## Feuille d'exercices n°4 : Ruin theory

*In all exercises that use the Cramer-Lundberg model, we denote by $c > 0$ the premimum rate, we denote by $\lambda > 0$ the intensity of the Poisson process that models the number of claims and we denote by $u \geq 0$ the initial wealth of the insurer.*

**Exercise 1.**

1. Show that the following distribution are thin tailed :
   (a) the distribution of a nonnegative bounded random variable.
   (b) the Gamma distribution.
   (c) the Weibull distribution, with parameters $C > 0, \gamma \geq 1$. The density function of a Weibull distribution with parameters $C, \gamma$ is

$$f(x) = C\gamma x^{\gamma-1} \exp(-Cx^{\gamma}) \mathbf{1}_{\{x>0\}}.$$

2. Show that the following distributions are sub-exponential :
   (a) the Pareto distribution with parameters $\alpha > 0, \beta > 0$ ($f(x) = \alpha\beta^{\alpha}/(\beta + x)^{\alpha+1}, x > 0$).
   (b) the Weibull distribution with parameters $C > 0, \gamma < 1$.

**Exercise 2.** The parameters $c > 0$, $\lambda > 0$ et $\beta > 0$ are fixed throughout. For every integer $k \in \mathbb{N}^*$, we consider the Cramer-Lundberg model, where the costs of the claims are distributed according to a $\Gamma(k, \beta)$ distribution. Set $\psi^{(k)}(u)$ for the ruin probability of this model. Show that for every $u > 0$ and every $k \in \mathbb{N}^*$,

$$\psi^{(k)}(u) \leq \psi^{(k+1)}(u).$$

**Exercise 3.** We condider the Cramér-Lundberg model, where the costs of the claims follow an exponential distribution with parameter $\gamma > 0$. The safety loading $\rho$ is positive. We wish to give en explicit formula for the ruin probability $\psi(u)$.

1. Show that the exponential distribution is thin tailed and compute the corresponding adjustment coefficient $R$.
2. Derive a "good" upper bound for the ruin probability thanks to Lundberg inequality.
3. Write the renewal equation satisfied by $u \mapsto e^{Ru}\psi(u)$.
4. Using the renewal theorem, solve the equation and compute $\psi(u)$ as a function of $\gamma, \rho$ and $u$.

**Exercise 4.** We consider the setting of the Cramer-Lundberg model, where the costs $X_i, i \geq 1$ follow a Pareto distribution with index $\alpha > 1$, $\beta = 1$, *i.e.*

$$\overline{F}_{X_1}(x) = (1+x)^{-\alpha}, \quad x \geq 0.$$

1. Compute $\mu = \mathbb{E}[X_1]$ and the associated safety loading $\rho$. For which values $c$ do we have $\rho > 0$?
2. Show that $\int_0^\infty e^{ux} F_{X_1,I}(\mathrm{d}x) = \infty$ for every $u > 0$. Derive that $F_{X_1,I}$ is not thin tailed.
3. Show that $F_{X_1,I}$ is subexponential. What can we say about the ruin probability $\psi(u)$ as $u \to \infty$?

**Exercise 5**. We work in the Cramer-Lundberg setting.

**Partie A.** The r.v. $X_i, i \geq 1$ that model the cost claims have a density

$$f(x) = \frac{1}{2\sqrt{x}} e^{-\sqrt{x}} \mathbf{1}_{\{x>0\}}.$$

1. Compute $\mu = \mathbb{E}[X_1]$ and $\overline{F}_{X_1}(x), x \geq 0$.
2. For every $x \geq 0$, set $F_{X_1,I}(x) = \mu^{-1} \int_0^x \overline{F}_{X_1}(y)\mathrm{d}y$ and

$$q(x) = \frac{\overline{F}_{X_1}(x)/\mu}{\overline{F}_{X_1,I}(x)}.$$

   (a) Show that

$$\int_x^\infty e^{-\sqrt{y}}\mathrm{d}y = 2e^{-\sqrt{x}}(\sqrt{x}+1), \quad \forall x \geq 0,$$

   and derive a simple expression for $q(x)$.

   (b) Derive that $F_{X_1,I}$ is the cumulative distribution function of a subexponential distribution.
3. Give an equivalent of the ruin probability $\psi(u)$ as $u \to \infty$. Express this equivalent as a function of $f$ and the parameters $c, \lambda$.

**Partie B.** We now assume that the $X_i, i \geq 1$ have density

$$g(x) = 2xe^{-x^2} \mathbf{1}_{\{x>0\}}.$$

1. Show that $\mu = \sqrt{\pi}/2$.
2. Show that $X_1$ is thin tailed.
3. Prove the existence of the adjustment coefficient $R$.
4. Express the integral $\int_0^\infty y e^{Ry-y^2}\mathrm{d}y$ as a function of $c, \lambda$ and $R$. Derive an expression for $\int_0^\infty e^{Ry-y^2}\mathrm{d}y$ as a function of $c, \lambda$ and $R$.
5. Compute $\mathrm{d}F_{X_1,I}$ and give the renewal equation satisfied by the function $u \mapsto e^{Ru}\psi(u)$, and check that the required conditions are satisfied here.

