Processus de Poisson et méthodes actuarielles (2014-2015)

Feuille d'exercices n°2 : Mixed Poisson process, total claim amount, renewal theory.

Exercise 1. Let $\tilde{N}$ be a mixed Poisson process and denote by

$$0 < \tilde{T}_1 < ... < \tilde{T}_n < ...$$

its jumps times. Prove that the conditional distribution of $(\tilde{T}_1, ..., \tilde{T}_n)$ given $\tilde{N}(t) = n$ ($n \in \mathbb{N} \setminus \{0\}$) coincides with the distribution of the order statistic of n independent random variables with common uniform distribution on $[0, t]$.

Exercise 2. A random variable X follows a negative binomial distribution on $\{0, 1, 2, ...\}$ with parameters $r > 0$ and $p \in (0, 1)$ if

$$\mathbb{P}(X = k) = \frac{\Gamma(r + k)}{\Gamma(r)k!} p^r (1 - p)^k, \quad \forall k \geq 0.$$

Let $\tilde{N}$ be a mixed Poisson process with mixture distribution $\Theta \sim \Gamma(\gamma, \beta)$. What is the distribution of $\tilde{N}(t)$? The process $\tilde{N}$ is called a *negative binomial process*. The negative binomial law is also called mixed Poisson or mixed Gamma-Poisson distribution.

Exercise 3. Let $N = (N_t, t \geq 0)$ be a standard Poisson process with intensity $\lambda > 0$. Let $f : [0, \infty) \to [0, \infty)$ be a locally bounded Borel function. Set

$$N(f)_t = \sum_{i \geq 1} f(T_i) \mathbf{1}_{\{T_i \leq t\}} \text{ for } t \geq 0,$$

where the $(T_i)_{i \geq 1}$ are the jump times of N.

1. Show that for all $t \geq 0$, we have $N(f)_t < \infty$ almost-surely.
2. If $f(s) = \mathbf{1}_{(a,b]}(s)$ where $[a, b] \subset [0, t]$, what is the distribution of $N(\mathbf{1}_{(a,b]})_t$?
3. Show that for $u \geq 0$, we have

$$\mathbb{E}\left[e^{-uN(f)_t} \middle| N_t = n\right] = \frac{1}{t^n} \Big(\int_0^t e^{-uf(s)} ds \Big)^n.$$

4. Derive $\mathbb{E}\left[e^{-uN(f)_t}\right]$ ad find back the result of Question 2.
5. Compute $\mathbb{E}\left[N(f)_t\right]$ and $\text{Var}[N(f)_t]$.
6. Prove that $N(f)_t - \lambda \int_0^t f(s)ds$ is a martingale.

Exercise 4. The total claim amount of a portfolio for a year is modelled by

$$X = \sum_{j=1}^{N} C_j$$

where N is the number of claims in the year and C_j is the cost of the j-th claim. Assume that N follows a mixed Poisson distribution with random parameter Λ, *i.e.* the conditional distribution of N given $\Lambda = \lambda$ is Poisson(λ). Assume moreover that Λ is distributed according to a $\Gamma(b, b)$ distribution, for some $b > 0$. Assume that the cost of the claims $(C_j)_{j\geq 1}$ are independent and identically distributed random variables, independent of N.

1. Compute $\mathbb{E}(\Lambda)$ and $\mathrm{Var}(\Lambda)$.
2. Compute $\mathbb{E}(N)$ and $\mathrm{Var}(N)$.
3. We assume that $C_1 \sim$ Exponential (α) for some $\alpha > 0$. Show that the conditional law of X given N is a Gamma distribution and identify its parameters. What is the pure premium ?
4. Show that the conditional law of Λ given (X, N) is independent of X and that it is a Gamma distribution. Identify its parameters.

Exercise 5. Let $(\xi_i, i \geq 1)$ be a sequence of i.i.d. real-valued random variables, with second-order moments, independent of the Poisson process $N = (N_t, t \geq 0)$ with intensity $\lambda > 0$. For $t \geq 0$, we set [1]

$$X_t = \sum_{i=1}^{N_t} \xi_i.$$

1. Show that $t^{-1}X_t$ converges almost surely as $t \to \infty$ and identify its limit.

We set [2], for $t \geq 0$

$$M_t = \sqrt{N_t}\Big(\frac{X_t}{N_t} - \mu\Big).$$

3. Compute $\mathbb{E}\left[e^{iuM_t}\right]$. and derive that M_t converges in distribution as $t \to \infty$. Identify its limit.
5. Show that

$$\sqrt{t}(\frac{X_t}{t} - \mu\frac{N_t}{t})$$

converges in distribution as $t \to \infty$ and identify its limit.

Exercise 6. Let $(T_n)_{n\geq 1}$ be a renewal process and let N_t denote its counting function. We assume that the common law of the interarrival times admits a density function f on $\mathbb{R}$ (with value 0 on $(-\infty, 0]$). For $n \geq 1$, we denote by f_n the density function of T_n and F_n its cumulative distribution function.

1. Putting $X_t = 0$ on $\{N_t = 0\}$.
2. Putting $M_t = 0$ on $\{N_t = 0\}$.

1. Check that $f_1(t) = f(t)$ and show that

$$f_{n+1}(t) = \int_{\mathbb{R}} f_n(t-s) f(s) ds.$$

Define $r(t) = \mathbb{E}[N_t]$ if $t \geq 0$ and $r(t) = 0$ for $t < 0$.

2. Show that $r(t) = \sum_{n \geq 1} F_n(t)$ and that for $n \geq 2$, we have $\mathbb{P}(T_n \leq t \,|\, T_1) = F_{n-1}(t - T_1)$.
3. Show that r satisfies the renewal equation

$$r(t) = F(t) + \int_0^t r(t-s) f(s) ds, \quad t \geq 0$$

   wher F denotes the cumulative distribution fcuntion of the common interarrival times.