Poisson processes — Year 2013/14
M1MMD

Mid-term exam: solutions

Exercise 1. 1. Since $N(t) \geq n \Leftrightarrow T_n \leq t$ we have

$$\mathbb{P}(N(t) \geq n) = \mathbb{P}(t - T_n \geq 0) = \mathbb{P}\left(\exp(t - T_n) \geq 1\right).$$

2. By Markov's inequality

$$\mathbb{P}\left(\exp(t - T_n) \geq 1\right) \leq \mathbb{E}\left[\exp(t - T_n)\right].$$

Moreover, since the variables τ_i are i.i.d.

$$\mathbb{E}\left[\exp(-T_n)\right] = \mathbb{E}\left[\exp\left(-\sum_{i=1}^{n} \tau_i\right)\right] = \left(\mathbb{E}\left[\exp(-\tau_1)\right]\right)^n.$$

Combining with the first question we get

$$\mathbb{P}(N(t) \geq n) \leq \mathrm{e}^t \left(\mathbb{E}\left[\exp(-\tau_1)\right]\right)^n.$$

3. We have $\tau_1 \geq 0$ hence $\mathrm{e}^{-\tau_1} \leq 1$ and thus

$$\mathbb{E}[\mathrm{e}^{-\tau_1}] \leq 1.$$

Moreover, if $\mathbb{E}[\mathrm{e}^{-\tau_1}] = 1$ then $\mathrm{e}^{-\tau_1} = 1$ hence $\tau_1 = 0$ almost surely, which contradicts the hypothesis. Therefore

$$\mathbb{E}[\mathrm{e}^{-\tau_1}] < 1.$$

The proof that $\mathbb{E}[\mathrm{e}^{-\tau_1}]$ is positive is similar.

4. Using the previous questions and letting $\alpha = \mathbb{E}[\mathrm{e}^{-\tau_1}]$ we get

$$\mathbb{E}\left[N_t^k\right] = \sum_{i=1}^{+\infty} i^k \ \mathbb{P}(N_t = i) \leq \sum_{i=1}^{+\infty} i^k \ \mathbb{P}(N_t \geq i) \leq \mathrm{e}^t \sum_{i=1}^{+\infty} i^k \alpha^i.$$

Since $\alpha \in (0,1)$ the series above is convergent, which shows that N_t^k is integrable.

Exercise 2. 1. The conditionnal law of $(T_1, \ldots, T_n)$ given $N(t) = n$ is the law of the increasing rearrangment of n independent variables uniform on $[0, t]$.

2. Since the variables D_i are independent of N_t and of the variables T_i and have the same law, we have

$$\begin{aligned}
\mathbb{E}[\mathcal{D}_t \mid N_t = n] &= \mathbb{E}\left[\sum_{i=1}^{n} D_i \mathrm{e}^{\alpha(t - T_i)} \mid N_t = n\right] \\
&= \sum_{i=1}^{n} \mathbb{E}[D_i] \ \mathbb{E}\left[\mathrm{e}^{\alpha(t-T_i)} \mid N_t = n\right] \\
&= \mathbb{E}[D_1] \ \mathbb{E}\left[\sum_{i=1}^{n} \mathrm{e}^{\alpha(t-T_i)} \mid N_t = n\right]
\end{aligned}$$

Let $U_1, \dots, U_n$ be independent variables uniform on $[0,t]$ and let $(U_1^*, \dots, U_n^*)$ be their increasing rearrangment. By question 1 and since the sum does not depend on the order of the terms

$$\mathbb{E}\left[\sum_{i=1}^{n} \mathrm{e}^{\alpha(t-T_i)} \mid N_t = n\right] = \mathbb{E}\left[\sum_{i=1}^{n} \mathrm{e}^{\alpha(t-U_i^*)}\right] = \mathbb{E}\left[\sum_{i=1}^{n} \mathrm{e}^{\alpha(t-U_i)}\right]$$
$$= n\mathbb{E}[\mathrm{e}^{-\alpha(t-U_1)}] = \frac{n(1-\mathrm{e}^{-\alpha t})}{\alpha t}.$$

Therefore

$$\mathbb{E}[\mathcal{D}_t \mid N_t = n] = \mathbb{E}[D_1]\,\frac{n(1-\mathrm{e}^{-\alpha t})}{\alpha t}.$$

3. By the previous question

$$\mathbb{E}[\mathcal{D}_t \mid N_t] = \mathbb{E}[D_1]\,\frac{N_t(1-\mathrm{e}^{-\alpha t})}{\alpha t}.$$

Taking expectation again we get

$$\mathbb{E}[\mathcal{D}_t] = \mathbb{E}[D_1]\,\frac{\lambda(1-\mathrm{e}^{-\alpha t})}{\alpha}.$$

Exercise 3. 1. $M_t = N_{F(t)}$ has Poisson distribution of parameter $\lambda F(t)$.

2. Let $0 = t_0 \le t_1 \le \dots \le t_n$ and set $s_i = F(t_i)$, so that

$$(M_{t_i} - M_{t_{i-1}})_{i=1}^{n-1} = (N_{s_i} - N_{s_{i-1}})_{i=1}^{n-1}.$$

Since F is increasing we have

$$ss_0 \le s_1 \le \dots \le s_n.$$

Therefore increments of M are increments of N, which are independent by hypothesis.

3. We consider the filtration $\mathcal{F}_t = \sigma(M_u, u \le t)$. Let $s \le t$, then M_s is $\mathcal{F}_s$-measurable and since M has independent increments, $M_t - M_s$ is independent of $\mathcal{F}_s$. Therefore

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}[M_s + (M_t - M_s) \mid \mathcal{F}_s] = M_s + \mathbb{E}[M_t - M_s].$$

Since $\mathbb{E}[M_t] = \lambda F(t)$ for all t, we get

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s + \lambda F(t) - \lambda F(s).$$

In other words

$$\mathbb{E}[M_t - \lambda F(t) \mid \mathcal{F}_s] = M_s - \lambda F(s),$$

which shows that $M_t - \lambda F(t)$ is an $\mathcal{F}_t$-martingale.

4. If M has stationary increments then $M_{t+s} - M_s$ and M_t have the same law, hence the same expectation. We obtain

$$F(s+t) = F(s) + F(t)$$

for all $s, t \ge 0$. This equality implies easily that $F(t) = tF(1)$ for all $t \ge 0$ Indeed one first shows the equality for rational t and then use the density of $\mathbb{Q}$ together with the continuity of F.
Conversely is $F(t) = ct$ for some $c \ge 0$ then M is a Poisson process of parameter $c\lambda$ so M has stationary increments.