Feuille d'exercices n°1 : Poisson processes

Exercise 1. Let $(\tau_n, n \geq 1)$ be a sequence of IID (independent and identically distributed) non-negative random variables. Set

$$T_n = \tau_1 + ... + \tau_n, \quad n \geq 1,$$

$(T_0 = 0)$ and

$$N_t = \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0.$$

1. Give a necessary and sufficient condition for having

$$\mathbb{P}(N \text{ only makes jumps of size } 1) > 0.$$

2. Show that under this condition

$$\mathbb{P}(N \text{ only makes jumps of size } 1) = 1.$$

3. Is it possible that $(T_n, n \geq 0)$ converges to a finite limit with positive [1] probability ?
4. Compute the probability of the event $\{\exists t \geq 0 : N(t) = \infty\}$.

Exercise 2. Let N be a Poisson process with intensity $\lambda > 0$. Prove and give an interpretation of the following properties

1. $\mathbb{P}(N_h = 1) = \lambda h + o(h)$ $(h \to 0)$
2. $\mathbb{P}(N_h \geq 2) = o(h)$ $(h \to 0)$
3. $\mathbb{P}(N_h = 0) = 1 - \lambda h + o(h)$ $(h \to 0)$.
4. $\forall t \geq 0$, $\mathbb{P}(N \text{ jumps at time } t) = 0$.
5. Compute $\mathrm{Cov}(N_s, N_t)$, $\forall s, t \geq 0$.

Exercise 3. Let N be a counting process with stationary and independent increments. Assume that there exists $\lambda > 0$ such that

$$\mathbb{P}(N_h = 1) = \lambda h + o(h), \;\; \mathbb{P}(N_h \geq 2) = o(h).$$

For $u \in \mathbb{R}$, let $g_t(u) = \mathbb{E}[e^{iuN_t}]$.

1. Prove that $g_{t+h}(u) = g_t(u) g_h(u)$ for every $t, h \geq 0$.

1. En anglais le mot *positive* signifie strictement positif. Pour dire positif au sens large on dit *non-negative*. De même les termes *negative, bigger, smaller* sont à prendre au sens strict.

2. Prove that

$$\frac{d}{dt}g_t(u) = \lambda(e^{iu} - 1)g_t, \quad g_0(u) = 1.$$

3. Conclude.

Exercise 4. Let N be a Poisson process with intensity $\lambda > 0$, modelling the arrival times of the claims for an insurance company. Let T_1 denote the arrival time of the first claim. Show that the conditional law of T_1 given $N_t = 1$ is uniformly distributed over $[0, t]$.

Exercise 5. Let $(T_n, n \geq 0)$ $(T_0 = 0)$ be a renewal process and N its associated counting process. Assume that N has independent and stationary increments.

1. Show that

$$\mathbb{P}(T_1 > s + t) = \mathbb{P}(T_1 > t)\mathbb{P}(T_1 > s), \quad \forall s, t \geq 0.$$

2. Derive that N is a Poisson process.

Exercise 6.

1. Show that two independent Poisson processes cannot jump simultaneously a.s.
2. Let N^1 and N^2 be two independent Poisson processes with parameters $\lambda_1 > 0$ and λ_2 respectively. Show that the process

$$N_t = N_t^1 + N_t^2, \quad t \geq 0$$

   is a Poisson process and give its intensity.
3. Derive that the sum of n independent Poisson processes with respective intensities $\lambda_1 > 0, ..., \lambda_n > 0$ is a Poisson process and give its intensity.

Exercise 7. Insects fall into a soup bowl according to a Poisson process N with intensity $\lambda > 0$ (the event $\{N_t = n\}$ means that there are n insects in the bowl at time t). Assume that every insect is green with probability $p \in (0, 1)$ and that its colour is independent of the colour of the other insects. Show that the number of green insects that fall into the bowl, as a function of time, is a Poisson process with intensity λp.

Exercise 8. Liver transplants arrive at an operating block following a Poisson process N with intensity $\lambda > 0$. Two patients wait for a transplant. The first patient has lifetime T (before the transplant) according to an exponential distribution with parameter μ_1. The second one has lifetime T'(before the transplant) according to an exponential distribution wity parameter μ_2. The rule is that the first transplant arrival to the hospital is given to the first patient if still alive, and to the second patient otherwise. Assume that T, T' and N are independent.

1. Compute the probability that the second patient is transplanted.
2. Compute the probability that the first patient is transplanted.
3. Let X denote the number of transplants arrived at the hospital during $[0, T]$. Compute the law of X.

Exercise 9. The Bus paradox. Buses arrive at a given bus stop according to a Poisson process with intensity $\lambda > 0$. You arrive at the bus stop at time t.