6. Give the asymptotic behaviour of the ruin probability $\psi(u)$ as $u \to \infty$ as a function of $c, \lambda, R$ and $\pi$.

**Exercise 6.**

1. **Part 1**

   An insurer has a risky portfolio with risks which are partitioned into two classes : the big claims, denoted by $X_i^1, i \geq 1$ and the small claims, denoted by $X_i^2, i \geq 1$. It is moreover assumed that the two kind of risks are independent. The total claim amount of the insurer at time $t$ is denoted by

   $$S_t = S_t^1 + S_t^2$$

   where $S_t^1 = \sum_{i=1}^{N_t^1} X_i^1$ is the total claim amount of the first kind (big claims) and $S_t^2 = \sum_{i=1}^{N_t^2} X_i^2$ is the total claim amount of the second kind (small claims).
   The processes $(N^i)_{i=1,2}$ are independent Poisson processes with intensities $\lambda^i$, and they are independent of the different costs $X_i^1, X_i^2, i \geq 1$. We assume that $(X_i^1, i \geq 1)$ are i.i.d. with distribution $F^1$ and that the $(X_i^2, i \geq 1)$ are i.i.d. with distribution $F^2$ .

   (a) Compute the value of the moment generating function $M_{S_t^1}$, of $S_t^1$, the moment generating function of $S_t^2$ and derive the moment generating function of $S_t$.

   (b) Check that $S$ is a compound Poisson process that will be written in the form

   $$S_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

   where $N$ is a Poisson process with intensity $\lambda = \lambda^1 + \lambda^2$ and $Y_i, i \geq 1$ are i.i.d. with distribution $F$ being a mixture of of $F^1$ and $F^2$. Compute the mixture coefficients explicitly.

   (c) We now assume that $F^1 = \mathcal{E}(\gamma)$ is the exponential distribution with parameter $\gamma > 0$ and $F^2 = \mathcal{P}ar(\alpha, 1)$ is the Pareto distribution with parameters $\alpha$, 1, with $\alpha > 1$. Compute in that case the density function $f_{Y_1,I}(y)$, the function $\bar{F}_{Y_1,I}(y)$, the expectation $\mathbb{E}[Y_1]$ and the coefficient $q(y) = \frac{f_{Y_1,I}(y)}{\bar{F}_{Y_1,I}(y)}$.

   (d) Consider the Cramer-Lundberg model

   $$U_t = u + ct - S_t, \quad t \geq 0$$

   where $u \geq 0$ is the initial wealth of the company. We assume that the safety loading coefficient $\rho$ is the same for each class and we take as premimu rate

   $$c := (1 + \rho)\mathbb{E}[Y_1]; \text{ with } \quad \rho > 0.$$

   Under the assumption of Question (c), compute $c$ as a function of the model parameters and compute an asymptotic equivalent $\psi(u)$.

2. **Part 2.**

The insurer decides to mix the two groups adding an insurance excess $a > 0$. This means that the insurer only pays for claims with a cost greater than a threshold $a > 0$, and for a claim with cost $Z > a$, the insurer only covers the amount $(Z - a)$ We consider the Cramer -Lundberg model

$$U_t = u + ct - S_t \quad \text{where} \quad S_t = \sum_{i=1}^{N_t} Y_i^a \quad \text{and} \quad Y_i^a = (Z_i - a)^+$$

$N$ being a Poisson process with intensity $\lambda$.

(a) Compute $\mu = \mathbb{E}[Y_1^a] = \mathbb{E}[(Z_1 - a)^+]$. when the claims have a cost $Z$ following a $\mathcal{E}(\gamma)$ distribution.

(b) Compute $M_{Y_1^a}$, the moment generating function of $Y_1^a$, and derive the moment generating function of $S_t$.

(c) Show that $M_{S_t}(u) = M_{S'_t}(u)$ where

$$S'_t = \sum_{i=1}^{N'_t} Z_i$$

$N'_t$ being a Poisson process with intensity $\lambda \exp(-\gamma a)$ indepenent of the $Z_i$'s.

(d) Derive that the processes $S$ and $S'$ have the same distribution.

(e) Derive that the risk process $U$ has the same distribution as $U'$ defined as

$$U'_t = u + ct - S'_t, \quad t \geq 0.$$

Show that $\psi(u) = \mathbb{P}[\inf_{t\geq 0} U_t < 0] = \mathbb{P}[\inf_{t \geq 0} U'_t < 0]$ and compute an asymptotic equivalent for $\psi(u)$.