Feuille d'exercices n°3 : Compound Poisson processes, renewal processes.

Exercise 1. Give a necessary and sufficient condition for a compound Poisson process to be a standard Poisson process.

Exercise 2. Find the renewal function and the renewal measure of a renewal process when the interarrival times are distributed according to a Gamma distribution with parameters 2, $\beta > 0$. Find back the renewal theorem and the key renewal theorem in that case.

Exercise 3. An appliance has repeated failures in time and is repaired after each failure. Denote by $(X_i, i \geq 1)$ the successive durations when the appliance is functioning, and $(Y_i, i \geq 1)$ the successive durations when the appliance is being repaired. In other words, the appliance is in order in the time interval $[0, X_1)$, it is being repaired in the time interval $[X_1, X_1 + Y_1)$, then it is in order again in the time interval $[X_1 + Y_1, X_1 + Y_1 + X_2)$ and so on. Assume that the $X_i's$ are i.i.d. and that their common law has no atom and satisfies $\mathbb{P}(X_1 > 0) = 1$ and $\mathbb{E}[X_1] < \infty$. Assume that the $Y_i's$ have the same properties and that they are moreover independent of the $X_i's$.

Denote by $p(t)$ the probability that the appliance is functioning at time t.

1. Let $Z_1 = X_1 + Y_1$ and let F be the cumulative distribution function of Z_1. Show that $p(t)$ satisfies the renewal equation
$$p(t) = \mathbb{P}(X_1 > t) + \int_0^t p(t-s)\mathrm{d}F(s), \quad \forall t \geq 0.$$
2. Derive
$$\lim_{t\to\infty} p(t) = \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_1 + Y_1]}.$$

Exercise 4. Let N be a renewal process and let $F(t) = T_{N(t)+1} - t$ denote the time elapsed between t and the time of the $N(t)+1$-th renewal, for $t \geq 0$. We denote by $\mathbb{P}_{T_1}$ the duration of the interarrival times and F_{T_1} its cumulative distribution function.

1. Show that for every $x \geq 0$, the function $\mathbb{P}(F(t) > x)$ satifies the renewal equation
$$\mathbb{P}(F(t) > x) = 1 - F_{T_1}(t+x) + \int_0^t \mathbb{P}(F(t-u) > x)\mathrm{d}\mathbb{P}_{T_1}(u), \quad t \geq 0.$$
(Hint : start by decomposing $\mathbb{P}(F(t) > x)$ in two terms, according to the event $\{T_1 > t\}$ or $\{T_1 \leq t\}$.)

2. Solve this equation when the interarrival times follow an exponential distribution.

Exercise 5. In the same setting as Exercise 4, we consider a renewal process with interarrival times distributed according to a Pareto distribution

$$\mathbb{P}(\tau_1 > x) = \frac{1}{(1+x)^\alpha}, \quad x \geq 0.$$

1. Let X be a nonnegative random variable. Show that for all $r > 0$,

$$\int_0^\infty r x^{r-1} \mathbb{P}(X > x) dx = \mathbb{E}[X^r].$$

2. Use the previous identity and the renewal equation satisfied by $F(t)$ to show that

$$\mathbb{E}[F(t)^2] = \int_0^t \Big(\int_0^\infty \frac{2x}{(1+t-u+x)^\alpha} dx \Big) dm(u),$$

where $m(t) = \mathbb{E}[N(t)]$ is the renewal measure of N.

3. Derive that for $\alpha > 3$, we have

$$\mathbb{E}[F(t)^2] \longrightarrow 2 \int_0^\infty x(1+x)^{1-\alpha} dx \ \text{ as } \ t \to \infty$$

and compute this limit explicitly.