall x_1 \ldots \forall x_m \exists y_1 \ldots \exists y_n B$ where B is quantifier-free and contains no n-ary function constants for $n > 0$ then it is decidable whether A is a theorem of L or not.

PROOF $H(A) = \exists y_1 \ldots \exists y_n B(c_1, \ldots, c_m, y_1, \ldots, y_n)$ where $c_i$ is a new individual constant. The terms that can be built up from symbols in A* are $c_1, \ldots, c_m$ all of degree one.

[1] See A. C. Leisenring, ***Mathematical Logic and Hilbert's $\varepsilon$-Symbol***, (London: McDonald & Co., 1969).

Hence, the l-reduction of **A** has mn disjuncts and so can be tested for tautologousness.

Finally, we mention a result used by Hilbert to prove constructively the consistency of various formal theories including a first order formalization of arithmetic restricted so that the induction axiom is quantifier-free.

Consistency theorem If $\Gamma$ is a set of wffs of L and a valuation can be given which makes each substitution instance of the matrix of a wff in $\Gamma$ true then $\Gamma$ is consistent (constructively).

Proof An easy generalization of the First $\varepsilon$-Theorem reads

If A and each wff in $\Gamma$ is in prenex normal form and $\Gamma \vdash A$ then $\Delta \vdash_q A_1 \vee \ldots \vee A_n$ where each wff in $\Delta$ is a substitution instance of the matrix of a wff in $\Gamma$ and each $A_i$ is a substitution instance of the matrix of A.

Using this if $\Gamma \vdash f$ where f is some false wff then $\vdash_q \sim A_1 \vee \ldots \vee \sim A_n$ where each $A_i$ is a substitution instance of the matrix of a wff in $\Gamma$. Hence, $\sim A_1 \vee \ldots \vee \sim A_n$ is a tautology so there is no valuation making each substitution of the matrix of each wff in $\Gamma$ true.