1. Give a first guess for the value of the average waiting time before the following bus arrives ?
2. Let $A_t = T_{N_t+1} - t$ be the waiting time before the next bus, and let $B_t = t - T_{N_t}$ denote the elapsed time since the last bus arrival. Compute the joint distribution of (A_t, B_t) (hint : compute first $\mathbb{P}(A_t \geq x_1, B_t \geq x_2)$ for $x_1, x_2 \geq 0$).
3. Derive that the random variables A_t and B_t are independent. What are their distributions ?
4. In particular, compute $\mathbb{E}[A_t]$. Compare with your initial first guess.

Exercise 10. Law of large numbers and central limit theorem.

1. Recall and prove a law of large numbers for a Poisson process with intensity $\lambda > 0$.
2. Prove that N satisfies the following central limit theorem

$$\frac{N_t - \lambda t}{\sqrt{\lambda t}} \xrightarrow{\text{law}} \mathcal{N}(0,1) \text{ as } t \to \infty,$$

   (a) by using characteristic functions

   (b) by showing first that $(N_n - \lambda n)/\sqrt{\lambda n}$ converges in distribution as $n \to \infty$ and then $\max_{t \in [n,n+1)} (N_t - N_n)/\sqrt{n} \to 0$ in probability.

Exercise 11.

1. Give an expression for the density function of the conditional distribution of

$$(T_1, ..., T_n) \text{ given } N_t = n$$

   when N is a Poisson process with intensity λ and $0 < T_1 < ... < T_n < ...$ are its jump times.
2. Derive an expression for the density of T_i given $N_t = n$, $\forall 1 \leq i \leq n$ and similarly for (T_i, T_j) given $N_t = n$, $\forall 1 \leq i < j \leq n$.
3. Set $U_{i,j} = T_j - T_i$, $1 \leq i < j \leq n$. Give an expression for the density of $U_{i,j}$ given $N_t = n$. Derive an expression for the density of $T_n - T_{n-1}$ given $N_t = n$.

Exercise 12. Martingales. Let $X = (X_t, t \geq 0)$ be a continuous time process and $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ a *filtration*, i.e. a nested family of sigma-fields $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{A}\ \forall s \leq t$, where $\mathcal{A}$ is the sigma-field on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ over which X is defined. The process X is a *martingale* with respect to the filtration $\mathcal{F}$ if X_t is $\mathcal{F}_t$-measurable and integrable $\forall t$ and

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \quad \forall 0 \leq s \leq t.$$

Let $N = (N_t, t \geq 0)$ be a Poisson process with intensity $\lambda > 0$. Show that the three processes

1. $(N_t - \lambda t, t \geq 0)$;
2. $((N_t - \lambda t)^2 - \lambda t), t \geq 0)$;
3. $(\exp(uN_t + \lambda t(1 - e^u)), t \geq 0)$ (for a given real number u) ;

are martingales with respect to the filtration generated by N, *i.e.* $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$.

Exercise 13. Let N be a Poisson process with intensity $\lambda > 0$ and let $0 < T_1 < ... < T_n < ...$ denote its jump times.

1. Show that T_n/n converges almost surely as $n \to \infty$ and identify its limit.
2. Show that $\sum_{i \geq 1} T_i^{-2}$ converges almost surely. Let X denote its limit.
3. Show that $X_{N_t} = \sum_{i=1}^{N(t)} T_i^{-2} \to X$ a.s. as $t \to \infty$.
4. Let $(U_i, i \geq 1)$ denote a sequence of independent uniform random variables on $[0, 1]$. We admit the following result

$$n^{-2} \sum_{i=1}^{n} U_i^{-2} \underset{n \longrightarrow \infty}{\overset{\text{law}}{\rightarrow}} Z,$$

where Z is a positive random variable, whose Laplace transform is given by $\mathbb{E}[\exp(-sZ)] = \exp(-c\sqrt{s}), \forall s \geq 0$, for some $c > 0$. **The goal is to show that X and $c'Z$ have same law for some c' that we will explicitly compute**.

We asssume moreover that $(U_i, i \geq 1)$ is independent of N.

   (a) Show that for every $n \geq 1$ and every $t > 0$, the law of X_{N_t} given $N_t = n$ is the same as the law of $t^{-2} \sum_{i=1}^{n} U_i^{-2}$.

   (b) Derive that $X_{N(t)}$ has same distribution as $t^{-2} \sum_{i=1}^{N(t)} U_i^{-2}$.

   (c) Prove that

$$N(t)^{-2} \sum_{i=1}^{N(t)} U_i^{-2} \xrightarrow{\text{law}} Z \quad \text{as } t \to \infty.$$

   (d) Recall the law of large numbers for Poisson processes and conclude.

5. Derive $\mathbb{E}[X] = \infty$.